MATHEMATICA, 65 (88), N° 2, 2023, pp. 219–228

VON NEUMANN LOCAL MATRICES

IULIA-ELENA CHIRU and SEPTIMIU CRIVEI

Abstract. We use our recent results on von Neumann regular matrices, strongly regular matrices and matrices having a non-zero outer inverse to derive applications to some generalizations of these concepts, called von Neumann local, strongly von Neumann local and outer von Neumann local matrices. Among other properties, we show that the t^{th} compound matrix of every matrix of determinantal rank t over a commutative local ring is strongly von Neumann local, and every matrix over an arbitrary semiperfect ring is outer von Neumann local.

MSC 2020. 15A09, 16E50.

Key words. (Strongly, outer) Von Neumann regular matrix, (strongly, outer) von Neumann local matrix.

1. INTRODUCTION

Von Neumann regular rings were introduced by von Neumann in the 1930s [12] as an algebraic tool for studying certain lattices, which were useful in the coordinatization of projective geometry. They have found many applications throughout the time, not only in algebra, but also in functional analysis, differential equations, statistics, probability or cryptography. Recall that a ring R is called *von Neumann regular* if every element $a \in R$ is *von Neumann regular*, i.e., there is an element $b \in R$, called an *inner inverse* or *generalized inverse* of a, such that $a = aba$. The definition may be adapted to matrices over a ring R as follows: an $m \times n$-matrix A is called *von Neumann regular* if there is an $n \times m$-matrix B such that $A = ABA$, and in this case B is called an *inner inverse* or *generalized inverse* of A.

More particularly, a ring R is called *strongly regular* if for every $a \in R$ there is $b \in R$ such that $a = a^2b$, and this definition turns out to be left-right symmetric [3]. Restricting it to elements, $a \in R$ is called *strongly regular* if $a \in a^2R \cap Ra^2$. Note that if a is strongly regular with $a = a^2c = da^2$ for some $c, d \in R$, then one may choose $b = ac^2$ and one has $a = a^2b = ba^2$ [4], and in this case b is called a *strong inner* (or *strong generalized) inverse* of a. Every strongly regular element is von Neumann regular. The definition may be adapted to matrices over a ring R as follows: an $n \times n$-matrix A is called *strongly regular* if there is an $n \times n$-matrix B such that $A = A^2B = BA^2$.

Corresponding author: Septimiu Crivei.

DOI: 10.24193/mathcluj.2023.2.08

Another important concept when talking about generalized inverses is the outer inverse of an element. An element $b \in R$ is called an *outer inverse* of $a \in R$ if $bab = b$. The same definition can be exteded to matrices: an $n \times m$-matrix B is called an *outer inverse* of an $m \times n$-matrix A if $BAB = B$. If A is a von Neumann regular $m \times n$-matrix with inner inverse $n \times m$-matrix B, then it is well known and easy to see that BAB is an outer inverse of A.

For general properties of von Neumann regular rings, strongly regular rings and various generalized inverses we refer to [5, 10, 11].

Throughout the paper $m, n \geq 2$ will be two integers, and R will be a ring with identity. We denote by $M_{m,n}(R)$ the set of all $m \times n$-matrices over R, and by $M_n(R)$ the set of all $n \times n$-matrices over R. Let $A \in M_{m,n}(R)$. Given subsets $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, m\}$ with $i_1 < \cdots < i_k$ and $J = \{j_1, \ldots, j_l\} \subseteq \{1, \ldots, n\}$ with $j_1 < \cdots < j_l$, we denote by $A_{I,J}$ the submatrix of A whose rows and columns are indexed by the sets I and J respectively.

