Laboratory 2: Difference Equations. Equilibrium Points. Periodic Points. Stability

1. Build a numerical solution for the following initial value problems. Plot your data to observe patterns in the solutions. Is there an equilibrium solution? Is it stable or unstable?

   (a) $a_{n+1} = -1.2a_n + 50,\ a_0 = 1000;$

   (b) $a_{n+1} = 0.8a_n - 100,\ a_0 = 500;$

   (c) $a_{n+1} = 0.8a_n - 100, a_0 = -500;$

   (d) $a_{n+1} = a_n - 100, a_0 = 1000;$

2. For the following problems find the solution to the difference equation and the equilibrium value if one exists. Discuss the long-term behaviour of the solutions for various initial data. Clasify the equilibrium values as stable or unstable. Draw the Cobweb diagram for each equation with different initial starting points.

   (a) $a_{n+1} = -a_n + 2;$

   (b) $a_{n+1} = a_n + 2;$

   (c) $a_{n+1} = a_n + 3.2;$

   (d) $a_{n+1} = -3a_n + 4;$

   (e) $a_{n+1} = a_n^2 + 3a_n;$

3. (Newton's Method of Computing the Square Root of a PositiveNumber)

   The equation $x^2 = a$ can be written in the form $x = \frac{1}{2}(x + \frac{a}{x})$. This form leads to Newton's method

   $$x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n})$$

   (a) Show that this difference equation has two equilibrium points $\sqrt{a}$ and $-\sqrt{a}$;

   (b) Sketch a cobweb diagram for $a = 3$, $x_0 = 1$ and $x_0 = -1$.

4. Let $f(x) = -\frac{1}{2}x^2 - x + \frac{1}{2}$. Show that 1 is an asymptotically stable 2-periodic point of f.

5. Assume we are considering the survival of the whales and if the number of whales falls below a minimum *survival level* m the species will become extinct. Assume also that the population is limited by the *carrying capacity* M of the environment. That is, if the whale population is above M then it will experience a decline because the environment cannot sustain that large population level. In the following model, a_n represent the whale population after n years. Build a numerical solution for $M = 5000$, $m = 100$, $k = 0.0001$ and $a_0 = 4000$.

   $$a_{n+1} - a_n = k(M - a_n)(a_n - m).$$

   Now experiment with different values for M, m and k. Try several starting values for a_0. What does this model predict?