NUMERICAL QUADRATURES AND MULTIPLE ORTHOGONAL POLYNOMIALS

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One of the most important uses of standard orthogonal polynomials on the real line is in the theory of quadratures, especially in the construction of quadrature formulae of maximum, or nearly maximum, algebraic degree of exactness for integrals involving a positive measure (cf. [4, Chapter 5]). In this lecture we consider applications of multiple orthogonal polynomials in some special type of quadratures. Otherwise, multiple orthogonal polynomials are intimately related to Hermite-Padé approximants and, because of that, they are known as Hermite-Padé polynomials. A nice survey on these polynomials, as well as some their applications to various fields of mathematics (number theory, special functions, etc.) and in the study of their analytic, asymptotic properties, was given by Aptekarev [1].

Multiple orthogonal polynomials are a generalization of standard orthogonal polynomials in the sense that they satisfy r orthogonality conditions.

Let $m \geq 1$ be an integer and let w_j, $j = 1, \ldots, m$, be weight functions on the real line so that the support of each w_j is a subset of an interval E_j. Let $\vec{n} = (n_1, n_2, \ldots, n_m)$ be a vector of m nonnegative integers, which is called a *multi-index* with the length $|\vec{n}| = n_1 +$

$n_2 + \cdots + n_m$. Here, we consider only the so-called *type II multiple orthogonal polynomials* $\pi_{\vec{n}}(t)$ of degree $|\vec{n}|$. Such a monic polynomial is defined by the m orthogonality relations

$$\int_{E_j} \pi_{\vec{n}}(t) t^\ell w_j(t)\, dt = 0, \quad 0 \le \ell \le n_j - 1, \quad j = 1, \ldots, m.$$

If the polynomial $\pi_{\vec{n}}\,(t)$ is unique, then $\vec{n}$ is a *normal multi-index*. When all multi-indices are normal, we have a *complete system*. One important complete system is the AT system, in which all weight functions are supported on the same interval E $(= E_1 = E_2 = \cdots = E_m)$ and the following $|\vec{n}|$ functions: $w_1(t)$, $tw_1(t)$, $\ldots$, $t^{n_1-1}w_1(t)$, $w_2(t)$, $tw_2(t)$, $\ldots$, $t^{n_2-1}w_2(t)$, $\ldots$, $w_m(t)$, $tw_m(t)$, $\ldots$, $t^{n_m-1}w_m(t)$ form a *Chebyshev system* on E for each multi-index $\vec{n}$. In 2001 Van Assche and E. Coussement [6] proved that *for an AT system, the type II multiple orthogonal polynomial has exactly $|\vec{n}|$ zeros on E*. These multiple orthogonal polynomials can be applied to some kinds of quadratures. We consider two applications.

1. In 1994 Borges [3] considered a problem that arises in evaluation of computer graphics illumination models. Starting with that problem, he examined the problem of numerically evaluating a set of m definite integrals taken with respect to distinct weight functions but related by a common integrand and interval of integration. Here, we show a direct connection with multiple orthogonal polynomias (see [5]).

2. Second application is related to a generalization of the Birkhoff-Young quadratures [2] for analytic functions in the complex plane. We will give a characterization of such generalized quadratures in terms of multiple orthogonal polynomials and prove the existence and uniqueness of these quadratures.

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