Now let R be a commutative ring. For each $k \in \{1, \ldots, \min(m, n)\}$, the *kth compound matrix* of A is defined as the matrix $C_k(A) \in M_{m',n'}(R)$, where $m' = \binom{m}{k}$ and $n' = \binom{n}{k}$, consisting of the $k \times k$-minors of A, where for every $I' = \{i'_1, \ldots, i'_k\}$ with $i'_1 < \cdots < i'_k$ and $J' = \{j'_1, \ldots, j'_k\}$ with $j'_1 < \cdots < j'_k$, the (I', J') entry of $C_k(A)$ is $\det(A_{I',J'})$. We denote $c_k = (-1)^k \mathrm{Tr}(C_k(A))$ for every $k \in \{1, \ldots, n\}$. For each $k \in \{1, \ldots, r = \min(m, n)\}$, $\mathcal{D}_k(A)$ will denote the ideal of R generated by all $k \times k$-minors of A (i.e., all entries of the compound matrix $C_k(A)$), and will be called the *kth determinantal ideal* of A. The *determinantal rank* of a non-zero $A \in M_{m,n}(R)$, denoted by $\rho(A)$, is defined as the maximal order of a submatrix of A with non-zero determinant. The determinantal rank of the zero matrix will be zero.

A ring R is called *local* if it has a unique maximal ideal. We denote by $\mathrm{rad}(R)$ the Jacobson radical of R, that is, the intersection of its maximal ideals, and by $U(R)$ the set of units of R.

2. VON NEUMANN LOCAL MATRICES

Contessa [9] has introduced *von Neumann local* rings as the rings R with the property that a or $1 - a$ is von Neumann regular for every $a \in R$. Clearly, every von Neumann regular ring and every local ring is von Neumann local. Also, every von Neumann local ring is an exchange ring. Von Neumann local rings have been also studied by Abu Osba, Henriksen and Alkam [1], and Anderson and Badawi [2], which specialized their definition to elements. Thus, an element $a \in R$ is called *von Neumann local* if a or $1 - a$ is von Neumann regular. Clearly, every von Neumann regular element is von Neumann local.

In particular, a matrix $A \in M_n(R)$ is von Neumann local if A or $I_n - A$ is von Neumann local. We show that there is a rich supply of von Neumann local matrices. But let us first recall the following theorem.

Theorem 2.1 ([8, Theorem 2.4]). *Let R be a commutative ring, and let $A \in M_n(R)$ be a non-zero matrix with $\rho(A) = t$.*

(1) *If A is strongly regular, then $c_t \notin \mathrm{rad}(R)$.*
(2) *If $c_t \in U(R)$, then A is strongly regular.*

THEOREM 2.2. *Let R be a commutative local ring, and let $A \in M_n(R)$ with $\rho(A) \leq 1$. Then A is von Neumann local.*

Proof. The case $\rho(A) = 0$ is clear. Next suppose that $\rho(A) = 1$ and A is not von Neumann local. Hence both A and $I_n - A$ are not von Neumann regular. Hence $\det(I_n - A) \notin U(R)$, and thus $\det(I_n - A) \in M$, where M is the maximal ideal of R. Since $\rho(A) = 1$, it follows that $\det(I_n - A) = 1 - \mathrm{Tr}(A) \in M$. This implies that $\mathrm{Tr}(A) \in U(R)$, because otherwise it follows that $1 \in M$, a contradiction. Then A is strongly regular by Theorem 2.1, hence A is von Neumann regular. This is a contradiction, and thus A is von Neumann local. □

COROLLARY 2.3. *Let R be a commutative local ring, and let $A \in M_n(R)$ with $\rho(A) = t$. Then $C_t(A)$ is von Neumann local.*

Proof. Since $\rho(A) = t$, we have $\rho(C_t(A)) = 1$ by [11, Theorem 2.5]. Hence $C_t(A)$ is von Neumann local by Theorem 2.2. □

Using our characterizations of von Neumann regular matrices, we may immediately deduce corresponding characterizations of von Neumann local matrices. Recall that a matrix $A \in M_n(R)$ will be denoted by $A_\mathfrak{p}$ when viewed over the localization $R_\mathfrak{p}$ of R at a prime ideal $\mathfrak{p}$.

