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SOME RESULTS ON BAIRENESS IN GENERALIZED TOPOLOGICAL SPACES

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Abstract. The aim of this paper is to extend some results on the Baire category in generalized topological spaces. We will apply the Banach-Mazur game to characterize Baireness in generalized topological spaces. Moreover, we will introduce a new separation axiom for generalized topological spaces which provides opportunity to generalize the Banach category theorem for locally compact generalized topological spaces.

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1. INTRODUCTION

Following [1, 2], a generalized topology on a nonempty set X is a collection of subsets of X which is closed under arbitrary unions and contains the empty set. A set X with a generalized topology τ, is called a generalized topological space and is denoted by (X, τ). As in usual topology, every element of a generalized topology is called an open subset of X and closed sets are the complements of open subsets of X. A generalized topological space (X, τ) is called strong [3], if $X \in \tau$. In this paper, we will assume that all generalized topological space are strong. Let us recall basic definitions from [8, 9].

DEFINITION 1.1. A set A in a generalized topological space (X, τ) is called

(i) τ-dense in X [4, 5] if $\overline{A} = X$.
(ii) τ-nowhere dense [4, 5] if $\text{int}\overline{A} = \emptyset$.
(iii) of the first category in X if there is a sequence $\{A_n\}$ of nowhere dense subsets of X such that $A = \bigcup_{n=1}^{\infty} A_n$.
(iv) of the second category in X if it is not of the first category in X.
(v) residual in X if $X \setminus A$ is of the first category.

DEFINITION 1.2. A generalized topological space (X, τ) is called Baire if for every sequence $\{A_n\}$ of dense open subsets of X, the set $\bigcap_{n=1}^{\infty} A_n$ is dense in X.

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Li and Lin in [9, Theorem 5.3] have shown that in a generalized topological space (X, τ), the following conditions are equivalent.

(i) X is Baire.
(ii) Every nonempty residual subset of X is dense in X.
(iii) Every nonempty open subset of X is of the second category in X.

In [8, Theorem 5.1], it is shown that in a special class of generalized metric spaces, Baireness is equivalent to not having a winning strategy for the first player in the Banach-Mazur game. In this paper, we extend this result by showing that the Banach-Mazur game can characterize Baireness in some generalized topological spaces.

The Baire category theorem is one of the fundamental results in general topology and functional analysis [6,10,11]. We propose a new separation axiom to provide a richer structure on generalized topological spaces, which help us extend some known results in analysis. In particular, we prove the Baire category theorem in the locally compact case. Moreover, we will show that under certain circumstances Baireness is preserved in generalized topological spaces satisfying our separation axiom. Our method suggests new separation axioms may be needed to extend some other results in topology and analysis for generalized topological spaces.

2. MAIN RESULTS

We start this section by introducing the following topological game, which is known as *"Banach-Mazur game"* or *"Choquet game"* (see e.g. [7, 11–13]).

Let X be a topological space. A Banach-Mazur game $BM(X)$ is played by two players α and β, who select nonempty open subsets of X. The player β starts a game by selecting a nonempty open subset V_1 of X. In return, the α-player chooses a nonempty open subset W_1 of V_1. In general, at the n-th stage of the game, $n \geq 1$, player β chooses a nonempty open subset $V_n \subset W_{n-1}$ and α answers by a nonempty open subset W_n of V_n. Proceeding in this fashion, the players generate a sequence $(V_n, W_n)_{n=1}^{\infty}$ which is called a *play*. The player α is said to be the *winner* of the play $(V_n, W_n)_{n=0}^{\infty}$ if $\bigcap_{n\geq 1} V_n = \bigcap_{n\geq 1} W_n \neq \emptyset$; otherwise the player β wins this play.

A *partial play* is a finite sequence of sets consisting of the first few moves of a play. A *strategy* for player α is a rule by means of which the player makes his/her choices. Here is a more formal definition of the notion strategy. A strategy s for the player α is a sequence of mappings $s = \{s_n\}$, which is inductively defined as follows:

The domain of s_1 is the set of all open subsets of X and s_1 assigns to each nonempty open set $V_1 \subset X$, a nonempty open subset $W_1 = s_1(V_1)$ of V_1.

In general, if a partial play $(V_1, \ldots, W_{n-1})$ has already been specified, where $W_i = s_i(V_1, \ldots, V_i)$, $1 \leq i \leq n-1$. Then the domain of s_n would be the set

$$\{(V_1, W_1, \ldots, W_{n-1}, V)\colon\ V \subset W_{n-1} \text{ can be the next move of } \beta\text{-player}\}$$

and it assigns to each choice $V_n \subset W_{n-1}$ some nonempty open subset

$$W_n = s_n(V_1, W_1, \dots, W_{n-1}, V_n)$$

of V_n.

