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AN IMPROVEMENT OF CAUCHY RADIUS FOR THE ZEROS OF A POLYNOMIAL

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Abstract. For a given polynomial

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

of degree n with complex coefficients, the Cauchy radius r_0 is a unique positive root of the equation

$$|a_n|\, t^n - \left(|a_{n-1}|\, t^{n-1} + |a_{n-2}|\, t^{n-2} + \cdots + |a_1|\, t + |a_0|\right) = 0.$$

It refers to a radius of the circular region $|z| \leq r_0$ in which all the zeros of $p(z)$ lie. The basic aim has been to determine the smallest radius, thereby, minimizing the area of the circular region. In this present paper, we have obtained a result which gives an improvement of the Cauchy radius. Also, we produce an annular region whose center is different from the origin in which the zeros of $p(z)$ lie. Moreover, in many cases, our results give better approximations for estimating the region of polynomial zeros than that obtained from many other well-known results.

MSC 2020. 30C15, 30C10, 26C10.

Key words. Zeros of polynomials, annular region, Cauchy radius.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

be a polynomial of degree n with complex coefficients. Concerning the bounds for the moduli of the zeros of $p(z)$, Cauchy [3] (see also [12, Chapter VII, Section 27, Theorem 27.1]) introduced the following result.

Theorem 1.1 ([12]). *All the zeros of $p(z)$ lie in the closed circular region*

$$|z| \leq r_0,$$

where r_0 (also called Cauchy radius of $p(z)$ and is denoted by $\rho[p(z)]$) is the only positive root of the equation

$$|a_n|\, t^n - \left(|a_{n-1}|\, t^{n-1} + |a_{n-2}|\, t^{n-2} + \cdots + |a_1|\, t + |a_0|\right) = 0.$$

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An improvement of Cauchy radius of $p(z)$ was found in [14, Theorem 8.3.1] by Rahman and Schmeisser that can be written as follows.

THEOREM 1.2 ([14]). *All the zeros of* $p(z)$ *lie in the closed circular region*

$$|z| \leq \rho\left[\left(a_n z^k - a_{n-k}\right) p\left(z\right)\right],$$

where k *is the smallest positive integer such that* $a_{n-k} \neq 0$*. Moreover*

$$\rho\left[\left(a_n z^k - a_{n-k}\right) p\left(z\right)\right] \leq \rho\left[p\left(z\right)\right].$$

In the literature there are some results (see [1, 4, 7–9, 13–15]) on the refinement and improvement of Theorem 1.1.

In this paper, we have obtained some new results on polynomial zeros. The upper bound of our first result gives an improvement of the Cauchy radius of $p(z)$ in Theorem 1.1. The second result produces a circular or an annular region whose center is different from the origin. More precisely, we prove the following.

THEOREM 1.3. *All the zeros of* $p(z)$ *lie in the closed circular region*

$$|z| \leq t_0,$$

where t_0 *is an unique positive root of the equation*

$$|a_n|^2 t^{2n} - \sum_{j=1}^{n} |c_{n-j}| t^{n-j} = 0, \quad \textit{where } c_{n-j} = \sum_{k=0}^{n-j} a_k \lambda_{n-j-k};\ j = 1, 2, \ldots, n$$

and the values $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ *are determined by solving the* n *equations*

$$\sum_{k=0}^{j} a_{n-j+k} \lambda_{n-k} = 0;\ j = 1, 2, \ldots, n$$

with $\lambda_n = a_n$*. Moreover, when* $a_0 \neq 0$*, the zeros of* $p\left(z\right)$ *lie in the closed annular region*

$$\frac{1}{t_0'} \leq |z| \leq t_0,$$

where t_0' *is an unique positive root of the equation*

$$|a_0|^2 t^{2n} - \sum_{j=1}^{n} \left|c_j^*\right| t^{n-j} = 0,$$

where

$$c_j^* = \sum_{k=0}^{n-j} a_{n-k} \lambda_{j+k}^*;\ j = 1, 2, \ldots, n$$

and the values $\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*$ *are determined by solving the* n *equations*

$$\sum_{k=0}^{j} a_{j-k} \lambda_k^* = 0;\ j = 1, 2, \ldots, n$$

with $\lambda_0^* = a_0$*.*

Using Theorem 1.3, we obtain the following result.