THEOREM 2.4. *Let R be a commutative ring, and let $A \in M_n(R)$. Then the following are equivalent:*

(1) *A is von Neumann local.*
(2) *For each $k \in \{1, \ldots, n\}$, $\mathcal{D}_k(A)$ or $\mathcal{D}_k(I_n - A)$ is generated by an idempotent of R.*
(3) *For every prime (maximal) ideal $\mathfrak{p}$ of R, $A_\mathfrak{p} \in M_n(R_\mathfrak{p})$ is von Neumann local.*
(4) *For every prime (maximal) ideal $\mathfrak{p}$ of R, $A_\mathfrak{p} = 0_n$ or $A_\mathfrak{p}$ has an invertible $\rho(A_\mathfrak{p}) \times \rho(A_\mathfrak{p})$-submatrix or $(I_n - A)_\mathfrak{p} = 0_n$ or $(I_n - A)_\mathfrak{p}$ has an invertible $\rho(I_n - A_\mathfrak{p}) \times \rho(I_n - A_\mathfrak{p})$-submatrix.*

Proof. This follows by [6, Theorems 4.1,4.3]. □

We recall the following characterization of von Neumann regular matrices, which will be needed several times.

THEOREM 2.5 ([6, Theorem 4.2]). *Let R be local, and let $A \in M_{m,n}(R)$. Then A is von Neumann regular if and only if A is either zero or A has an invertible $\rho(A) \times \rho(A)$-submatrix.*

THEOREM 2.6. *Let R be a commutative local ring, and let $A \in M_n(R)$ with $\rho(A) = t$ and $\rho(I_n - A) = s$. Then A is von Neumann local if and only if $A = 0_n$ or A has an invertible $t \times t$-submatrix or $A = I_n$ or $I_n - A$ has an invertible $s \times s$-submatrix.*

Proof. This follows by Theorem 2.5. □

Next we characterize von Neumann local 2×2-matrices over commutative local rings only in terms of determinants. We show that von Neumann local and strongly von Neumann local matrices coincide in this case.

THEOREM 2.7. *Let R be a commutative local ring, and let $A \in M_2(R)$. Then the following are equivalent:*

(1) *A is von Neumann local.*
(2) $\det(A) \in U(R) \cup \{0\}$ *or* $\det(I_2 - A) \in U(R) \cup \{0\}$.

If R is not a field, then they are further equivalent to:

(3) $\det(A) \notin \mathrm{rad}(R) \setminus \{0\}$ *or* $\det(I_2 - A) \notin \mathrm{rad}(R) \setminus \{0\}$.
(4) $\det(A) \in U(R) \cup \{0\}$ *or* $1 - \mathrm{Tr}(A) \in U(R)$.

Proof. (1) $\Longrightarrow$ (2) We show that the negation of (2) implies the negation of (1). To this end, assume that $\det(A) \notin U(R) \cup \{0\}$ and $\det(I_2 - A) \notin U(R) \cup \{0\}$. Suppose that A is von Neumann local. Then A or $I_2 - A$ is von Neumann regular. Since $\det(A) \neq 0$ and $\det(I_2 - A) \neq 0$, we have $\rho(A) = 2$ and $\rho(I_2 - A) = 2$. If A is von Neumann regular, then $\det(A) \in U(R)$, while if $I_2 - A$ is von Neumann regular, then $\det(I_2 - A) \in U(R)$ by Theorem 2.5. In both cases, we have a contradiction. Consequently, A is not von Neumann local, as needed.

(2) $\Longrightarrow$ (1) Assume that $\det(A) \in U(R) \cup \{0\}$ or $\det(I_2 - A) \in U(R) \cup \{0\}$. Suppose that A is not von Neumann local. Hence both A and $I_2 - A$ are not von Neumann regular. By Theorem 2.2, we have $\rho(A) = 2$ and $\rho(I_2 - A) = 2$. Now this together with Theorem 2.5 imply that $\det(A) \notin U(R) \cup \{0\}$ and $\det(I_2 - A) \notin U(R) \cup \{0\}$, a contradiction. Hence A is von Neumann local.

(2) $\Longleftrightarrow$ (3) This is clear.

