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MODULAR HADAMARD, RIEMANN-LIOUVILLE AND WEYL FRACTIONAL INTEGRALS

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Abstract. This paper establishes the modular inequalities for the Hadamard fractional integrals, the Riemann-Liouville fractional integrals and the Weyl fractional integrals.

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Key words. Riemann-Liouville fractional integral, Weyl fractional integral, Orlicz space, Hadamard fractional integral.

1. INTRODUCTION

This paper aims to investigate the mapping properties of the Hadamard fractional integrals $J^{\alpha}_{0+,\mu;\gamma,\sigma}f$, $J^{\alpha}_{-,\mu;\gamma,\sigma}f$, $I^{\alpha}_{0+,\mu;\gamma,\sigma}f$ and $I^{\alpha}_{-,\mu;\gamma,\sigma}f$, the Riemann-Liouville fractional integrals $R_\alpha f$ and the Weyl fractional integrals $W_\alpha f$ when $\Phi(|f(x)|)$ is integrable where Φ is a modular function. That is, we obtain the modular inequalities in the sense of [19] for the above operators.

The Riemann-Liouville fractional integrals $R_\alpha f$ were introduced to study the α^{th} order antiderivative of f when α is not an positive integer. Precisely, when α is a positive integer, the α^{th} order derivative of $R_\alpha f$ is f. In addition, the Weyl fractional integral, roughly speaking, is the dual operator of the Riemann-Liouville fractional integrals. That is, for any nonnegative Lebesgue measurable functions f and g, we have

$$\int_0^\infty R_\alpha f(x)g(x)\mathrm{d}x = \int_0^\infty f(x)W_\alpha g(x)\mathrm{d}x.$$

The reader is referred to [22] for the applications of the above integral on fractional calculus.

The Hadamard fractional integrals were introduced by Hadamard in [9]. It is related with the fractional calculus in the framework of Mellin transform, see [5, p. 388]. The studies of Hadamard fractional integrals was extended by Butzer, Kilbas and Trujillo, where a number of Hadamard type fractional integrals $J^{\alpha}_{0+,\mu;\gamma,\sigma}f$, $J^{\alpha}_{-,\mu;\gamma,\sigma}f$, $I^{\alpha}_{0+,\mu;\gamma,\sigma}f$ and $I^{\alpha}_{-,\mu;\gamma,\sigma}f$ were introduced [4, 5].

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The mapping properties for the Riemann-Liouville fractional integrals and the Weyl fractional integrals on Lebesgue spaces were established in [10] and these results were extended to weighted Lebesgue spaces in [1, 21].

The reader is referred to [4, 5] for the mapping properties of the Hadamard fractional integrals and the Hadamard type fractional integrals $J^{\alpha}_{0+,\mu;\gamma,\sigma}f$, $J^{\alpha}_{-,\mu;\gamma,\sigma}f$, $I^{\alpha}_{0+,\mu;\gamma,\sigma}f$ and $I^{\alpha}_{-,\mu;\gamma,\sigma}f$ on Lebesgue spaces endowed with the measure $\frac{\mathrm{d}t}{t}$. Moreover, for the mapping properties of the Hadamard fractional integrals and the Hadamard type fractional integrals on function space of bounded mean oscillation, amalgam spaces and function spaces of q-integral p-variation, see [13, 15, 17], respectively.

This paper is devoted to extending the mapping properties for the above fractional integrals when $\Phi(|f(x)|)$ is integrable, where Φ is a modular function. Notice that the investigations for the Hardy-Littlewood maximal operator, the Marcinkiewicz interpolation, the Fourier transform and the k-plane transform for f satisfying

$$\int_{\mathbb{R}^n} \Phi(|f(x)|)\mathrm{d}x < \infty$$

had been conducted in [3, 7, 11, 12].

We obtain our results by using the decreasing rearrangement of Lebesgue measurable function [2, Chapter 2, Section 1] and the operators of joint weak type [2, Chapter 3, Section 5]. Both notions are important tools for the study of mapping properties of operators on function spaces [2, Chapter 3, Section 5] and modular spaces [11, 12, 18]. Moreover, we also use the Hardy inequalities [6] on modular spaces to obtain our desired results.

This paper is organized as follows. The definitions and the mapping properties of the Hadamard fractional integrals $J^{\alpha}_{0+,\mu;\gamma,\sigma}f$, $J^{\alpha}_{-,\mu;\gamma,\sigma}f$, $I^{\alpha}_{0+,\mu;\gamma,\sigma}f$ and $I^{\alpha}_{-,\mu;\gamma,\sigma}f$, the Riemann-Liouville fractional integrals $R_{\alpha}f$ and the Weyl fractional integrals $W_{\alpha}f$ on Lebesgue spaces are presented in Section 2. The definition of operators of joint weak type and the Hardy inequalities on modular spaces are given in Section 3. Our main results are established in Section 4. An application of our main result on the mapping properties of the Riemann-Liouville fractional integrals, the Weyl fractional integrals and the Hadamard fractional integrals on Orlicz spaces are presented at the end of Section 4.