THEOREM 1.4. *All the zeros of* $p(z)$ *lie in the closed circular region*

$$|z - z_0^*| \le \rho_0,$$

where ρ_0 *is an unique positive root of the equation*

$$|a_n|^2 t^{2n} - \sum_{j=1}^{n} |c_{n-j}| t^{n-j} = 0, \text{ where } c_{n-j} = \sum_{k=0}^{n-j} d_k \lambda_{n-j-k};\ j = 1, 2, \ldots, n$$

and the values $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ *are determined by solving the* n *equations*

$$\sum_{k=0}^{j} d_{n-j+k}\lambda_{n-k} = 0;\ j = 1, 2, \ldots, n \text{ with } \lambda_n = d_n = a_n,$$

$$\begin{aligned} d_p = &\binom{n}{n-p} a_n (z_0^*)^{n-p} + \binom{n-1}{n-p-1} a_{n-1} (z_0^*)^{n-p-1} + \cdots \\ &+ \binom{n-k}{n-p-k} a_{n-k} (z_0^*)^{n-p-k} + \cdots + \binom{p+1}{1} a_{p+1} z_0^* + \binom{p}{0} a_p; \\ &p = 0, 1, 2, \ldots, n \end{aligned}$$

with

$$\binom{0}{0} = 1,\ (z_0^*)^0 = 1,\ \binom{r}{-j} = 0,\ \binom{r}{0} = 1;\ r, j = 1, 2, \ldots, n,$$

and

$$z_0^* = -\frac{a_{n-1}}{n a_n}.$$

Moreover, when $d_0 \neq 0$*, the zeros of* $p(z)$ *lie in the closed annular region*

$$\frac{1}{\rho_0'} \le |z - z_0^*| \le \rho_0,$$

where ρ_0' *is an unique positive root of the equation*

$$|d_0|^2 t^{2n} - \sum_{j=1}^{n} |c_j^*| t^{n-j} = 0,$$

where

$$c_j^* = \sum_{k=0}^{n-j} d_{n-k}\lambda_{j+k}^*;\ j = 1, 2, \ldots, n$$

and the values $\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*$ *are determined by the* n *equations*

$$\sum_{k=0}^{j} d_{j-k}\lambda_k^* = 0;\ j = 1, 2, \ldots, n$$

with $\lambda_0^* = d_0$*.*

In the particular case when $a_{n-1} = 0$, Theorem 1.4 reduces to Theorem 1.3.

REMARK 1.5. The result in Theorem 1.3 attains its limits (both lower and upper bounds) by the polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, where $a_0 = a_1 = \cdots = a_{n-1} = a_n = 1$ which can be seen by observing that $\lambda_n = \lambda_n^* = 1, \lambda_{n-1} = \lambda_{n-1}^* = -1, \lambda_j = \lambda_j^* = 0; j = 0, 1, \ldots, n-2$ and $|a_n| = |a_0| = 1, |c_{n-1}| = |c_1^*| = 1, |c_{n-j}| = \left|c_j^*\right| = 0; j = 2, 3, \ldots, n$, and therefore, $t_0 = t_0' = 1$.

REMARK 1.6. For $a_0 = 0$, Theorem 1.3 and Theorem 1.1 never produce an annular region with the center in the origin for the location of zeros of $p(z)$. Here, both theorems give a circular region for polynomial zeros with center in the origin. However, when $a_{n-1} \neq 0$, Theorem 1.4 produces an annular region of zeros of $p(z)$ with center

$$z_0^* = -\frac{a_{n-1}}{n a_n},$$

even though $a_0 = 0$.

REMARK 1.7. In many cases, our results give better regions related with the Cauchy region and its subsequent refinements. To illustrate this, we consider two types of polynomial, one having no gaps and the other whose second highest term is absent i.e., Lacunary type [12, Ch. VIII, Sect. 34, p. 156] defined by

$$p(z) = z^6 + 6z^5 - 49z^4 - 374z^3 + 216z^2 + 5600z + 9600$$

and

$$P(z) = z^5 + 2z^3 + 3z^2 + 4z + 5,$$

respectively.

We see that all the zeros of $p(z)$ lie in the following regions.