(3) $\Longleftrightarrow$ (4) If $\det(A) \in U(R) \cup \{0\}$, then there is nothing to prove. Next assume that $\det(A) \notin U(R) \cup \{0\}$, hence $\det(A) \in \mathrm{rad}(R) \setminus \{0\}$. Note that

$$\det(I_2 - A) = 1 - \mathrm{Tr}(A) + \det(A).$$

Then $\det(I_2 - A) \in \mathrm{rad}(R) \setminus \{0\}$ if and only if $1 - \mathrm{Tr}(A) + \det(A) \in \mathrm{rad}(R) \setminus \{0\}$ if and only if $1 - \mathrm{Tr}(A) \in \mathrm{rad}(R)$. □

EXAMPLE 2.8. (1) $A = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \in M_2(\mathbb{Z}_4)$ is von Neumann local (since $I_2 - A$ is invertible), but not von Neumann regular by Theorem 2.5.

(2) $A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \in M_2(\mathbb{Z}_4)$ is not von Neumann local by Theorem 2.7.

Unlike the case of von Neumann regularity, the property of being von Neumann local is not well behaved with respect to direct products (e.g., see [1, p.2644]). By [1, Theorem 3.1], a direct product $R = \prod_{k \in K} R_k$ is von Neumann local if and only if there is $l \in K$ such that R_l is von Neumann local and R_k is von Neumann regular for every $k \in K \setminus \{l\}$. Next we state an element-wise version of this result, whose commutative version was given by Anderson and Badawi [2, Theorem 5.1]. Let us denote by $\mathrm{vnr}(R)$ (respectively $\mathrm{vnl}(R)$) the set of von Neumann regular (respectively von Neumann local) elements of a ring R.

THEOREM 2.9. *Let $R = \prod_{k\in K} R_k$ be a product of arbitrary rings. Then $\mathrm{vnl}(R) = \prod_{k\in K} \mathrm{vnl}(R_k)$ if and only if $\mathrm{vnl}(R_k) = \mathrm{vnr}(R_k)$ for all but at most one $k \in K$. In particular, R is a von Neumann local ring if and only if there is at most one $k \in K$ such that R_k is not von Neumann regular, but R_k is von Neumann local.*

Proof. The proof is essentially the same as the proof of [2, Theorem 5.1] from the commutative case. □

COROLLARY 2.10. *Let $R = \prod_{k\in K} R_k$ be a product of local commutative rings such that $\mathrm{vnl}(R_k) = \mathrm{vnr}(R_k)$ for all but at most one $k \in K$, and let $0_n \neq A \in M_n(R)$. For every $k \in K$, denote by $h_k : M_n(R) \to M_n(R_k)$ the canonical projection, and $t_k = \rho(h_k(A))$, $s_k = \rho(I_n - h_k(A))$. Then the following are equivalent:*

(1) *A is von Neumann local.*
(2) *For every $k \in K$, $h_k(A)$ is von Neumann local.*
(3) *For every $k \in K$, $h_k(A) = 0_n$ or A has an invertible $t_k \times t_k$-submatrix or $h_k(A) = I_n$ or $I_n - h_k(A)$ has an invertible $s_k \times s_k$-submatrix.*

Proof. This follows by Theorems 2.9 and 2.6. □

EXAMPLE 2.11. Consider the ring $\mathbb{Z}_{12} \cong \mathbb{Z}_3 \times \mathbb{Z}_4$. We have $\mathrm{vnl}(\mathbb{Z}_3) = \mathrm{vnr}(\mathbb{Z}_3) = \mathbb{Z}_3$. Let

$$A = \begin{pmatrix} 10 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{Z}_{12}).$$

We can easily prove that A is not von Neumann regular. Consider $B = I_2 - A = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$, $B_1 = (B \bmod 3) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{Z}_3)$ and $B_2 = (B \bmod 4) = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{Z}_4)$. By Theorem 2.5, both B_1 and B_2 are von Neumann regular, hence they are von Neumann local. Then A is von Neumann local by Corollary 2.10.