An *s-play* is a play in which α selects his/her moves according to the strategy s. The strategy s for the player α is said to be a *winning strategy* if every s-play is won by α. A space X is called *α-favorable* if there exists a winning strategy for α in $BM(X)$.

The concept of a Banach-Mazur game is closely related to the notion of Baire spaces [11, 12]. It is easy to verify that every α-favorable topological space is a Baire space. The following example shows that this is not true in generalized topological spaces.

Example 2.1. Let $X = \{1, 2, 3\}$ and $\tau = \{\emptyset, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$. One can easily verify that α has a winning strategy in $BM(X)$. However, $\{2, 3\}$ is union of nowhere dense sets $\{2\}$ and $\{3\}$. So the generalized topological space (X, τ) is not Baire.

In general topology, it is known that X is a Baire space if and only if the player β does not have a winning strategy in the game $BM(X)$ [11]. In spite of that fact, α-favorability does not imply Baireness in generalized topological spaces. The following result shows that Baire spaces are β-unfavorable.

The proof of the following result may be known in the literature, however, we give it here for the sake of completion.

Theorem 2.2. *Suppose that (X, τ) is a generalized topological space which is also a Baire space. Then the player β has no winning strategy in $BM(X)$.*

Proof. Let t be a strategy for the player β and $U_1 = t(X)$ be the first choice of the player β. Let $\mathcal{P}$ denote the set of all t-plays $p = (U_{i,p}, V_{i,p})$ such that for each distinct elements $p, p' \in \mathcal{P}$, $V_{i,p} \cap V_{i,p'} = \emptyset$ for some $i \geq 1$. Suppose that $W_n := \bigcup\{V_{n,p} : p \in \mathcal{P}\}$ for $n \in \mathbb{N}$. We claim that each W_n is dense in U_1. Suppose that $V_0 \cap W_n = \emptyset$ for some nonempty open subset V_0 of U_1. Then the partial play (U_1, V_0) has an extension to some $p_0 \in \mathcal{P}$. But then $V_{n,p_0} \subset W_n \cap V_0$. This contradiction proves our claim. Since X is Baire, $\bigcap_{n=1}^{\infty} W_n$ is dense in X. Let $x \in \bigcap_{n=1}^{\infty} W_n$. Then there is an unique element $p = (U_{i,p}, V_{i,p})$ such that $x \in V_{n,p}$ for each $n \in \mathbb{N}$. But then $x \in \bigcap_{n=1}^{\infty} V_{n,p} \neq \emptyset$. Therefore t is not a winning strategy for the player β. □

In the next theorem, we will show that the converse of the above result holds under certain circumstances.

Theorem 2.3. *Let (X, τ) be a generalized topological space with the following property*

$$V_1, V_2 \in \tau \text{ and } V_1 \cap V_2 \neq \emptyset \text{ implies that } \operatorname{int}(V_1 \cap V_2) \neq \emptyset. \tag{1}$$

If β has no winning strategy in $BM(X, \tau)$, then (X, τ) is a Baire space.

Proof. Suppose, to the contrary, some $U \in \tau$ is of the first category. There is a sequence $\{E_n\}$ of nowhere dense subsets of X such that $U \subseteq \bigcup_{n=1}^{\infty} \overline{E_n}$. We define a strategy t for the player β as follows. The first move of β is $s_1(X) = V_1 = \mathrm{int}(U \cap (X \setminus \overline{E_1})$. In general, in step $n > 1$, when $V_1, U_1, \ldots, V_n, U_{n-1}$ are specified the player β chooses the nonempty open set $V_n = s_n(U_1, \ldots, U_{n-1}) = \mathrm{int}(U_{n-1} \cap (X \setminus \overline{E_n})$. Then

$$\bigcap_{n=1}^{\infty} U_n \subseteq \bigcap_{n=1}^{\infty} U \cap (X \setminus \overline{E_n}) = U \setminus \bigcup_{n=1}^{\infty} \overline{E_n} = \emptyset.$$

Therefore t is a winning strategy for the player β, which is a contradiction. $\square$

The following result follows immediately from Theorem 2.2 and Theorem 2.3.

COROLLARY 2.4. *Let (X, τ) be a generalized topological space which satisfies* (1)*. Then (X, τ) is Baire if and only if the player β has no winning strategy in $BM(X)$.*

Since the structure of generalized topological spaces is weaker than the structure of topological spaces, in some situations, the same separation axioms are not useful. In the following, we define a new separation axiom which coincides with T_2 in topological spaces but is different in generalized topological spaces.