(i) $|z| \leq 12.6028$, by Theorem 1.1,
(ii) $|z| \leq 10.8869$, by Theorem 1.2,
(iii) $|z| \leq 9.8240$, by Melman [13, Theorem 3.1] (which is the smallest),
(iv) $|z| < 24.5705$, by Sun and Hsieh [15, Theorem 1],
(v) $|z| < 31.9259$, by Bairagi, Jain, Mishra and Saha [1, Corollary 1.1],
(vi) $|z| \leq 14.5184$, by Jain [9, Theorem 1],
(vii) $|z| < 14.2048$, by Lagrange [11] (see also [2, Theorem 1.1]),
(viii) $|z| \leq 14.1031$, by Batra, Mignotte and Ștefănescu [2, Theorem 3.1],
(ix) $|z| \leq 101.511$, by Dehmer and Mowshowitz [7, Theorem 2],
(x) $|z| < 240.6768$, by Dehmer and Mowshowitz [7, Theorem 3],
(xi) $2.1576 \leq |z| \leq 7.9261$, by Theorem 1.3,
(xii) $1.5866 \leq |z+1| \leq 8.1813$, by Theorem 1.4.

Again, using either Theorem 1.3 or Theorem 1.4, all the zeros of $P(z)$ lie in

$$1.0278 \leq |z| \leq 1.7803,$$

whereas the regions obtained by previous well-known results are as follows:

(i) $|z| \leq 2.1719$, by Theorem 1.1,
(ii) $|z| \leq 1.7863$, by Theorem 1.2,
(iii) $|z| \leq 1.8932$, by Melman [13, Theorem 3.1] (which is the smallest),
(iv) $|z| \leq 5.9993$, by Datt and Govil [5, Theorem 1],
(v) $|z| \leq 2.8564$, by Lagrange [11] (see also [2, Theorem 1.1]),
(vi) $|z| \leq 2.8425$, by Batra, Mignotte and Ştefănescu [2, Theorem 3.1],
(vii) $|z| \leq 5.9482$, by Jain [8, Theorem 1],
(viii) $|z| < 2.2566$, by Jain [9, Theorem 1],
(ix) $|z| < 2.3507$, by Bairagi, Jain, Mishra and Saha [1, Theorem 1.5],
(x) $|z| \leq 2.3744$, by Sun and Hsieh [15, Theorem 1],
(xi) $|z| \leq 5.9993$, by Dehmer [6, Theorem 3.2],
(xii) $|z| \leq 5.9998$, by Dehmer [6, Theorem 3.3].

2. PROCEDURE TO DETERMINE T_0 AND T_0' IN THEOREM 1.3

We can easily determine the values of t_0 and t_0' in Theorem 1.3 by the following steps:

Step 1. For a given polynomial

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

with $a_0 \neq 0$, let us construct two lower triangular matrices $\overline{A}$ and $\overline{A}^*$ with diagonal elements a_n and a_0 respectively as follows:

$$\overline{A} = \begin{pmatrix} a_n & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_{n-1} & a_n & 0 & \cdots & 0 & 0 & 0 \\ a_{n-2} & a_{n-1} & a_n & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_2 & a_3 & a_4 & \cdots & a_n & 0 & 0 \\ a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n & 0 \\ a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} & a_n \end{pmatrix}_{(n+1)\times(n+1)}$$

and

$$\overline{A}^* = \begin{pmatrix} a_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_1 & a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-2} & a_{n-3} & a_{n-4} & \cdots & a_0 & 0 & 0 \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 & 0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 & a_0 \end{pmatrix}_{(n+1)\times(n+1)}$$

Step 2. Determine

$$\overline{\lambda} = [\lambda_n, \lambda_{n-1}, \cdots, \lambda_0]_{(n+1)\times 1} \text{ and } \overline{\lambda}^* = [\lambda_0^*, \lambda_1^*, \cdots, \lambda_n^*]_{(n+1)\times 1}$$

by solving the equations

$$\overline{A}.\overline{\lambda} = \overline{b} \text{ and } \overline{A}^*.\overline{\lambda}^* = \overline{b}^*,$$

where

$$\overline{b} = \left[a_n^2, 0, \cdots, 0\right]_{(n+1)\times 1} \text{ and } \overline{b}^* = \left[a_0^2, 0, \cdots, 0\right]_{(n+1)\times 1}$$

respectively.