3. STRONGLY VON NEUMANN LOCAL MATRICES

We consider a specialization of the notion of von Neumann local element of a ring. Thus, an element $a \in R$ is called *strongly von Neumann local* if a or $1 - a$ is strongly regular. Clearly, every strongly regular element is strongly von Neumann local, and every strongly von Neumann local element is von Neumann local. In particular, a matrix $A \in M_n(R)$ is strongly von Neumann local if A or $I_n - A$ is strongly von Neumann local. Note that our concept of strongly von Neumann local element is different of the one with the same name from [1].

Next we improve some results from von Neumann local matrices.

Theorem 3.1. *Let R be a commutative local ring, and let $A \in M_n(R)$ with $\rho(A) \leq 1$. Then A is strongly von Neumann local.*

Proof. An easy adaptation of Theorem 2.2 yields the result. $\square$

Corollary 3.2. *Let R be a commutative local ring, and let $A \in M_n(R)$ with $\rho(A) = t$. Then $C_t(A)$ is strongly von Neumann local.*

Proof. Since $\rho(A) = t$, we have $\rho(C_t(A)) = 1$ by [11, Theorem 2.5]. Hence $C_t(A)$ is strongly von Neumann local by Theorem 3.1. $\square$

Our previous results on strongly regular matrices from [8] may be immediately applied to derive corresponding properties of strongly von Neumann local matrices.

Theorem 3.3. *Let R be a commutative ring, and let $A \in M_n(R)$. Then the following are equivalent:*

(1) *A is strongly von Neumann local.*
(2) *For every prime (maximal) ideal $\mathfrak{p}$ of R, $A_\mathfrak{p}$ is strongly von Neumann local.*
(3) *For every prime (maximal) ideal $\mathfrak{p}$ of R, $A_\mathfrak{p} = 0_n$ or $c_t \in U(R_\mathfrak{p})$ or $A_\mathfrak{p} = I_n$ or $d_s \in U(R_\mathfrak{p})$, where $t = \rho(A_\mathfrak{p})$, $s = \rho(I_n - A_\mathfrak{p})$, $c_t = (-1)^t \mathrm{Tr}(C_t(A))$ and $d_s = (-1)^s \mathrm{Tr}(C_s(I_n - A))$.*

Theorem 3.4. *Let R be a commutative local ring, and let $A \in M_n(R)$ with $\rho(A) = t$ and $\rho(I_n - A) = s$. Then the following are equivalent:*

(1) *A is strongly von Neumann local.*
(2) *$A = 0_n$ or $c_t \in U(R)$ or $A = I_n$ or $d_s \in U(R)$, where we denote $c_t = (-1)^t \mathrm{Tr}(C_t(A))$ and $d_s = (-1)^s \mathrm{Tr}(C_s(I_n - A))$.*

Next we show that von Neumann local and strongly von Neumann local matrices coincide in this case of 2×2-matrices over commutative local rings. But first, let us recall the next theorem.

Theorem 3.5 ([8, Theorem 2.7]). *Let R be local and let $A \in M_n(R)$ be a non-zero matrix with $\rho(A) = t$. Then A is strongly regular if and only if $c_t \in U(R)$.*

Theorem 3.6. *Let R be a commutative local ring, and let $A \in M_2(R)$. Then A is strongly von Neumann local if and only if A is von Neumann local.*

Proof. For the non-trivial implication assume that A is von Neumann local. Then $\det(A) \in U(R) \cup \{0\}$ or $\det(I_2 - A) \in U(R) \cup \{0\}$ by Theorem 2.7. Suppose that A is not strongly von Neumann local. Hence both A and $I_2 - A$ are not strongly regular. By Theorem 3.1, we have $\rho(A) = 2$ and $\rho(I_2 - A) = 2$.

Now this together with Theorem 3.5 imply that $\det(A) \notin U(R) \cup \{0\}$ and $\det(I_2 - A) \notin U(R) \cup \{0\}$, a contradiction. Hence A is strongly von Neumann local. $\square$

EXAMPLE 3.7. (1) $A = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \in M_2(\mathbb{Z}_4)$ is strongly von Neumann local (since $I_2 - A$ is invertible), but not strongly regular by Theorem 3.5.

