MATHEMATICA, 65 (88), N° 2, 2023, pp. 207–214

ON THE GOLDIE DIMENSION OF FINITELY GENERATED LOCALLY CYCLIC MODULES

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Abstract. Let R be a commutative ring with identity. In this paper we investigate the Goldie dimension of finitely generated locally cyclic R-modules. Then, we give a characterization of rings whose finitely generated locally cyclics have finite Goldie dimension.

MSC 2020. 13C15, 13C13.

Key words. Commutative rings, Goldie dimension, finitely generated modules.

1. INTRODUCTION

For physicists, dimensions in the universe are viewed as independent directions. In order to show if our R-module M contains only one independent physical direction, let us say that this R-module cannot contain any direct sum of two nonzero submodules, otherwise these direct factors will present two independent physical directions. In particular, if N and L are two nonzero submodules of M, then $L \cap N \neq (0)$, otherwise their sum will be a direct sum. Recall that a submodule N of M is called an essential submodule of M (or a large submodule of M) if for any submodule L of M, $L \cap N \neq \{0\}$ and M is called a uniform R-module if every nonzero submodule of M is essential [8]. Then, an R-module which contains only one independent physical direction is exactly a uniform R-module.

Now, if we have a direct sum of n uniform submodules in M then, we get n independent physical directions in M. In order to ensure that all directions in our universe are covered, it is enough that this direct sum of uniform submodules be an essential submodule in M. In other words, if $N = \bigoplus_{k=1}^{n} N_k$ is an essential submodule of M such that for each $k \in \{1, ..., n\}$, N_k is a uniform submodule of M, then n is the number of independent physical directions in our universe M, and it is called the Goldie dimension (or the uniform dimension) of the R-module M, denoted by $\mathrm{Gdim}(M) = n$ (see [7]).

The authors thank the referee for his helpful comments and suggestions.

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DOI: 10.24193/mathcluj.2023.2.06

Throughout this paper all rings will be commutative with identity. An R-module M is called a multiplication module if for any submodule N of M, there is an ideal I of R such that $N = I.M$, in this case we can take $I = [N : M] = \{r \in R \mid (\forall m \in M), r.m \in N\}$ (see for example [1,5]).

Our aim in this paper is to investigate the Goldie dimension of finitely generated locally cyclic R-modules. Some important lemmas on locally cyclic modules and multiplication modules are given in Section 2, then we prove several results in Section 3, which characterize uniform finitely generated locally cyclic modules and Goldie dimensions of finitely generated locally cyclic modules. Furthermore, we give a condition under which any finitely generated locally cyclic R-module has a finite Goldie dimension. This generalizes partially the results obtained by Al-Huzali, Jain, and Lopez-Permouth in [2].

2. PRELIMINARIES AND LEMMAS

By the following fact, finitely generated multiplication modules and finitely generated locally cyclic modules are the same.

LEMMA 2.1 ([5, Proposition 5]). *A finitely generated module is a multiplication module if and only if it is locally cyclic.*

Recall that an R-module M has cancellation if $I.M = J.M$ implies that $I = J$ for any ideals I and J of R. The R-module M has weak cancellation if $I.M = J.M$ implies that $I + \mathrm{Ann}_R(M) = J + \mathrm{Ann}_R(M)$ for any ideals I and J of R.

LEMMA 2.2 ([3, Theorem 5.16]).

(i) *A module is a multiplication module and has weak cancellation if and only if it is finitely generated and locally cyclic.*

(ii) *A module is a multiplication module and has cancellation if and only if it is finitely generated and a faithful locally cyclic module.*

LEMMA 2.3. *Let M be a multiplication R-module and N_1 and N_2 two submodules of M. Then, the following statements are equivalent:*

(i) $N_1 \cap N_2 = (0)$.

(ii) $[N_1 : M] \cap [N_2 : M] \subseteq \mathrm{Ann}_R(M)$.

Proof. **Step 1.** Let $r \in [N_1 \cap N_2 : M]$. Then, $r.m \in N_1 \cap N_2$ for any $m \in M$. Namely, $r.m \in N_1$ and $r.m \in N_2$ for any $m \in M$. Thus, $r \in [N_1 : M]$ and $r \in [N_2 : M]$. This proves that $r \in [N_1 : M] \cap [N_2 : M]$.