Step 3. Determine the co-factors of $(1,1)$ entries with respect to $\overline{A}$ and $\overline{A}^*$, which are denoted by $\overline{A}_{a_n(1,1)}$ and $\overline{A}^*_{a_0(1,1)}$, respectively, and obtain the transpose of $\overline{A}_{a_n(1,1)}$ and $\overline{A}^*_{a_0(1,1)}$ given by

$$\overline{A}^T_{a_n(1,1)} = \begin{pmatrix} a_n & a_{n-1} & a_{n-2} & \cdots & a_3 & a_2 & a_1 \\ 0 & a_n & a_{n-1} & \cdots & a_4 & a_3 & a_2 \\ 0 & 0 & a_n & \cdots & a_5 & a_4 & a_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_n & a_{n-1} & a_{n-2} \\ 0 & 0 & 0 & \cdots & 0 & a_n & a_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 & a_n \end{pmatrix}_{n\times n}$$

and

$$\overline{A}^{*T}_{a_0(1,1)} = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-3} & a_{n-2} & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-4} & a_{n-3} & a_{n-2} \\ 0 & 0 & a_0 & \cdots & a_{n-5} & a_{n-4} & a_{n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_0 & a_1 & a_2 \\ 0 & 0 & 0 & \cdots & 0 & a_0 & a_1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & a_0 \end{pmatrix}_{n\times n},$$

respectively.

Step 4. Determine

$$\overline{c} = [c_{n-1}, c_{n-2}, \cdots, c_0]_{n\times 1} \quad \text{and} \quad \overline{c}^* = [c_1^*, c_2^*, \cdots, c_n^*]_{n\times 1}$$

which are obtained from

$$\overline{c} = \overline{A}^{*T}_{a_{0(1,1)}}.\overline{\lambda}_n \quad \text{and} \quad \overline{c}^* = \overline{A}^T_{a_n(1,1)}.\overline{\lambda}^*_0,$$

where

$$\overline{\lambda}_n = [\lambda_{n-1}, \lambda_{n-2}, \cdots, \lambda_0]_{n\times 1} \quad \text{and} \quad \overline{\lambda}^*_0 = [\lambda_1^*, \lambda_2^*, \cdots, \lambda_n^*]_{n\times 1},$$

respectively.

Step 5. Determine an unique positive roots t_0 and t_0' by solving the equations

$$|a_n|^2 t^{2n} - \sum_{j=1}^{n} |c_{n-j}| t^{n-j} = 0 \quad \text{and} \quad |a_0|^2 t^{2n} - \sum_{j=1}^{n} |c_j^*| t^{n-j} = 0,$$

respectively.

Also, for finding ρ_0 and ρ_0' from Theorem 1.4, at first, we calculate the d_j's by the following rule

$$\begin{aligned} d_p = &\binom{n}{n-p} a_n (z_0^*)^{n-p} + \binom{n-1}{n-p-1} a_{n-1} (z_0^*)^{n-p-1} + \cdots \\ &+ \binom{n-k}{n-p-k} a_{n-k} (z_0^*)^{n-p-k} + \cdots + \binom{p+1}{1} a_{p+1} z_0^* + \binom{p}{0} a_p; \\ &p = 0, 1, 2, \ldots, n, \end{aligned}$$

with

$$\binom{0}{0} = 1, \ (z_0^*)^0 = 1, \ \binom{r}{-j} = 0, \ \binom{r}{0} = 1; \ r, j = 1, 2, \ldots, n,$$

and

$$z_0^* = -\frac{a_{n-1}}{n a_n}.$$

Now, set

$$a_j = d_j; j = 1, 2, \ldots, n$$

and obtain t_0 and t_0' by using the above steps (Step 1 to Step 5), and set $\rho_0 = t_0$ and $\rho_0' = t_0'$ respectively.