2. PRELIMINARIES AND DEFINITIONS

Let μ be a totally σ-finite measure on $(0, \infty)$, Let $\mathcal{M}(\mu)$ and $\mathcal{M}$ be the set of μ-measurable functions and the Lebesgue measurable functions on $(0, \infty)$, respectively.

For $0 < \alpha < 1$, the Riemann-Liouville fractional integral $R_{\alpha}f$ and the Weyl fractional integral $W_{\alpha}f$ for a locally integrable function f on $(0, \infty)$ are defined as

$$R_\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) \mathrm{d}t, \quad x > 0,$$
$$W_\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t) \mathrm{d}t, \quad x > 0,$$

respectively, where $\Gamma(\cdot)$ is the Gamma function.

In view of [10, Theorem 383], we have the following results for the Riemann-Liouville fractional integral and the Weyl fractional integral.

THEOREM 2.1. *Let $0 < \alpha < 1$ and $1 < p < \frac{1}{\alpha}$. If*

$$\frac{1}{q} = \frac{1}{p} - \alpha,$$

there exists a constant $C > 0$ such that for any $f \in L^p$, we have

$$\left(\int_0^\infty |R_\alpha f(x)|^q \mathrm{d}x \right)^{1/q} \leq C \left(\int_0^\infty |f(x)|^p \mathrm{d}x \right)^{1/p},$$
$$\left(\int_0^\infty |W_\alpha f(x)|^q \mathrm{d}x \right)^{1/q} \leq C \left(\int_0^\infty |f(x)|^p \mathrm{d}x \right)^{1/p}.$$

We also have the corresponding result for the Hadamard fractional integrals. Let $\alpha > 0$, the Hadamard fractional integral $J_{0+}^\alpha f$ for locally integrable function f is defined as

$$(J_{0+}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \left(\log \frac{x}{u} \right)^{\alpha-1} f(t) \frac{\mathrm{d}t}{t},$$

where $\Gamma(\alpha)$ is the Euler gamma function. The Hadamard fractional integral was introduced by Hadamard in [9]. The reader is also referred to [22, Section 18.3], for the applications of the Hadamard fractional integrals.

The Hadamard fractional integrals had been generalized by using the confluent hypergeometric function, which is also named as a Kummer function. The confluent hypergeometric function $\Phi[a, c; z]$ is defined for $|z| < 1$, $c > 0$ and $a \neq -j$, $j \in \mathbb{N} \cup \{0\}$ by

$$\Phi[a, c; z] = \sum_{k=0}^\infty \frac{(a)_k}{(c)_k} \frac{z^k}{k!}$$

where $(a)_k$, $k \in \mathbb{N} \cup \{0\}$ is the Pochhammer symbol [8, Section 6.1] given by

$$(a)_0 = 1, \quad (a)_k = a(a+1)\cdots(a+k-1), \quad k \in \mathbb{N}.$$

For $\beta > 0$, $\gamma \in \mathbb{R}$ and $\mu, \sigma \in \mathbb{C}$, the generalized Hadamard fractional integrals $J^{\alpha}_{0+,\mu;\gamma,\sigma}f$, $J^{\alpha}_{-,\mu;\gamma,\sigma}f$, $I^{\alpha}_{0+,\mu;\gamma,\sigma}f$ and $I^{\alpha}_{-,\mu;\gamma,\sigma}f$ are defined as

$$J^{\alpha}_{0+,\mu;\gamma,\sigma}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \left(\frac{t}{x}\right)^{\mu}\left(\log\frac{x}{t}\right)^{\alpha-1}\Phi\left[\gamma,\alpha;\sigma\log\frac{x}{t}\right]f(t)\frac{\mathrm{d}t}{t},$$
$$J^{\alpha}_{-,\mu;\gamma,\sigma}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^{\infty} \left(\frac{x}{t}\right)^{\mu}\left(\log\frac{t}{x}\right)^{\alpha-1}\Phi\left[\gamma,\alpha;\sigma\log\frac{t}{x}\right]f(t)\frac{\mathrm{d}t}{t},$$
$$I^{\alpha}_{0+,\mu;\gamma,\sigma}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \left(\frac{t}{x}\right)^{\mu}\left(\log\frac{x}{t}\right)^{\alpha-1}\Phi\left[\gamma,\alpha;\sigma\log\frac{x}{t}\right]f(t)\frac{\mathrm{d}t}{x}$$

and

$$I^{\alpha}_{-,\mu;\gamma,\sigma}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^{\infty} \left(\frac{x}{t}\right)^{\mu}\left(\log\frac{t}{x}\right)^{\alpha-1}\Phi\left[\gamma,\alpha;\sigma\log\frac{t}{x}\right]f(t)\frac{\mathrm{d}t}{x},$$

respectively.