(2) $A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \in M_2(\mathbb{Z}_4)$ is not strongly von Neumann local, because it is not von Neumann local by Example 2.8.

(3) By Theorem 3.6, in order to find an example of a von Neumann local matrix which is not strongly von Neumann local over a commutative local ring we need to look for some matrix having a larger size than 2×2. Let us take

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 3 \end{pmatrix} \in M_3(\mathbb{Z}_4).$$

Since $\rho(A) = 2$ and A has an invertible 2×2-submatrix, A is von Neumann regular by Theorem 2.5, and consequently, A is von Neumann local. Since the sum of diagonal 2×2-submatrices of A is 0, A is not strongly regular by Theorem 3.5. Now consider

$$I_3 - A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 3 & 0 & 2 \end{pmatrix}.$$

Since $\rho(I_3 - A) = 3$ and $\det(I_3 - A) = 2 \notin U(\mathbb{Z}_4)$, $I_3 - A$ is not strongly regular by Theorem 3.5, and consequently, A is not strongly von Neumann local.

We have already seen that in general the property of being von Neumann local is not well behaved with respect to direct products, and we have given Theorem 2.9. Now we deal with the similar problem for strongly von Neumann local elements in an arbitrary ring R. Let us denote by $\mathrm{svnr}(R)$ (respectively $\mathrm{svnl}(R)$) the set of strongly regular (respectively strongly von Neumann local) elements of R.

THEOREM 3.8. *Let $R = \prod_{k \in K} R_k$ be a product of arbitrary rings. Then* $\mathrm{svnl}(R) = \prod_{k \in K} \mathrm{svnl}(R_k)$ *if and only if* $\mathrm{svnl}(R_k) = \mathrm{svnr}(R_k)$ *for all but at most one $k \in K$. In particular, R is a strongly von Neumann local ring if and only if there is at most one $k \in K$ such that R_k is not strongly regular, but R_k is strongly von Neumann local.*

Proof. We follow a similar approach as in the proof of the corresponding result for von Neumann local matrices.

Suppose first that there exist $a_i \in \mathrm{svnl}(R_i) \setminus \mathrm{svnr}(R_i)$ and $b_j \in \mathrm{svnl}(R_j) \setminus \mathrm{svnr}(R_j)$ for some distinct $i, j \in K$. Putting $a_j = 1 - b_j$ and $a_k = 1$ for every $k \in K \setminus \{i, j\}$, we have $(a_k)_{k \in K} \in \prod_{k \in K} \mathrm{svnl}(R_k)$. On the other hand, we have $(a_k)_{k\in K} \notin \prod_{k\in K} \mathrm{svnr}(R_k) = \mathrm{svnr}(R)$, because $a_i \notin \mathrm{svnr}(R_i)$. Hence $\mathrm{svnl}(R) \neq \prod_{k\in K} \mathrm{svnl}(R_k)$.