Conversely, if $r \in [N_1 : M] \cap [N_2 : M]$, then $r.m \in N_1$ and $r.m \in N_2$ for any $m \in M$. Namely, $r \in [N_1 : M]$ and $r \in [N_2 : M]$. This proves that $[N_1 \cap N_2 : M] = [N_1 : M] \cap [N_2 : M]$.

Step 2. We know that $N_1 = [N_1 : M].M$ and $N_2 = [N_2 : M].M$, then $N_1 \cap N_2 = ([N_1 : M].M) \cap ([N_2 : M].M)$. As well as, $N_1 \cap N_2 = [N_1 \cap N_2 : M].M = ([N_1 : M] \cap [N_2 : M]).M$. Therefore, $N_1 \cap N_2 = ([N_1 : M].M) \cap ([N_2 : M].M) = ([N_1 : M] \cap [N_2 : M]).M$.

Step 3. $N_1 \cap N_2 = (0)$ means that $([N_1 : M] \cap [N_2 : M]).M = (0)$ which is equivalent to saying that $[N_1 : M] \cap [N_2 : M] \subseteq \mathrm{Ann}_R(M)$. □

LEMMA 2.4. *Any integral domain is uniform.*

Proof. Let R be an integral domain and I and J be two ideals of R such that $I \cap J = (0)$ and $I \neq (0)$. Let a be a nonzero element in I and b be an element in J. Then, $ab \in I \cap J = (0)$. Namely, $ab = 0$. However, $a \neq 0$ then $b = 0$. This proves that $J = (0)$. thus, R is uniform. □

LEMMA 2.5. *Let M be an R-module, $\mathfrak{S}$ be the set of nonzero submodules of M and* $X = \{n \in \mathbb{N} \mid (\exists (N_k)_{1 \leq k \leq n} \subset \mathfrak{S}), N_1 + N_2 + ... + N_n \text{ is a direct sum}\}$. *Then,* $\mathrm{Gdim}(M) = \sup X$.

Proof. If $\mathrm{Gdim}(M)$ is infinite, then there exist an infinite family $(N_k)_{k \in \mathbb{N}}$ of nonzero uniform submodules of M such that $\sum_{k \in \mathbb{N}} N_k$ is a direct sum. Then, $\sup X$ is infinite.

Suppose now that $\mathrm{Gdim}(M) = n$ for an integer n. Then, there exists a family $(N_k)_{1 \leq k \leq n}$ of nonzero uniform submodules of M such that $\sum_{k=1}^{n} N_k$ is a direct sum. It follows that $\sup X \geq n$.

Suppose that there exists a family $(N'_k)_{1 \leq k \leq n+1}$ of nonzero submodules of M such that $\sum_{k=1}^{n+1} N'_k$ is a direct sum. Since any nonzero R-module contains a nonzero uniform submodule, $\mathrm{Gdim}(M) \geq n + 1$ which is a contradiction. Then, $\sup X = n$. □

LEMMA 2.6. *Let m be an integer. Then,* $\mathrm{Gdim}(\mathbb{Z}/m\mathbb{Z}) = n$ *where n is the number of prime divisors of m.*

Proof. Let $m = p_1^{v_1}.p_2^{v_2}...p_n^{v_n}$ be the prime factorization of the integer m. Put $q_i = \frac{m}{p_i}$ for each i and $N = q_1\mathbb{Z}/m\mathbb{Z} + q_2\mathbb{Z}/m\mathbb{Z} + ... + q_n\mathbb{Z}/m\mathbb{Z}$.

Notice that

$$q_2\mathbb{Z}/m\mathbb{Z} + q_3\mathbb{Z}/m\mathbb{Z} + ... + q_n\mathbb{Z}/m\mathbb{Z} = \gcd(q_2, q_3, ..., q_n)\mathbb{Z}/m\mathbb{Z} \subseteq p_1^{v_1}\mathbb{Z}/m\mathbb{Z}$$

and

$$q_1\mathbb{Z}/m\mathbb{Z} \cap p_1^{v_1}\mathbb{Z}/m\mathbb{Z} = \mathrm{lcm}(q_1, p_1^{v_1})\mathbb{Z}/m\mathbb{Z} = m\mathbb{Z}/m\mathbb{Z} = (0).$$

Likewise, we see that $q_1\mathbb{Z}/m\mathbb{Z} + q_2\mathbb{Z}/m\mathbb{Z} + ... + q_n\mathbb{Z}/m\mathbb{Z}$ is a direct sum.