Since $\Phi[a, c; 0] = 1$, when $\sigma = 0$, the above Hadamard fractional integral $J^{\alpha}_{0+,\mu;\gamma,0}$ reduces to the Hadamard fractional integral J^{α}_{0+}. Moreover, $J^{\alpha}_{-,\mu;\gamma,\sigma}$, $I^{\alpha}_{0+,\mu;\gamma,\sigma}$ and $I^{\alpha}_{-,\mu;\gamma,\sigma}$ become the Hadamard type fractional integrals introduced and studied in [4].

We now give the mapping properties for $J^{\alpha}_{0+,\mu;\gamma,0}$, $J^{\alpha}_{-,\mu;\gamma,\sigma}$, $I^{\alpha}_{0+,\mu;\gamma,\sigma}$ and $I^{\alpha}_{-,\mu;\gamma,\sigma}$. Let $\mathrm{d}\omega = \frac{\mathrm{d}t}{t}$ and

$$L^p_{\omega} = \left\{ f \in \mathcal{M} : \|f\|_{L^p_{\omega}} = \left(\int_0^{\infty} |f(t)|^p \frac{\mathrm{d}t}{t}\right)^{\frac{1}{p}} < \infty \right\}$$

where $\mathrm{d}t$ is the Lebesgue measure on $(0, \infty)$.

According to [5, Theorems 4 and 5], we have the following results.

Theorem 2.2. *Let $c, \gamma \in \mathbb{R}$, $\alpha > 0$ and $\mu, \sigma \in \mathbb{C}$. Suppose that $1 \leq p \leq q$ satisfy $\frac{1}{p} - \frac{1}{q} < \alpha$.*

(1) *If $\mathrm{Re}(\mu - \sigma) > c$ or $\mathrm{Re}(\mu - \sigma) = c$ and $\gamma < \frac{1}{p} - \frac{1}{q}$, then the operator $J^{\alpha}_{0+,\mu-c;\gamma,0}$ is bounded from L^p_{ω} to L^q_{ω}.*

(2) *If $\mathrm{Re}(\mu - \sigma) > -c$ or $\mathrm{Re}(\mu - \sigma) = -c$ and $\gamma < \frac{1}{p} - \frac{1}{q}$, then the operator $J^{\alpha}_{-,\mu+c;\gamma,\sigma}$ is bounded from L^p_{ω} to L^q_{ω}.*

(3) *If $\mathrm{Re}(\mu - \sigma) > c - 1$ or $\mathrm{Re}(\mu - \sigma) = c - 1$ and $\gamma < \frac{1}{p} - \frac{1}{q}$, then the operator $I^{\alpha}_{0+,\mu-c;\gamma,\sigma}$ is bounded from L^p_{ω} to L^q_{ω}.*

(4) *If $\mathrm{Re}(\mu - \sigma) > 1 - c$ or $\mathrm{Re}(\mu - \sigma) = 1 - c$ and $\gamma < \frac{1}{p} - \frac{1}{q}$, then the operator $I^{\alpha}_{-,\mu+c;\gamma,\sigma}$ is bounded from L^p_{ω} to L^q_{ω}.*

The reader is reminded that we present the mapping properties for the operators $J^{\alpha}_{0+,\mu-c;\gamma,0}$, $J^{\alpha}_{-,\mu+c;\gamma,\sigma}$, $I^{\alpha}_{0+,\mu-c;\gamma,\sigma}$ and $I^{\alpha}_{-,\mu+c;\gamma,\sigma}$ while the results in [5, Theorems 4 and 5] are mapping properties for $J^{\alpha}_{0+,\mu;\gamma,0}$, $J^{\alpha}_{-,\mu;\gamma,\sigma}$, $I^{\alpha}_{0+,\mu;\gamma,\sigma}$ and $I^{\alpha}_{-,\mu;\gamma,\sigma}$. That is, in Theorem 2.2, the parameter c appears in the indices of the operators while for the results in [5, Theorems 4 and 5], the parameter c appears in the definition of weighted Lebesgue spaces [5, (1.10)].