Conversely, suppose that $\mathrm{svnl}(R_k) = \mathrm{svnr}(R_k)$ for all but at most one $k \in K$. We claim first that $\mathrm{svnl}(R) \subseteq \prod_{k\in K} \mathrm{svnl}(R_k)$. To this end, let $a = (a_k)_{k\in K} \in \mathrm{svnl}(R)$ and denote $p_k : R \to R_k$ be the canonical projections for every $k \in K$. Then $(p_k(a))_{k\in K} \in \mathrm{svnl}\left(\prod_{k\in K} R_k\right)$, hence $p_k(a) \in \mathrm{svnr}(R_k)$ or $1 - p_k(a) \in \mathrm{svnr}(R_k)$ for every $k \in K$, which implies that $p_k(a) \in \mathrm{svnl}(R_k)$ for every $k \in K$, and finally $(a_k)_{k\in K} \in \prod_{k\in K} \mathrm{svnl}(R_k)$. For the other inclusion, let $(a_k)_{k\in K} \in \prod_{k\in K} \mathrm{svnl}(R_k)$. If $\mathrm{svnl}(R_k) = \mathrm{svnr}(R_k)$ for every $k \in K$, then $(a_k)_{k\in K} \in \prod_{k\in K} \mathrm{svnr}(R_k) = \mathrm{svnr}(R) \subseteq \mathrm{svnl}(R)$. Next assume that $\mathrm{svnl}(R_k) = \mathrm{svnr}(R_k)$ for every $k \in K \setminus \{l\}$ for some $l \in K$. Take $a_l \in \mathrm{svnl}(R_l) \setminus \mathrm{svnr}(R_l)$. Then $1 - a_l \in \mathrm{svnr}(R_l)$ and for every $k \in K \setminus \{l\}$ we have $1 - a_k \in \mathrm{svnl}(R_k) = \mathrm{svnr}(R_k)$, whence $1 - (a_k)_{k\in K} \in \prod_{k\in K} \mathrm{svnr}(R_k) = \mathrm{svnr}(R)$, and thus $(a_k)_{k\in K} \in \mathrm{svnl}(R)$. This shows that $\mathrm{svnl}(R) = \prod_{k\in K} \mathrm{svnl}(R_k)$. □

COROLLARY 3.9. *Let $R = \prod_{k\in K} R_k$ be a product of local commutative rings such that $\mathrm{svnl}(R_k) = \mathrm{svnr}(R_k)$ for all but at most one $k \in K$, and let $0_n \neq A \in M_n(R)$. For every $k \in K$, denote by $h_k : M_n(R) \to M_n(R_k)$ the canonical projection, and $t_k = \rho(h_k(A))$, $s_k = \rho(I_n - h_k(A))$. Then the following are equivalent:*

(1) *A is strongly von Neumann local.*
(2) *For every $k \in K$, $h_k(A)$ is strongly von Neumann local.*
(3) *For every $k \in K$, $h_k(A) = 0_n$ or $c_{t_k} \in U(R_k)$ or $h_k(A) = I_n$ or $d_{s_k} \in U(R)$, where $c_t = (-1)^{t_k}\mathrm{Tr}(C_{t_k}(h_k(A)))$ and $d_{s_k} = (-1)^{s_k}\mathrm{Tr}(C_{s_k}(I_n - h_k(A)))$.*

Proof. This follows by Theorems 3.8 and 3.4. □

EXAMPLE 3.10. Consider the ring $\mathbb{Z}_{12} \cong \mathbb{Z}_3 \times \mathbb{Z}_4$. We have $\mathrm{svnl}(\mathbb{Z}_3) = \mathrm{svnr}(\mathbb{Z}_3) = \mathbb{Z}_3$. Let $A = \begin{pmatrix} 0 & 0 \\ 7 & 0 \end{pmatrix} \in M_2(\mathbb{Z}_{12})$. Since $(A \bmod 3) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{Z}_3)$ is not strongly regular by Theorem 3.5, A is not strongly regular. Consider $B = I_2 - A = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$, $B_1 = (B \bmod 3) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \in M_2(\mathbb{Z}_3)$ and $B_2 = (B \bmod 4) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in M_2(\mathbb{Z}_4)$. By Theorem 3.5, both B_1 and B_2 are strongly regular, hence they are strongly von Neumann local. Then A is strongly von Neumann local by Corollary 3.9.

4. OUTER VON NEUMANN LOCAL MATRICES

Generalizing von Neumann local elements of a ring, an element $a \in R$ is called *outer von Neumann local* if a or $1 - a$ has a non-zero outer inverse. An element having a non-zero outer inverse will also be called an *outer von Neumann regular* element. Clearly, every outer von Neumann regular is outer von Neumann local, and every von Neumann local element is outer von Neumann local. In particular, a matrix $A \in M_n(R)$ is outer von Neumann local if A or $I_n - A$ is outer von Neumann local. Throughout this section the ring R need not be commutative.

In our paper [7] we have given a characterization of matrices having a non-zero outer inverse over semiperfect rings.