3. AN IMPROVEMENT OF THE CAUCHY RADIUS

In this section, we establish that the new upper bounds of Theorem 1.3 are an improvement of the Cauchy radius of $p(z)$. For this, it is sufficient to prove $\rho[\lambda(z) p(z)] \leq \rho[p(z)]$, i.e., $t_0 \leq r_0$. From Theorem 1.3, it is clear that the upper bound is the Cauchy radius t_0 of $\lambda(z) p(z)$ which is an unique positive root of the equation $H(t) = 0$, where

$$H(t) = |a_n|^2 t^{2n} - R(t), \qquad R(t) = \sum_{j=1}^{n} |c_{n-j}| t^{n-j}.$$

Now

$$\begin{aligned} R(r_0) &= \sum_{j=1}^{n} |c_{n-j}| r_0^{n-j} = \sum_{j=1}^{n} \left| \sum_{k=0}^{n-j} a_k \lambda_{n-j-k} \right| r_0^{n-j} \\ &\leq \sum_{j=1}^{n} \left(\sum_{k=0}^{n-j} |a_k| |\lambda_{n-j-k}| \right) r_0^{n-j} = \sum_{k=0}^{n-1} |\lambda_k| r_0^k \left(\sum_{j=0}^{n-k-1} |a_j| r_0^j \right). \end{aligned}$$

So,

$$\left|a_n\right|^2 r_0^{2n} - R\left(r_0\right) \geq \left|a_n\right|^2 r_0^{2n} - \sum_{k=0}^{n-1} \left|\lambda_k\right| r_0^k \left(\sum_{j=0}^{n-k-1} \left|a_j\right| r_0^j\right).$$

Again the values λ_k, for $k = 0, 1, \ldots, n-1$, are satisfying the following n equations

$$\sum_{k=0}^{j} a_{n-j+k} \lambda_{n-k} = 0; \;\; j = 1, 2, \ldots, n$$

with $\lambda_n = a_n$. These give

$$\left|a_n\right| \left|a_{n-j}\right| = \left|\sum_{k=1}^{j} a_{n-j+k} \lambda_{n-k}\right|; \;\; j = 1, 2, \ldots, n \;\; (\text{as } \lambda_n = a_n),$$

and so

$$\left|a_n\right| \left|a_{n-j}\right| \geq \left(\left|a_n\right| \left|\lambda_{n-j}\right| - \sum_{k=1}^{j-1} \left|a_{n-j+k}\right| \left|\lambda_{n-k}\right|\right); \;\; j = 1, 2, \ldots, n.$$

Now, we multiply the above inequalities by r_0^{n-j} for $j = 1, 2, \ldots, n$ successively and by adding them we get

$$\left|a_n\right| \sum_{j=0}^{n-1} \left|a_j\right| r_0^j \geq \left|a_n\right| \sum_{j=0}^{n-1} \left|\lambda_j\right| r_0^j - \sum_{k=1}^{n-1} \frac{\left|\lambda_k\right|}{r_0^{n-k}} \left(\sum_{j=n-k}^{n-1} \left|a_j\right| r_0^j\right),$$

$$\left|a_n\right|^2 r_0^n \geq \left|a_n\right| \sum_{j=0}^{n-1} \left|\lambda_j\right| r_0^j - \frac{1}{r_0^n} \sum_{k=1}^{n-1} \left|\lambda_k\right| r_0^k \left(\left|a_n\right| r_0^n - \sum_{j=0}^{n-k-1} \left|a_j\right| r_0^j\right),$$

$$\left|a_n\right|^2 r_0^{2n} \geq \left|a_n\right| r_0^n \sum_{j=0}^{n-1} \left|\lambda_j\right| r_0^j - \left|a_n\right| r_0^n \sum_{k=1}^{n-1} \left|\lambda_k\right| r_0^k + \sum_{k=1}^{n-1} \left|\lambda_k\right| r_0^k \left(\sum_{j=0}^{n-k-1} \left|a_j\right| r_0^j\right),$$

or

$$\left|a_n\right|^2 r_0^{2n} \geq \left|a_n\right| r_0^n \left|\lambda_0\right| + \sum_{k=1}^{n-1} \left|\lambda_k\right| r_0^k \left(\sum_{j=0}^{n-k-1} \left|a_j\right| r_0^j\right).$$