3. DECREASING REARRANGEMENTS AND MODULAR FUNCTIONS

In this section, we first present the notion of decreasing rearrangement of Lebesgue measurable function. This notion is related with operators of joint weak type. We also recall the definition of modular functions. It is an extension of Young's function used in the study of Orlicz spaces. At the end of this section, we present some results on the modular estimates of the Hardy operators.

We begin with a review on the notion of decreasing rearrangement. Let μ be a totally σ-finite measure on $(0,\infty)$. For any $f \in \mathcal{M}(\mu)$ and $s > 0$, write

$$d_f^\mu(s) = \mu(\{x \geq 0 : |f(x)| > s\})$$

and

$$f_\mu^*(t) = \inf\{s > 0 : d_f^\mu(s) \leq t\}, \quad t > 0.$$

We call f_μ^* the decreasing rearrangement of f with respect to μ. We say that f and g are μ-equimeasurable if $d_f^\mu(s) = d_g^\mu(s)$ for all $s > 0$.

When μ is the Lebesgue measure, we write f_μ^* by f^*. When μ is the measure $\mathrm{d}\omega = \frac{\mathrm{d}t}{t}$ on $(0,\infty)$ where $\mathrm{d}t$ is the Lebesgue measure on $(0,\infty)$, we write the decreasing rearrangement of f with respect to $\mathrm{d}\omega$ by f_ω^*.

We write $f \approx g$ if $Bf \leq g \leq Cf$, for some constants $B, C > 0$ independent of appropriate quantities involved in the expressions of f and g.

We recall the definition of linear operator of joint weak type from [2, Chapter 3, Definitions 5.1, 5.3 and 5.4].

Definition 3.1. Let μ_0, μ_1 be totally σ-finite measures on $(0,\infty)$. Suppose $1 \leq p_0 < p_1 \leq \infty$ and $1 \leq q_0, q_1 \leq \infty$ with $q_0 \neq q_1$. Let T be a linear operator whose domain is some linear subspace of $\mathcal{M}(\mu_0)$ and whose range is contained in $\mathcal{M}(\mu_1)$.

We say that T is of joint weak type $(p_0, q_0; p_1, q_1)$ if

$$\int_0^1 s^{\frac{1}{p_0}} f_{\mu_0}^*(s) \frac{\mathrm{d}s}{s} + \int_1^\infty s^{\frac{1}{p_1}} f_{\mu_0}^*(s) \frac{\mathrm{d}s}{s} < \infty$$

and there is a constant $C > 0$ such that

$$(Tf)_{\mu_1}^* \leq C \left(t^{-\frac{1}{q_0}} \int_0^{t^m} s^{\frac{1}{p_0}} f_{\mu_0}^*(s) \frac{\mathrm{d}s}{s} + t^{-\frac{1}{q_1}} \int_0^{t^m} s^{\frac{1}{p_1}} f_{\mu_0}^*(s) \frac{\mathrm{d}s}{s} \right)$$

where

$$m = \frac{\frac{1}{q_0} - \frac{1}{q_1}}{\frac{1}{p_0} - \frac{1}{p_1}}.$$

Next, we turn to the definition of modular function.

DEFINITION 3.2. A Lebesgue measurable function $\Phi : [0,\infty) \to [0,\infty)$ is called a modular function if Φ is a non-decreasing function with

$$\lim_{t\to 0^+} \Phi(t) = 0.$$

A Young's function is a modular function. For the definition of Young's function, the reader is referred to [2, Chapter 4, Definition 8.1].

For any modular function Φ and $f \in \mathcal{M}(\omega)$, we have

$$\int_0^\infty \Phi(f^*_\omega(t))\mathrm{d}t = \int_0^\infty \Phi(|f(x)|)\frac{\mathrm{d}x}{x}. \tag{1}$$

A modular function Φ is said to satisfy the $\triangle_2$ condition if there exists a constant $K > 0$ such that

$$\Phi(2t) \le K\Phi(t), \quad t > 0. \tag{2}$$

We write $\Phi \in \triangle_2$ if it satisfies the $\triangle_2$ condition. If $\Phi \in \triangle_2$, then

$$\Phi(a+b) \le \Phi(2\max(a,b)) \le C\max(\Phi(a),\Phi(b)) \le C(\Phi(a)+\Phi(b)). \tag{3}$$

We recall the definitions of two Hardy type operators used in [6]. They are related to the estimates for the Hadamard, Riemann-Liouville and Weyl fractional integrals by using the operator of joint weak type.