Moreover, for each i, if $kq_i\mathbb{Z}/m\mathbb{Z} \cap hq_i\mathbb{Z}/m\mathbb{Z} = (0)$ for some integers k and h then $\mathrm{lcm}(k, h)q_i\mathbb{Z}/m\mathbb{Z} = (0)$. Namely, p_i divides $\mathrm{lcm}(k, h)$. It follows that either p_i divides k or p_i divides h, then either $kq_i\mathbb{Z}/m\mathbb{Z} = (0)$ or $hq_i\mathbb{Z}/m\mathbb{Z} = (0)$. This proves that $q_i\mathbb{Z}/m\mathbb{Z}$ is uniform.

It suffices now to prove that N is essential in $\mathbb{Z}/m\mathbb{Z}$. Let k be an integer such that $k\mathbb{Z}/m\mathbb{Z} \cap N = (0)$. Notice that

$$N = \gcd(q_1, q_2, ..., q_n)\mathbb{Z}/m\mathbb{Z} = p_1^{v_1-1}.p_2^{v_2-1}...p_n^{v_n-1}\mathbb{Z}/m\mathbb{Z}$$

then

$$k\mathbb{Z}/m\mathbb{Z} \cap N = \mathrm{lcm}(k, p_1^{v_1-1}.p_2^{v_2-1}...p_n^{v_n-1})\mathbb{Z}/m\mathbb{Z} = (0).$$

Therefore, m divides $\mathrm{lcm}(k, p_1^{v_1-1}.p_2^{v_2-1}...p_n^{v_n-1})$, it follows that m divides k. Hence $k\mathbb{Z}/m\mathbb{Z} = (0)$. This proves that N is a direct sum of n nonzero uniform submodules of $\mathbb{Z}/m\mathbb{Z}$ which is essential. As well as, $\mathrm{Gdim}(\mathbb{Z}/m\mathbb{Z}) = n$. □

Recall that a q.f.d. R-module is an R-module whose every quotient has finite Goldie dimension.

LEMMA 2.7 ([6]). *An R-module M is q.f.d. if and only if every submodule N of M contains a finitely generated submodule T such that N/T has no maximal submodules.*

3. MAIN RESULTS

Recall that an ideal I of R is said to be irreducible, if $I = J_1 \cap J_2$ implies that either $J_1 = I$ or $J_2 = I$ (see for example [4]).

THEOREM 3.1. *Let M be a finitely generated locally cyclic R-module. Then, the following statements are equivalent:*

(i) *M is uniform.*
(ii) *$R/\mathrm{Ann}_R(M)$ is uniform.*
(iii) *$\mathrm{Ann}_R(M)$ is irreducible.*

Proof. (i) ⇒ (ii). Assume that M is uniform. Let I and J be two ideals in R such that $\mathrm{Ann}_R(M) \subseteq I \cap J$.

Suppose $(I/\mathrm{Ann}_R(M)) \cap (J/\mathrm{Ann}_R(M)) = (0)$. Then, $(I \cap J)/\mathrm{Ann}_R(M) = (0)$. Namely, $I \cap J \subseteq \mathrm{Ann}_R(M)$. Let $N_1 = I.M$ and $N_2 = J.M$. Since M has weak cancellation, $I + \mathrm{Ann}_R(M) = [N_1 : M] + \mathrm{Ann}_R(M)$ and $J + \mathrm{Ann}_R(M) = [N_2 : M] + \mathrm{Ann}_R(M)$. It follows that $[N_1 : M]/\mathrm{Ann}_R(M) = I/\mathrm{Ann}_R(M)$ and $[N_2 : M]/\mathrm{Ann}_R(M) = J/\mathrm{Ann}_R(M)$, then $([N_1 : M] \cap [N_2 : M])/\mathrm{Ann}_R(M) = (I \cap J)/\mathrm{Ann}_R(M)$. However, $I \cap J \subseteq \mathrm{Ann}_R(M)$, then $([N_1 : M] \cap [N_2 : M])/\mathrm{Ann}_R(M) = (0)$ and $[N_1 : M] \cap [N_2 : M] \subseteq \mathrm{Ann}_R(M)$.