Using the above inequality, we have

$$\left|a_n\right|^2 r_0^{2n} - R\left(r_0\right) \geq \\ \left|a_n\right| r_0^n \left|\lambda_0\right| + \sum_{k=1}^{n-1} \left|\lambda_k\right| r_0^k \left(\sum_{j=0}^{n-k-1} \left|a_j\right| r_0^j\right) - \sum_{k=0}^{n-1} \left|\lambda_k\right| r_0^k \left(\sum_{j=0}^{n-k-1} \left|a_j\right| r_0^j\right),$$

or

$$|a_n|^2 r_0^{2n} - R(r_0) \geq |a_n| r_0^n |\lambda_0| - |\lambda_0| \left(\sum_{j=0}^{n-1} |a_j| r_0^j \right),$$

or

$$|a_n|^2 r_0^{2n} - R(r_0) \geq |a_n| r_0^n |\lambda_0| - |\lambda_0| |a_n| r_0^n \left(\text{as } |a_n| r_0^n = \sum_{j=0}^{n-1} |a_j| r_0^j \right).$$

Clearly,

$$|a_n|^2 r_0^{2n} - R(r_0) \geq 0.$$

Therefore, we conclude that

$$t_0 \leq r_0, \text{ i.e., } \rho[\lambda(z) p(z)] \leq \rho[p(z)].$$

4. PROOF OF THEOREMS

Proof of Theorem 1.3. At first, we consider a polynomial $g(z)$ defined by

$$g(z) = \lambda(z) p(z),$$

where

$$\lambda(z) = \lambda_n z^n + \lambda_{n-1} z^{n-1} + \cdots + \lambda_1 z + \lambda_0$$

is a complex polynomial of degree n whose coefficients λ_k, for $k = 0, 1, \ldots, n-1$ are to be determined.

Clearly,

$$g(z) = a_n \lambda_n z^{2n} + \sum_{j=1}^{n} b_{2n-j} z^{2n-j} + \sum_{j=1}^{n} c_{n-j} z^{n-j},$$

where

$$b_{2n-j} = \sum_{k=0}^{j} a_{n-j+k} \lambda_{n-k}, \; c_{n-j} = \sum_{k=0}^{n-j} a_k \lambda_{n-j-k}; \; j = 1, 2, \ldots, n.$$

Now, we choose the values of $\lambda_n, \lambda_{n-1}, \ldots, \lambda_0$ for which

$$a_n \lambda_n = a_n^2 \text{ and } b_{2n-j} = 0; j = 1, 2, \ldots, n.$$

Determine $c_{n-1}, c_{n-2}, \ldots, c_0$ by using the known values of $\lambda_n, \lambda_{n-1}, \ldots, \lambda_0$ from the relations

$$c_{n-j} = \sum_{k=0}^{n-j} a_k \lambda_{n-j-k}; \; j = 1, 2, \ldots, n.$$

So $g(z)$ becomes

$$g(z) = a_n^2 z^{2n} + \sum_{j=1}^{n} c_{n-j} z^{n-j}.$$

Now given $|z| > 0$,

$$|g(z)| \geq |a_n|^2 |z|^{2n} - \sum_{j=1}^{n} |c_{n-j}| |z|^{n-j}.$$

Clearly, the equation

$$|a_n|^2 t^{2n} - \sum_{j=1}^{n} |c_{n-j}| t^{n-j} = 0$$

has exactly one positive root, say t_0, and so

$$|a_n|^2 t^{2n} - \sum_{j=1}^{n} |c_{n-j}| t^{n-j} > 0, \text{ if } t > t_0,$$

which implies

$$|g(z)| > 0, \text{ if } |z| > t_0.$$

Therefore, all the zeros of $g(z)$ lie in the disc $|z| \leq t_0$, and it follows that the zeros of $p(z)$ lie in

$$|z| \leq t_0.$$

In case if $a_0 \neq 0$, we choose a polynomial

$$\lambda^*(z) = \lambda_0^* z^n + \lambda_1^* z^{n-1} + \cdots + \lambda_{n-1}^* z + \lambda_n^*$$

for which the first n terms latter of the leading term of

$$g^*(z) = \lambda^*(z) p^*(z)$$

vanish with $\lambda_0^* = a_0$, where

$$p^*(z) = z^n p\left(\frac{1}{z}\right).$$

Now

$$g^*(z) = a_0 \lambda_0^* z^{2n} + \sum_{j=1}^{n} b_j^* z^{2n-j} + \sum_{j=1}^{n} c_j^* z^{n-j},$$

where

$$b_j^* = \sum_{k=0}^{j} a_{j-k} \lambda_k^*, \;\; c_j^* = \sum_{k=0}^{n-j} a_{n-k} \lambda_{j+k}^*; \;\; j = 1, 2, \ldots, n.$$