Let $0 < a, b < \infty$. For any $f \in \mathcal{M}$, define

$$\begin{aligned} S_a f(t) &= \frac{1}{t^{1/a}}\int_0^t f(s)s^{1/a}\frac{\mathrm{d}s}{s}, \\ \tilde{S}_b f(t) &= \frac{1}{t^{1/b}}\int_t^\infty f(s)s^{1/b}\frac{\mathrm{d}s}{s}. \end{aligned}$$

We now present the modular inequalities for the Hardy type operators S_a and $\tilde{S}_b$.

THEOREM 3.3. *Let $0 < a \le 1$ and Φ be a modular function. There exist constants $B, C > 0$ such that for any decreasing nonnegative function f,*

$$\int_0^\infty \Phi(S_a f(t))\mathrm{d}t \le B\int_0^\infty \Phi(Cf(t))\mathrm{d}t$$

if and only if there exist constants $H, K > 0$ such that

$$t^a\int_0^t \frac{\Phi(y)}{y^{a+1}}\mathrm{d}y \le H\Phi(Kt), \quad \forall t > 0. \tag{4}$$

We write $\Phi \in C_a$ if Φ satisfies (4).

Let $0 < a_0 < a_1 \le 1$. We find that for any $t > 0$

$$\begin{aligned} t^{a_0}\int_0^t \frac{\Phi(y)}{y^{a_0+1}}\mathrm{d}y = t^{a_0}\int_0^t y^{a_1-a_0}\frac{\Phi(y)}{y^{a_1+1}}\mathrm{d}y \le t^{a_0}t^{a_1-a_0}\int_0^t \frac{\Phi(y)}{y^{a_1+1}}\mathrm{d}y \\ = t^{a_1}\int_0^t \frac{\Phi(y)}{y^{a_1+1}}\mathrm{d}y. \end{aligned}$$

Therefore,

$$\Phi \in C_{a_1} \Rightarrow \Phi \in C_{a_0}. \tag{5}$$

We also have the modular inequality for $\tilde{S}_b$.

THEOREM 3.4. *Let $0 < b < \infty$ and Φ be a modular function. There exists a constant $C > 0$ such that for any decreasing nonnegative function f,*

$$\int_0^\infty \Phi(\tilde{S}_b f(t))\mathrm{d}t \le C \int_0^\infty \Phi(f(t))\mathrm{d}t$$

if and only if there exists a constant $B > 0$ such that

$$t^b \int_t^\infty \frac{\Phi(y)}{y^{b+1}}\mathrm{d}y \le B\Phi(t), \quad \forall t > 0. \tag{6}$$

We write $\Phi \in \tilde{C}_b$ if Φ satisfies (6).

Let $0 < b_0 < b_1 \le 1$. We find that for any $t > 0$

$$t^{b_1} \int_t^\infty \frac{\Phi(y)}{y^{b_1+1}}\mathrm{d}y = t^{b_1} \int_t^\infty y^{b_0-b_1} \frac{\Phi(y)}{y^{b_0+1}}\mathrm{d}y \le t^{b_1} t^{b_0-b_1} \int_t^\infty \frac{\Phi(y)}{y^{b_0+1}}\mathrm{d}y$$
$$= t^{b_0} \int_t^\infty \frac{\Phi(y)}{y^{b_0+1}}\mathrm{d}y.$$

Therefore,

$$\Phi \in \tilde{C}_{b_0} \Rightarrow \Phi \in \tilde{C}_{b_1}. \tag{7}$$

For the proofs of Theorems 3.3 and 3.4, the reader is referred to [6, Theorems 2.1 and 2.3] and [6, Theorem 4.5 (iii)], respectively. Theorems 3.3 and 3.4 are used in the following section to obtain the main results of this paper.

4. MAIN RESULT

In this section, we establish the main results of this paper, the modular estimates of the Hadamard, the Riemann-Liouville and the Weyl fractional integrals.

THEOREM 4.1. *Let $c, \gamma \in \mathbb{R}$, $\alpha > 0$, $0 \le \theta < \alpha$ and $\mu, \sigma \in \mathbb{C}$. Let $\Phi \in \triangle_2$ be a modular function. Suppose that there exist $1 < \beta < \kappa < \infty$ such that $\Phi \in C_\beta \cap \tilde{C}_\kappa$.*

(1) *If $\mathrm{Re}(\mu - \sigma) > c$ or $\mathrm{Re}(\mu - \sigma) = c$ and $\gamma < \theta$, then there exists a constant $H_\theta > 0$ depending on θ such that*

$$\int_0^\infty \Phi(t^{-\theta}(J^\alpha_{0+,\mu-c;\gamma,0} f)^*_\omega(t))\mathrm{d}t \le H_\theta \int_0^\infty \Phi(|f(t)|)\frac{\mathrm{d}t}{t}.$$