By Lemma 2.3 $N_1 \cap N_2 = ([N_1 : M] \cap [N_2 : M]).M$, then $N_1 \cap N_2 = (0)$. Since M is uniform, either $N_1 = (0)$ or $N_2 = (0)$. Namely, either $[N_1 : M] \subseteq \mathrm{Ann}_R(M)$ or $[N_2 : M] \subseteq \mathrm{Ann}_R(M)$. As well as, either $I \subseteq \mathrm{Ann}_R(M)$ or $J \subseteq \mathrm{Ann}_R(M)$ since $I + \mathrm{Ann}_R(M) = [N_1 : M] + \mathrm{Ann}_R(M)$ and $J + \mathrm{Ann}_R(M) = [N_2 : M] + \mathrm{Ann}_R(M)$. Therefore, either $I/\mathrm{Ann}_R(M) = (0)$ or $J/\mathrm{Ann}_R(M) = (0)$. This proves that $R/\mathrm{Ann}_R(M)$ is uniform.

(ii) ⇒ (i). Suppose $R/\mathrm{Ann}_R(M)$ is uniform. Let N_1 and N_2 be two submodules of M such that $N_1 \cap N_2 = (0)$.

By Lemma 2.3, $[N_1 : M] \cap [N_2 : M] \subseteq \mathrm{Ann}_R(M)$, then

$$([N_1 : M] + \mathrm{Ann}_R(M))/\mathrm{Ann}_R(M) \cap ([N_2 : M] + \mathrm{Ann}_R(M))/\mathrm{Ann}_R(M) = (0)$$

in $R/\mathrm{Ann}_R(M)$, which is uniform.

It follows that either $([N_1 : M] + \mathrm{Ann}_R(M))/\mathrm{Ann}_R(M) = (0)$ or $([N_2 : M] + \mathrm{Ann}_R(M))/\mathrm{Ann}_R(M) = (0)$. Namely, either $[N_1 : M] \subseteq \mathrm{Ann}_R(M)$ or

$[N_2 : M] \subseteq \mathrm{Ann}_R(M)$. Thus, either $N_1 = [N_1 : M].M = (0)$ or $N_2 = [N_2 : M].M = (0)$. This proves that M is uniform.

(ii) $\Rightarrow$ (iii). Suppose $R/\mathrm{Ann}_R(M)$ is uniform. Let I and J be two ideals of R such that $\mathrm{Ann}_R(M) = I \cap J$. Then, $I/\mathrm{Ann}_R(M) \cap J/\mathrm{Ann}_R(M) = (0)$. Since $R/\mathrm{Ann}_R(M)$ is uniform, either $I/\mathrm{Ann}_R(M) = (0)$ or $J/\mathrm{Ann}_R(M) = (0)$. It follows that either $I = \mathrm{Ann}_R(M)$ or $J = \mathrm{Ann}_R(M)$. Thus $\mathrm{Ann}_R(M)$ is irreducible.

(iii) $\Rightarrow$ (ii). Suppose $\mathrm{Ann}_R(M)$ is irreducible. Let

$$I/\mathrm{Ann}_R(M) \cap J/\mathrm{Ann}_R(M) = (0)$$

for some ideals I and J in R. Then, $I \cap J = \mathrm{Ann}_R(M)$. However $\mathrm{Ann}_R(M)$ is irreducible, then either $I = \mathrm{Ann}_R(M)$ or $J = \mathrm{Ann}_R(M)$. Namely, either $I/\mathrm{Ann}_R(M) = (0)$ or $J/\mathrm{Ann}_R(M) = (0)$. Hence $R/\mathrm{Ann}_R(M)$ is uniform. □

Corollary 3.2. *Let M be a finitely generated R-module, Γ be the set of ideals I of R such that $\mathrm{Ann}_R(M) \subsetneq I$. Suppose that $R/\mathrm{Ann}_R(M)$ is Artinian. Then, the following statements are equivalent:*