DEFINITION 2.5. A generalized topological space (X, τ) is called a T_k space if for every compact subset K of X and a point $p \in X \setminus K$ there exist subsets $U_1, U_2 \in \tau$ such that $p \in U_1$, $K \subset U_2$ and $U_1 \cap U_2 = \emptyset$.

Clearly, every T_k generalized topological space is Hausdorff and every Hausdorff topological space is a T_k space. As it is shown in the following example, these concepts are different in generalized topological spaces.

EXAMPLE 2.6. Let $X = \{a, b, c, d\}$ and

$$\begin{aligned}\tau \;=\; & \Big\{\emptyset, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \\ & \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, \{a,b,c,d\}\Big\}.\end{aligned}$$

Then (X, τ) is a Hausdorff generalized topological space but it is not a T_k-space, since, for example, the point a and compact set $\{b, c, d\}$ can not be separated by elements of τ.

DEFINITION 2.7. A generalized topological space (X, τ) is called locally compact if every point of X has a neighborhood with compact closure.

In the following, we will give an example of a generalized topological locally compact space which is not a topological space.

Example 2.8. Let $X = \mathbb{N}$ and τ be the generalized topology generated by $\mathcal{B} = \{\{m, n\} : m, n \in \mathbb{N}\}$. Obviously every compact subset of $(\mathbb{N}, \tau)$ is finite. Let K be a nonempty compact subset of $(\mathbb{N}, \tau)$ and $n \in \mathbb{N} \setminus K$. Let $m_1, m_2 \in \mathbb{N} \setminus (K \cup \{n\})$ be such that $m_1 \neq m_2$. Set $U_1 = \{n, m_1\}, U_2 = K \cup \{m_2\}$. Then $U_1, U_2 \in \tau$, $U_1 \cap U_2 = \emptyset$, $n \in U_1$ and $K \subset U_2$. Therefore (X, τ) satisfies T_k. Moreover, each nonempty open set G of X has at least two elements, say $n_1, n_2 \in G$. Then $K = \{n_1, n_2\}$ is an open compact subset of G. Therefore $(\mathbb{N}, \tau_2)$ is a locally compact T_k generalized topological space.

It is known that in a Hausdorff topological spaces, every compact set is closed. However, this is not true in generalized topological space. For example, the set $\{a, b, c\}$ in Example 2.6, is a compact subset of X which is not closed. The next result shows that in T_k generalized topological spaces, compact sets are closed.

Proposition 2.9. *If (X, τ) is a generalized T_k topological space, then any compact subset of X is closed.*

Proof. Let K be a compact subset of a generalized T_k topological space (X, τ). We will show that $X \setminus K$ is open. If $x \in X \setminus K$, there exist subsets $U_1, U_2 \in \tau$ such that $x \in U_1$, $K \subset U_2$ and $U_1 \cap U_2 = \emptyset$. Therefore $U_1 \subset X \setminus K$, which means that x is an interior point of $X \setminus K$. Since x was arbitrary, $X \setminus K$ is open. $\square$

The following result states that the Cantor intersection theorem is true in T_k generalized topological spaces.

Proposition 2.10. *Let (X, τ) be a generalized topological space. Then the following statements hold.*

(i) *If A and B are two subset of X such that $\overline{A}$ and $\overline{B}$ are compact, then $\overline{A \cup B} = \overline{A} \cup \overline{B}$.*
(ii) *If (X, τ) is a T_k space and $\{K_\alpha : \alpha \in A\}$ is a collection of compact subsets of X which has the finite intersection property (i.e. $\bigcap_{\alpha \in F} K_\alpha \neq \emptyset$, for every finite subset $F \subseteq A$), then $\bigcap_{\alpha \in A} K_\alpha \neq \emptyset$.*

Proof. To prove (i), note first that $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. Let $x \in \overline{A \cup B}$, but $x \notin \overline{A} \cup \overline{B}$. There exists two open sets U_1, U_2 such that $x \in U_1$, $\overline{A} \cup \overline{B} \subseteq U_2$ and $U_1 \cap U_2 = \emptyset$. Since $x \in \overline{A \cup B}$ then $U_1 \cap (A \cup B) \neq \emptyset$, which is a contradiction.