(2) *If $\mathrm{Re}(\mu - \sigma) > -c$ or $\mathrm{Re}(\mu - \sigma) = -c$ and $\gamma < \theta$, then there exists a constant $H_\theta > 0$ depending on θ such that*

$$\int_0^\infty \Phi(t^{-\theta}(J^\alpha_{-,\mu+c;\gamma,\sigma} f)^*_\omega(t))\mathrm{d}t \le H_\theta \int_0^\infty \Phi(|f(t)|)\frac{\mathrm{d}t}{t}.$$

(3) *If* $\operatorname{Re}(\mu-\sigma) > c-1$ *or* $\operatorname{Re}(\mu-\sigma) = c-1$ *and* $\gamma < \theta$*, then there exists a constant* $H_\theta > 0$ *depending on* θ *such that*

$$\int_0^\infty \Phi(t^{-\theta}(I^\alpha_{0+,\mu-c;\gamma,\sigma}f)^*_\omega(t))\mathrm{d}t \le H_\theta \int_0^\infty \Phi(|f(t)|)\frac{\mathrm{d}t}{t}.$$

(4) *f* $\operatorname{Re}(\mu-\sigma) > 1-c$ *or* $\operatorname{Re}(\mu-\sigma) = 1-c$ *and* $\gamma < \theta$*, then there exists a constant* $H_\theta > 0$ *depending on* θ *such that*

$$\int_0^\infty \Phi(t^{-\theta}(I^\alpha_{-,\mu+c;\gamma,\sigma}f)^*_\omega(t))\mathrm{d}t \le H_\theta \int_0^\infty \Phi(|f(t)|)\frac{\mathrm{d}t}{t}.$$

Proof. We just present the proof for (1) as the proofs for (2)-(4) follow similarly.

Let $1 < p_i < q_i < \infty$, $i = 0, 1$ be selected so that $1 < p_0 < \beta < \kappa < p_1 < \infty$ and

$$\frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{p_1} - \frac{1}{q_1} = \theta.$$

Since

$$\frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{p_1} - \frac{1}{q_1} = \theta < \alpha,$$

Theorem 2.2 guarantees that $J^\alpha_{0+,\mu-c;\gamma,0} : L^{p_i}_\omega \to L^{q_i}_\omega$, $i = 0, 1$ are bounded. Therefore, $J^\alpha_{0+,\mu-c;\gamma,0}$ is of joint weak type $(p_0, q_0, p_1.q_1)$ [2, Chapter 4, Theorem 4.11].

Since

$$m = \frac{\frac{1}{q_0} - \frac{1}{q_1}}{\frac{1}{p_0} - \frac{1}{p_1}} = 1,$$

we have

$$(J^\alpha_{0+,\mu-c;\gamma,0}f)^*_\omega(t) \le K_\theta \left(t^{-\frac{1}{q_0}} \int_0^t s^{\frac{1}{p_0}} f^*_\omega(s)\frac{\mathrm{d}s}{s} + t^{-\frac{1}{q_1}} \int_0^t s^{\frac{1}{p_1}} f^*_\omega(s)\frac{\mathrm{d}s}{s} \right) \tag{8}$$

for some K_θ depending on θ.

We find that

$$\begin{aligned} &t^{-\frac{1}{q_0}} \int_0^t s^{\frac{1}{p_0}} f^*_\omega(s)\frac{\mathrm{d}s}{s} + t^{-\frac{1}{q_1}} \int_0^t s^{\frac{1}{p_1}} f^*_\omega(s)\frac{\mathrm{d}s}{s} \\ &= t^{\frac{1}{p_0}-\frac{1}{q_0}} \frac{1}{t^{\frac{1}{p_0}}} \int_0^t s^{\frac{1}{p_0}} f^*_\omega(s)\frac{\mathrm{d}s}{s} + t^{\frac{1}{p_1}-\frac{1}{q_1}} \frac{1}{t^{\frac{1}{p_1}}} \int_0^t s^{\frac{1}{p_1}} f^*_\omega(s)\frac{\mathrm{d}s}{s} \\ &= t^\theta S_{\frac{1}{p_0}} f^*_\omega(t) + t^\theta \tilde{S}_{\frac{1}{p_1}} f^*_\omega(t). \end{aligned} \tag{9}$$

Therefore, (8) and (9) yield

$$t^{-\theta}(J^\alpha_{0+,\mu-c;\gamma,0}f)^*_\omega(t) \le K_\theta \left(S_{\frac{1}{p_0}} f^*_\omega(t) + \tilde{S}_{\frac{1}{p_1}} f^*_\omega(t) \right).$$