Theorem 4.1 ([7, Theorem 4.1]). *Let R be a semiperfect ring. Then the following are equivalent for $A = (a_{ij}) \in M_{m,n}(R)$:*

(1) *There exists some a_{ij} having a non-zero outer inverse.*
(2) *A has a non-zero outer inverse.*
(3) *$A \notin M_{m,n}(\mathrm{rad}(R))$.*

We may use it to obtain the following result.

Theorem 4.2. *Let R be a semiperfect ring. Then every matrix $A = (a_{ij}) \in M_n(R)$ is outer von Neumann local.*

Proof. Assume that none of A and $I_n - A$ has a non-zero outer inverse. Then $A, I_n - A \in M_n(\mathrm{rad}(R))$ by Theorem 4.1. In particular, we have $a_{11}, 1 - a_{11} \in \mathrm{rad}(R)$, which is a contradiction. Hence A is von Neumann local. □

Example 4.3. Let us give an example of outer von Neumann local matrix which is not von Neumann local. By Theorem 4.2, one should look for a matrix over a non-semiperfect ring. Consider the semilocal ring $R = \mathbb{Z}_{(2)} \cap \mathbb{Z}_{(3)}$ (which is not semiperfect), and the matrix $A = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(R)$. Direct calculations show that A does not have a non-zero outer inverse. But we have $B(I_2 - A)B = B$ for $B = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} \in M_2(R)$, hence $I_2 - A = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$ has a non-zero outer inverse, and consequently, A is outer von Neumann local. On the other hand, direct calculations show that neither A nor $I_2 - A$ is von Neumann regular. Hence A is not von Neumann local.

Let us now recall the following characterization of matrices having a non-zero outer inverse over local rings.

Theorem 4.4 ([7, Theorem 3.1]). *Let R be local and let $A = (a_{ij}) \in M_{m,n}(R)$. Then the following are equivalent:*

(1) *There exists some $a_{ij} \in U(R)$.*
(2) *There exists some a_{ij} having a non-zero outer inverse.*
(3) *A has a non-zero outer inverse.*
(4) *$A \notin M_{m,n}(\mathrm{rad}(R))$.*

Theorem 4.5. *Let R be an arbitrary local ring and let $A \in M_n(R)$. Then the following are equivalent:*

(1) *A is outer von Nemann local.*
(2) *A or $I_n - A$ has an invertible entry.*
(3) *A or $I_n - A$ has an entry with a non-zero outer inverse.*

(4) $A \notin M_n(\mathrm{rad}(R))$ *or* $I_n - A \notin M_n(\mathrm{rad}(R))$.

Proof. This follows from Theorem 4.4. □

Finally, let us see how the property of being outer von Neumann local behaves with respect to direct products. We obtain some similar result as in the case of (strongly) von Neumann local property. Let us denote by $\mathrm{ovnr}(R)$ (respectively $\mathrm{ovnl}(R)$) the set of outer von Neumann regular (respectively outer von Neumann local) elements of a ring R.

THEOREM 4.6. *Let* $R = \prod_{k \in K} R_k$ *be a product of arbitrary rings. Then* $\mathrm{ovnl}(R) = \prod_{k \in K} \mathrm{ovnl}(R_k)$ *if and only if* $\mathrm{ovnl}(R_k) = \mathrm{ovnr}(R_k)$ *for all but at most one* $k \in K$. *In particular,* R *is an outer von Neumann local ring if and only if there is at most one* $k \in K$ *such that* R_k *does not have a non-zero outer inverse, but* R_k *is outer von Neumann local.*

Proof. Use the same path as in the proofs of Theorems 2.9 and 3.8. □

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Received March 5, 2023
Accepted June 15, 2023

Babeş-Bolyai University
Faculty of Mathematics and Computer Science
Department of Mathematics
Cluj-Napoca, Romania
E-mail: iulia.chiru@ubbcluj.ro
E-mail: septimiu.crivei@ubbcluj.ro
https://orcid.org/0000-0001-6382-9148