We prove (ii) by contraposition. Suppose that $\{K_\alpha : \alpha \in A\}$ is a collection of compact subsets of X and $\bigcap_{\alpha \in A} K_\alpha = \emptyset$. Fix one of the K_α's, say K_{α_1}. Since $\bigcap_{\alpha \in A} K_\alpha = \emptyset$, every element of K_{α_1} is in the complement K_α^c of K_α for some α. This means that $\{K_\alpha^c\}_{\alpha \in A}$ covers K_{α_1}. By Proposition 2.9, each K_α^c is open, so by the compactness of K_{α_1}, there is a finite subset $F_0 \subseteq A$ such that $K_{\alpha_1} \subseteq \bigcup_{\alpha \in F_0} K_\alpha^c$. Put $F = \{\alpha_1\} \cup F_0$. Then $\bigcap_{\alpha \in F} K_\alpha = \emptyset$. $\square$

The following result is a special case of Proposition 2.10.

COROLLARY 2.11. *Let A, B be two subsets of a compact T_k generalized topological space (X, τ). Then $\overline{A \cup B} = \overline{A} \cup \overline{B}$.*

THEOREM 2.12. *Every compact T_k generalized topological space is a compact Hausdorff topological space.*

Proof. Let (X, τ) be a compact T_k generalized topological space and let $U_1, U_2 \in \tau$. So $X \setminus U_1$ and $X \setminus U_2$ are closed. Then by DeMorgan's law and Corollary 2.11,

$$\begin{aligned} \overline{X \setminus (U_1 \cap U_2)} &= \overline{(X \setminus U_1) \cup (X \setminus U_2)} = \overline{X \setminus U_1} \cup \overline{X \setminus U_2} \\ &= (X \setminus U_1) \cup (X \setminus U_2) = X \setminus (U_1 \cap U_2) \end{aligned}$$

Hence $U_1 \cap U_2$ is open. Therefore τ is closed under finite union. □

Let (X, τ) be a generalized topological space. Then (X, τ) is called locally compact if every point x of X has an open set with compact closure. We also need the following.

LEMMA 2.13. *Let (X, τ) be a locally compact generalized topological space with the following property:*

$$U_1 \cap U_2 \neq \emptyset \quad \textit{implies} \quad \operatorname{int}(U_1 \cap U_2) \neq \emptyset$$

where $U_1, U_2 \in \tau$. Suppose that U is a nonempty open subset of X, then there is a nonempty open set V with compact closure such that $V \subset \overline{V} \subset U$.

Proof. Let $x \in U$. By local compactness, there exists an open set W such that $x \in W$ and $\overline{W}$ is compact. Set $V = \operatorname{int}(U \cap W)$, then $\overline{V} \subseteq \overline{W}$ and since $(\overline{W}, \tau|_{\overline{W}})$ is compact, $\overline{V}$ is also compact in (X, τ). □

Now, we are ready to state Baire's category theorem in locally compact generalized topological spaces.

THEOREM 2.14. *If (X, τ) is a locally compact T_k generalized topological space which satisfies the following property:*

$$U_1 \cap U_2 \neq \emptyset \quad \textit{implies} \quad \operatorname{int}(U_1 \cap U_2) \neq \emptyset$$

for every $U_1, U_2 \in \tau$, then player α has a winning strategy in $BM(X)$.

Proof. We define inductively a strategy $t = \{t_n\}$ for the payer α as follows. Let the player β start a play by choosing a nonempty open set V_1. By Lemma 2.13, we can find a nonempty open set U_1 with compact closure such that $U_1 \subseteq \overline{U}_1 \subseteq V_1$. Define $U_1 = t_1(V_1)$ as the first choice of the player α.

In step $n > 1$, when the partial play $p_n = (V_1, U_1, \cdots, V_n)$ is determined, by applying Lemma 2.13 once again, the player α chooses a set U_n with compact closure such that $U_n \subseteq \overline{U}_n \subseteq V_n$. Define $U_n = t_n(V_1, \ldots, V_n)$. By Proposition 2.10, $\bigcap_{n=1}^{\infty} \overline{U_n} \neq \emptyset$. Therefore

$$\emptyset \neq \bigcap_{n=1}^{\infty} \overline{U}_n \subseteq \bigcap_{n=1}^{\infty} V_n = \bigcap_{n=1}^{\infty} U_n.$$

Therefore t is a winning strategy for the player α. □

The following result follows from Theorem 2.14 and Corollary 2.4.

Corollary 2.15 (The Baire Category Theorem). *If (X, τ) is a locally compact T_k generalized topological space which satisfies the following property:*

$$U_1 \cap U_2 \neq \emptyset \quad \textit{implies} \quad \mathrm{int}(U_1 \cap U_2) \neq \emptyset$$

for every $U_1, U_2 \in \tau$, then X is a Baire space.

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