By applying the modular $\int_0^\infty \Phi(\cdot)\mathrm{d}t$ on both sides of the above inequality, (2) and (3) give

$$\begin{aligned}&\int_0^\infty \Phi(t^{-\theta}(J^\alpha_{0+,\mu-c;\gamma,0}f)^*_\omega(t))\mathrm{d}t\\&\le K_\theta\left(\int_0^\infty \Phi(S_{\frac{1}{p_0}}f^*_\omega(t))\mathrm{d}t+\int_0^\infty \Phi(\tilde{S}_{\frac{1}{p_1}}f^*_\omega(t))\mathrm{d}t\right).\end{aligned}$$

As $1<p_0<\beta<\kappa<p_1<\infty$, (5) and (7) assure that $\Phi\in C_{p_0}\cap\tilde{C}_{p_1}$. Consequently, Theorem 3.3 and 3.4 show that

$$\int_0^\infty \Phi(t^{-\theta}(J^\alpha_{0+,\mu-c;\gamma,0}f)^*_\omega(t))\mathrm{d}t\le K_\theta\int_0^\infty \Phi(Df^*_\omega(t))\mathrm{d}t$$

for some $D>0$. As $\Phi\in\triangle_2$, (1) yields

$$\int_0^\infty \Phi(t^{-\theta}(J^\alpha_{0+,\mu-c;\gamma,0}f)^*_\omega(t))\mathrm{d}t\le H_\theta\int_0^\infty \Phi(|f(x)|)\frac{\mathrm{d}x}{x}$$

for some constant $H_\theta>0$ depending on θ. □

We have similar results for the Riemann-Liouville integrals and the Weyl fractional integrals.

THEOREM 4.2. *Let $0<\alpha<1$ and let $\Phi\in\triangle_2$ be a modular function. Suppose that there exist $1<\beta<\kappa<\frac{1}{\alpha}$ such that $\Phi\in C_\beta\cap\tilde{C}_\kappa$. We have a constant $C>0$ such that*

$$\begin{aligned}\int_0^\infty \Phi(t^{-\alpha}(R_\alpha f)^*(t))\mathrm{d}t&\le C\int_0^\infty \Phi(|f(x)|)\mathrm{d}x,\\\int_0^\infty \Phi(t^{-\alpha}(W_\alpha f)^*(t))\mathrm{d}t&\le C\int_0^\infty \Phi(|f(x)|)\mathrm{d}x.\end{aligned}$$

As the proof for the preceding theorem follows from the proof of Theorem 4.1, for brevity, we skip the details and leave it to the reader.

As an application of Theorem 4.2, we have the modular inequalities for one-sided fractional maximal operators. For any $0<\alpha<1$, the one-sided maximal operators are defined as

$$\begin{aligned}(M^-_\alpha)f(x)&=\sup_{0<h<x}h^{\alpha-1}\int_{x-h}^x|f(t)|\mathrm{d}t,\\(M^+_\alpha)f(x)&=\sup_{0<h}h^{\alpha-1}\int_x^{x+h}|f(t)|\mathrm{d}t.\end{aligned}$$

Since $(M^-_\alpha)f(x)\le(R_\alpha|f|)(x)$ and $(M^+_\alpha)f(x)\le(W_\alpha|f|)(x)$, we have the following results.

Corollary 4.3. *Let* $0 < \alpha < 1$ *and let* $\Phi \in \triangle_2$ *be a modular function. Suppose that there exist* $1 < \beta < \kappa < \frac{1}{\alpha}$ *such that* $\Phi \in C_\beta \cap \tilde{C}_\kappa$. *We have a constant* $C > 0$ *such that*

$$\int_0^\infty \Phi(t^{-\alpha}(M_\alpha^- f)^*(t))\mathrm{d}t \leq C \int_0^\infty \Phi(|f(x)|)\mathrm{d}x,$$
$$\int_0^\infty \Phi(t^{-\alpha}(M_\alpha^+ f)^*(t))\mathrm{d}t \leq C \int_0^\infty \Phi(|f(x)|)\mathrm{d}x.$$

We now present another application of Theorems 4.1 and 4.2 on the mapping properties of the Hadamard, the Riemann-Liouville and the Weyl fractional integrals on Orlicz spaces.

We recall the definition of Orlicz spaces [2, Chapter 4, Definition 8.6].

Definition 4.4. Let μ be a totally σ-finite measure on $(0, \infty)$ and let Φ be a Young's function. The Orlicz space $L_\Phi(\mu)$ consists of those $f \in \mathcal{M}$ such that

$$\|f\|_{L_\Phi(\mu)} = \inf\{\lambda > 0 : \int_0^\infty \Phi(|f(x)|/\lambda)\mathrm{d}\mu \leq 1\} < \infty.$$

The reader is referred to [20] for the study and applications of Orlicz spaces.