(i) *M is uniform.*
(ii) *Γ has a unique minimal element by inclusion.*

Proof. By Theorem 3.1, it is enough to prove that $R/\mathrm{Ann}_R(M)$ is uniform if and only if Γ has a unique minimal element by inclusion. Suppose that $R/\mathrm{Ann}_R(M)$ is uniform. Let Γ' be the set of nonzero ideals of $R/\mathrm{Ann}_R(M)$. Since $R/\mathrm{Ann}_R(M)$ is Artinian, every chain in Γ' is finite and has a minimal element. Then, by Zorn's lemma Γ' has a minimal element. If $I/\mathrm{Ann}_R(M) \neq (0)$ and $J/\mathrm{Ann}_R(M) \neq 0$ are two minimal elements in Γ' then either $(I \cap J)/\mathrm{Ann}_R(M) = I/\mathrm{Ann}_R(M)$ or $(I \cap J)/\mathrm{Ann}_R(M) = (0)$. However, $R/\mathrm{Ann}_R(M)$ is uniform, then $(I \cap J)/\mathrm{Ann}_R(M) \neq (0)$. Then, $(I \cap J)/\mathrm{Ann}_R(M) = I/\mathrm{Ann}_R(M)$ which proves that $I/\mathrm{Ann}_R(M) = J/\mathrm{Ann}_R(M)$. So that Γ' has a unique minimal element $I/\mathrm{Ann}_R(M)$. Set the map:

$$\varphi : \Gamma' \to \Gamma, \qquad J/\mathrm{Ann}_R(M) \mapsto J.$$

φ is a one-to-one correspondence verifying $J_1/\mathrm{Ann}_R(M) \subset J_2/\mathrm{Ann}_R(M)$ if and only if $\varphi(J_1/\mathrm{Ann}_R(M)) \subset \varphi(J_2/\mathrm{Ann}_R(M))$. Then, $\varphi(I/\mathrm{Ann}_R(M)) = I$ is the unique minimal element in Γ.

Conversely, assume that Γ has a unique minimal element I. Consider $J_1/\mathrm{Ann}_R(M)$ and $J_2/\mathrm{Ann}_R(M)$ to be two nonzero ideals in $R/\mathrm{Ann}_R(M)$. Then

$$I \subseteq \varphi(J_1/\mathrm{Ann}_R(M)) = J_1 \text{ and } I \subseteq \varphi(J_2/\mathrm{Ann}_R(M)) = J_2.$$

Namely, $I \subseteq J_1 \cap J_2$. As well as, $(0) \neq I/\mathrm{Ann}_R(M) \subseteq (J_1/\mathrm{Ann}_R(M)) \cap (J_2/\mathrm{Ann}_R(M))$. This proves that $R/\mathrm{Ann}_R(M)$ is uniform. □

Corollary 3.3. *Let M be a finitely generated locally cyclic R-module. If $\mathrm{Ann}_R(M)$ is a prime ideal of R then, M is uniform.*

Proof. Since $\mathrm{Ann}_R(M)$ is prime, $R/\mathrm{Ann}_R(M)$ is integral. By Lemma 2.4, $R/\mathrm{Ann}_R(M)$ is uniform. As well as, by Theorem 3.1 M is uniform. □

THEOREM 3.4. *Let M be a finitely generated locally cyclic R-module. Then,* $\mathrm{Gdim}(M) = \mathrm{Gdim}(R/\mathrm{Ann}_R(M))$.

Proof. **Step 1.** Let N_1 and N_2 be two nonzero submodules of M such that $N_1 \cap N_2 = (0)$. By Lemma 2.3, $[N_1 : M] \cap [N_2 : M] \subseteq \mathrm{Ann}_R(M)$. Then, $[N_1 : M]/\mathrm{Ann}_R(M) \cap [N_2 : M]/\mathrm{Ann}_R(M) = (0)$. This proves that $[N_1 : M]/\mathrm{Ann}_R(M) + [N_2 : M]/\mathrm{Ann}_R(M)$ is a direct sum of nonzero ideals of $R/\mathrm{Ann}_R(M)$.