As for

$$\lambda_0^* = a_0 \quad \text{ and } \quad b_j^* = 0; j = 1, 2, \ldots, n,$$

we can uniquely determine $\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*$ and consequently we determined the polynomial $\lambda^*(z)$. Now we obtain c_j^* for $j = 1, 2, \ldots, n$ by using the relations

$$c_j^* = \sum_{k=0}^{n-j} a_{n-k} \lambda_{j+k}^* \text{ for } j = 1, 2, \ldots, n.$$

So

$$g^*(z) = a_0^2 z^{2n} + \sum_{j=1}^{n} c_j^* z^{n-j}.$$

For $|z| > 0$,

$$|g^*(z)| \geq |a_0|^2 |z|^{2n} - \sum_{j=1}^{n} \left|c_j^*\right| |z|^{n-j}.$$

As the equation

$$|a_0|^2 t^{2n} - \sum_{j=1}^{n} \left|c_j^*\right| t^{n-j} = 0,$$

has exactly one positive root, say t_0', which gives

$$|g^*(z)| > 0, \text{ for } |z| > t_0'.$$

This leads us to the desired result. □

Proof of Theorem 1.4. First of all, we construct a transformation from the z-plane to the ω-plane defined by

$$\omega = L(z) = z - z_0^*,$$

where z_0^* is a complex number, which needs to be determined so that the polynomial $p(z)$ in the z-plane transforms to $T(\omega)$ in the ω-plane with the property that the coefficient of ω^{n-1} in $T(\omega)$ is absent.

Clearly,

$$\begin{aligned} T(\omega) &= p(\omega + z_0^*) \\ &= d_n \omega^n + d_{n-1}\omega^{n-1} + d_{n-2}\omega^{n-2} + \cdots + d_1\omega + d_0, \end{aligned}$$

where

$$\begin{aligned} d_p = &\binom{n}{n-p} a_n (z_0^*)^{n-p} + \binom{n-1}{n-p-1} a_{n-1} (z_0^*)^{n-p-1} + \cdots \\ &+ \binom{n-k}{n-p-k} a_{n-k} (z_0^*)^{n-p-k} + \cdots + \binom{p+1}{1} a_{p+1} z_0^* + \binom{p}{0} a_p; \\ & p = 0, 1, 2, \ldots, n, \end{aligned}$$

with

$$\binom{0}{0} = 1, \ \ (z_0^*)^0 = 1, \ \ \binom{r}{-j} = 0, \ \ \binom{r}{0} = 1; \ \ r, j = 1, 2, \ldots, n.$$

As the coefficient of ω^{n-1} in $T(\omega)$ is absent, i.e., $d_{n-1} = 0$, i.e.,

$$\binom{n}{1} a_n z_0^* + \binom{n}{0} a_{n-1} (z_0^*)^0 = 0,$$

which can determine the value of

$$z_0^* = -\frac{a_{n-1}}{n a_n}.$$

Using Theorem 1.3 on $T(\omega)$ in the ω-plane, we obtain that all the zeros of $T(\omega)$ lie in the circular region

$$|\omega| \leq \rho_0,$$

and when $d_0 \neq 0$, the zeros of $T(\omega)$ lie in the annular region

$$\frac{1}{\rho_0'} \leq |\omega| \leq \rho_0$$

in the ω-plane.

Clearly $L^{-1} : z = L^{-1}(\omega) = \omega + z_0^*$ is the inverse mapping of L from the ω-plane to the z-plane and it is an Entire Linear Transformation. So it preserves the shape.

Now we consider a circle in the ω-plane defined by

$$\Omega : |\omega| = \rho_0.$$

Because

$$L^{-1}(\Omega) : |z - z_0^*| = \rho_0$$

and

$$L^{-1}(0) = z_0^* \in \operatorname{Int}\left(L^{-1}(\Omega)\right) : |z - z_0^*| < \rho_0,$$

we obtain that the zeros of $p(z)$ must lie in the circular region

$$|z - z_0^*| \leq \rho_0,$$

in the z-plane. Also, for $d_0 \neq 0$, by applying a similar argument, we get the desired result. □

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