The following gives the definition of the Lorentz-Orlicz space [12, Definition 4.4]. The Lorentz-Orlicz space is used to characterize the mapping properties of the Hadamard, the Riemann-Liouville and the Weyl fractional integrals on Orlicz spaces.

Definition 4.5. Let $a \in \mathbb{R}$, μ be a totally σ-finite measure on $(0, \infty)$ and Φ be a Young's function. The Lorentz-Orlicz space $L_{\Phi,a}(\omega)$ consists of those $f \in \mathcal{M}$ such that

$$\|f\|_{L_{\Phi,a}(\mu)} = \inf\{\lambda > 0 : \int_0^\infty \Phi(t^a f_\mu^*(t)/\lambda)\mathrm{d}t \leq 1\} < \infty.$$

When μ is the Lebesgue measure, we write $L_\Phi(\mu) = L_\Phi$. $L_{\Phi,a}(\mu) = L_{\Phi,a}$. The Lorentz-Orlicz space is a generalization of the Lorentz spaces, see [12].

According to Theorems 4.1, 4.2 and Definition 4.5, we establish the following mapping properties for the Hadamard, the Riemann-Liouville and the Weyl fractional integrals on Orlicz spaces.

Corollary 4.6. *Let* $c, \gamma \in b$, $\alpha > 0$, $0 \leq \theta < \alpha$ *and* $\mu, \sigma \in \mathbb{C}$. *Let* $\Phi \in \triangle_2$ *be a modular function. Suppose that there exist* $1 < \beta < \kappa < \infty$ *such that* $\Phi \in C_\beta \cap \tilde{C}_\kappa$.

(1) *If* $\mathrm{Re}(\mu - \sigma) > c$ *or* $\mathrm{Re}(\mu - \sigma) = c$ *and* $\gamma < \theta$, *then there exists a constant* $H_\theta > 0$ *depending on* θ *such that*

$$\|J^\alpha_{0+,\mu-c;\gamma,0} f\|_{L_{\Phi,-\theta}(\omega)} \leq H_\theta \|f\|_{L_\Phi(\omega)}.$$

(2) *If* $\mathrm{Re}(\mu - \sigma) > -c$ *or* $\mathrm{Re}(\mu - \sigma) = -c$ *and* $\gamma < \theta$, *then there exists a constant* $H_\theta > 0$ *depending on* θ *such that*

$$\|J^\alpha_{-,\mu+c;\gamma,\sigma} f\|_{L_{\Phi,-\theta}(\omega)} \leq H_\theta \|f\|_{L_\Phi(\omega)}.$$

(3) *If* $\operatorname{Re}(\mu-\sigma) > c-1$ *or* $\operatorname{Re}(\mu-\sigma) = c-1$ *and* $\gamma < \theta$, *then there exists a constant* $H_\theta > 0$ *depending on* θ *such that*

$$\|I^{\alpha}_{0+,\mu-c;\gamma,\sigma} f\|_{L_{\Phi,-\theta}(\omega)} \le H_\theta \|f\|_{L_\Phi(\omega)}.$$

(4) *f* $\operatorname{Re}(\mu-\sigma) > 1-c$ *or* $\operatorname{Re}(\mu-\sigma) = 1-c$ *and* $\gamma < \theta$, *then there exists a constant* $H_\theta > 0$ *depending on* θ *such that*

$$\|I^{\alpha}_{-,\mu+c;\gamma,\sigma} f\|_{L_{\Phi,-\theta}(\omega)} \le H_\theta \|f\|_{L_\Phi(\omega)}.$$

We have the corresponding results for the Riemann-Liouville and Weyl fractional integrals.

COROLLARY 4.7. *Let* $0 < \alpha < 1$ *and let* $\Phi \in \triangle_2$ *be a modular function. Suppose that there exist* $1 < \beta < \kappa < \frac{1}{\alpha}$ *such that* $\Phi \in C_\beta \cap \tilde{C}_\kappa$. *We have a constant* $C > 0$ *such that*

$$\|R_\alpha f\|_{L_{\Phi,-\alpha}} \le C\|f\|_{L_\Phi}, \|W_\alpha f\|_{L_{\Phi,-\alpha}} \le C\|f\|_{L_\Phi}.$$

The above corollaries can also be obtained by using the interpolation developed in [14, 18]. Furthermore, Corollary 4.7 also yields the mapping properties for the one-sided maximal operators M^-_α and M^+_α on Orlicz spaces and Lorentz-Orlicz spaces. In addition, we can also obtain the mapping properties for R_α and W_α by using the extrapolation such as the results in [16]. For simplicity, we skip the details and leave it to the reader.

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