Suppose that if $N_1 + N_2 + ... + N_n$ is a direct sum of nonzero submodules of M then $[N_1 : M]/\mathrm{Ann}_R(M)+[N_2 : M]/\mathrm{Ann}_R(M)+...+[N_n : M]/\mathrm{Ann}_R(M)$ is a direct sum of nonzero ideals of $R/\mathrm{Ann}_R(M)$ for an integer n. Let $N_1 + N_2 + ...+N_{n+1}$ be a direct sum of nonzero submodules of M. In particular, $N_1+N_2+ ...+N_n$ is a direct sum of nonzero submodules of M, then $[N_1 : M]/\mathrm{Ann}_R(M)+ [N_2 : M]/\mathrm{Ann}_R(M)+...+[N_n : M]/\mathrm{Ann}_R(M)$ is a direct sum of nonzero ideals of $R/\mathrm{Ann}_R(M)$. Set $N = N_1 + N_2 + ... + N_n$, we have that $N + N_{n+1}$ is a direct sum of nonzero submodules of M, then $[N : M]/\mathrm{Ann}_R(M) + [N_{n+1} : M]/\mathrm{Ann}_R(M)$ is a direct sum of nonzero ideals of $R/\mathrm{Ann}_R(M)$. Since $[N_1 : M] + [N_2 : M] + ... + [N_n : M] \subseteq [N : M]$, the sum $[N_1 : M]/\mathrm{Ann}_R(M) + [N_2 : M]/\mathrm{Ann}_R(M) + ... + [N_{n+1} : M]/\mathrm{Ann}_R(M)$ is a direct sum in $R/\mathrm{Ann}_R(M)$. By induction, we see that any direct sum of n nonzero submodules of M gives a direct sum of n nonzero ideals of $R/\mathrm{Ann}_R(M)$. By Lemma 2.5, this proves that $\mathrm{Gdim}(M) \leq \mathrm{Gdim}(R/\mathrm{Ann}_R(M))$.

Step 2. Consider $I_1/\mathrm{Ann}_R(M)$ and $I_2/\mathrm{Ann}_R(M)$ to be two nonzero ideals of $R/\mathrm{Ann}_R(M)$ such that $I_1/\mathrm{Ann}_R(M)\cap I_2/\mathrm{Ann}_R(M) = (0)$. Then, $I_1 \cap I_2 = \mathrm{Ann}_R(M)$. Put $N_1 = I_1.M = [N_1 : M].M$ and $N_2 = I_2.M = [N_2 : M].M$. Since M has weak cancellation, $I_1 + \mathrm{Ann}_R(M) = [N_1 : M] + \mathrm{Ann}_R(M)$ and $I_2 + \mathrm{Ann}_R(M) = [N_2 : M] + \mathrm{Ann}_R(M)$. However, $\mathrm{Ann}_R(M) \subseteq I_1 \cap I_2$ and $\mathrm{Ann}_R(M) \subseteq [N_1 : M]\cap[N_2 : M]$ then $I_1 = [N_1 : M]$ and $I_2 = [N_2 : M]$. Then, $N_1 \cap N_2 = (I_1 \cap I_2).M = (0)$ since $I_1 \cap I_2 \subseteq \mathrm{Ann}_R(M)$.

Suppose that for an integer n, if

$$I_1/\mathrm{Ann}_R(M) + I_2/\mathrm{Ann}_R(M) + ... + I_n/\mathrm{Ann}_R(M)$$

is a direct sum of nonzero ideals of $R/\mathrm{Ann}_R(M)$ then $N_1 + N_2 + ... + N_n$ is a direct sum of nonzero submodules of M where $N_k = I_k.M$ for each k. Let $I_1/\mathrm{Ann}_R(M) + I_2/\mathrm{Ann}_R(M) + ... + I_{n+1}/\mathrm{Ann}_R(M)$ be a direct sum of $n + 1$ nonzero ideals of $R/\mathrm{Ann}_R(M)$. Then, $I_1/\mathrm{Ann}_R(M) + I_2/\mathrm{Ann}_R(M) + ... + I_n/\mathrm{Ann}_R(M)$ is a direct sum of n nonzero ideals of $R/\mathrm{Ann}_R(M)$. It follows that $N_1+N_2+...+N_n$ is a direct sum of nonzero submodules of M, where $N_k = I_k.M$ for each k. Let $I = I_1 + I_2 + ... + I_n$, then $I/\mathrm{Ann}_R(M) + I_{n+1}/\mathrm{Ann}_R(M)$ is a direct sum of two nonzero ideals of $R/\mathrm{Ann}_R(M)$. Therefore, $N + N_{n+1}$ is a direct sum of nonzero submodules of M where $N = I.M = N_1 + N_2 + ... + N_n$

and $N_{n+1} = I_{n+1}.M$. Namely, $N_1 + N_2 + ... + N_{n+1}$ is a direct sum of nonzero submodules of M.

By induction, we see that any direct sum of n nonzero ideals of $R/\mathrm{Ann}_R(M)$ gives a direct sum of n nonzero submodules of M. By Lemma 2.5, this proves that $\mathrm{Gdim}(R/\mathrm{Ann}_R(M)) = \mathrm{Gdim}(M)$. □

Corollary 3.5. *Let M be a finitely generated locally cyclic $\mathbb{Z}$-module. Then, there are two possible cases:*

(i) *If M is faithful, then* $\mathrm{Gdim}(M) = 1$.
(ii) *If M is not faithful, then* $\mathrm{Gdim}(M) = n$ *where n is the number of prime divisors of* $|\mathbb{Z}/\mathrm{Ann}_R(M)|$.

Proof. If M is faithful then, $\mathrm{Ann}_R(M) = (0)$ and $\mathbb{Z}/\mathrm{Ann}_R(M) = \mathbb{Z}$ which is integral. By Lemma 2.4, $\mathbb{Z}$ is uniform and by Theorem 3.1 M is uniform. Thus, $\mathrm{Gdim}(M) = 1$.

If M is not faithful then, $\mathrm{Ann}_R(M) = m\mathbb{Z}$ for a nonzero integer m. By Theorem 3.4 $\mathrm{Gdim}(M) = \mathrm{Gdim}(\mathbb{Z}/m\mathbb{Z})$ and by Lemma 2.6 $\mathrm{Gdim}(M) = n$ where n is the number of prime divisors of $m = |\mathbb{Z}/m\mathbb{Z}|$. □

Corollary 3.6. *Every faithful finitely generated locally cyclic module has finite Goldie dimension if and only if R has finite Goldie dimension.*

Proof. Let M be a faithful finitely generated locally cyclic module. Then, by Theorem 3.4, $\mathrm{Gdim}(M) = \mathrm{Gdim}(R/\mathrm{Ann}_R(M))$. However M is faithful, then $\mathrm{Ann}_R(M) = (0)$ and $\mathrm{Gdim}(M) = \mathrm{Gdim}(R)$. Then, $\mathrm{Gdim}(M)$ is finite if and only if $\mathrm{Gdim}(R)$ is finite. □

Al-Hazali et al. [2] studied the rings whose cyclics have finite Goldie dimension. In the following theorem we characterize also the rings whose finitely generated locally cyclics have finite Goldie dimension.

Theorem 3.7. *The following statements are equivalent:*

(i) *Every finitely generated locally cyclic R-module has finite Goldie dimension.*
(ii) *R is q.f.d.*
(iii) *Every ideal I of R contains a finitely generated ideal J such that I/J is a simple R-module.*

Proof. (i) ⇒ (ii). Let I be an ideal of R. Then, $M = R/I$ is a finitely generated locally cyclic R-module since it is cyclic. It follows that $\mathrm{Gdim}(M)$ is finite. Thus, R is q.f.d.

(ii) ⇒ (i). Let M be a finitely generated locally cyclic R-module. By Lemma 2.1, M is a finitely generated multiplication R-module and by Theorem 3.4 $\mathrm{Gdim}(M) = \mathrm{Gdim}(R/\mathrm{Ann}_R(M))$. Since R is q.f.d., $\mathrm{Gdim}(R/\mathrm{Ann}_R(M))$ is finite and $\mathrm{Gdim}(M)$ is finite.

(ii) ⇔ (iii). Obtained from Lemma 2.7. It suffices to notice that for any ideals I and J of R, the fact that I/J has no maximal ideals is equivalent to saying that I/J is a simple R-module. □

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Received June 25, 2022
Accepted January 27, 2023

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