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LEFT MULTIPLIERS AND COMMUTATIVITY OF 3-PRIME NEAR-RINGS

ABDELKARIM BOUA and BIJAN DAVVAZ

Abstract. Our objective in this paper is to study the structure of 3-prime near-rings satisfying some algebraic properties.

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1. INTRODUCTION

Throughout this paper, $\mathcal{N}$ will be a right near-ring with multiplicative center $Z(\mathcal{N})$. A near-ring is called zero symmetric if $x0 = 0$, for all $x \in \mathcal{N}$ (recall that the right distributive law yields $0x = 0$), and usually $\mathcal{N}$ will be 3-prime if for all $x, y \in \mathcal{N}$, $x\mathcal{N}y = \{0\}$ implies $x = 0$ or $y = 0$.

Further $\mathcal{N}$ is called 2-torsion free if $2x = 0$ implies $x = 0$, for all $x \in \mathcal{N}$. An additive mapping $d : \mathcal{N} \to \mathcal{N}$ is a derivation if $d(xy) = xd(y) + d(x)y$, for all $x, y \in \mathcal{N}$, or equivalently, as noted in [12], if $d(xy) = d(x)y + xd(y)$, for all $x, y \in \mathcal{N}$.

Let d be a derivation of $\mathcal{N}$, an additive mapping $F : \mathcal{N} \to \mathcal{N}$ is said to be a right generalized derivation associated with d if $F(xy) = F(x)y + xd(y)$, for all $x, y \in \mathcal{N}$. F is said to be a left generalized derivation associated with d if $F(xy) = d(x)y + xF(y)$, for all $x, y \in \mathcal{N}$. Moreover, F is said to be a generalized derivation associated with d if it is both a left and a right generalized derivation of $\mathcal{N}$, associated with d.

An additive mapping $H : \mathcal{N} \to \mathcal{N}$ is said to be a left (right) multiplier if $H(xy) = H(x)y$ $(H(xy) = xH(y))$, for all $x, y \in \mathcal{N}$. F is said to be a multiplier if it is both left and right multiplier. Notice that a right (resp. left) generalized derivation with associated derivation $d = 0$ is a left (resp. a right) multiplier. The notion of generalized derivation is introduced in [5] by Bresar. It is clear that, every derivation is a generalized derivation, but the converse is not true in general. Hence generalized derivation covers both the concepts of derivation and left multiplier maps.

For any pair of elements $x, y \in \mathcal{N}$, $[x, y] = xy - yx$ and $x \circ y = xy + yx$ will denote the well-known Lie product and Jordan product, respectively.

Corresponding author: Abdelkarim Boua.

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Let α and β be two left multipliers of $\mathcal{N}$, for any $x, y \in \mathcal{N}$,

$$[x, y]_{\alpha,\beta} = \alpha(x)y - y\beta(x), \text{ and } (x \circ y)_{\alpha,\beta} = \alpha(x)y + y\beta(x).$$

A number of authors have proved commutativity theorems for prime rings admitting derivations, generalized derivations or left multipliers (see [1–4, 6–9, 11], where further references can be found).

In 2009, the authors with M. Ashraf [1] established that a prime ring $\mathcal{R}$ with a nonzero ideal I must be commutative if it admits a nonzero-generalized derivation F satisfying either one of the properties: $F([x, y]) - [x, y] \in Z(\mathcal{R})$, $F(x \circ y) - x \circ y \in Z(\mathcal{R})$, for all $x, y \in I$.

In 2013, B. Dhara et al. proved that $\mathcal{R}$ must be commutative, if it admits a non-trivial left multiplier F such that $F([x, y]) - [x, y] \in Z(\mathcal{R})$, for all $x, y \in \mathcal{R}$. After, in [10], M. Samman et al. have also studied the following identities $F([x, y]) \in Z(\mathcal{N})$ and $F(x \circ y) \in Z(\mathcal{N})$, for all $x, y \in \mathcal{N}$, which generalize the results mentioned previously in the case of a ring (it suffices to replace F by $F \pm \mathrm{id}_{\mathcal{N}}$). If the underlying derivation d is zero, then the problem is still open.

In this section, we continue this study and obtain similar results in the setting of left multipliers. In this line of investigation, it is more interesting to study this identities involving left multipliers. In this paper, our main object is to investigate the cases when left multipliers H, α, and β satisfy the identities:

(i) $H([x, y]_{\alpha,\beta}) \in Z(\mathcal{N})$,
(ii) $H((x \circ y)_{\alpha,\beta}) \in Z(\mathcal{N})$,
(iii) $H([x, [y, z]_{\alpha,\beta}]_{\alpha,\beta}) \in Z(\mathcal{N})$, and
(iv) $H\big((x \circ (y \circ z)_{\alpha,\beta})_{\alpha,\beta}\big) \in Z(\mathcal{N})$,

for all $x, y \in \mathcal{N}$.

2. SOME PRELIMINARIES

In this section, we give some lemmas which are crucial in the development of our main results.

Lemma 2.1. *Let $\mathcal{N}$ be 3-prime near-ring.*

(i) [3, Lemma 1.2 (iii)] *If $z \in Z(\mathcal{N})$ and x is an element of $\mathcal{N}$ such that $xz \in Z(\mathcal{N})$ or $zx \in Z(\mathcal{N})$, then $x \in Z(\mathcal{N})$.*
(ii) [3, Lemma 1.3 (i)] *If U is a nonzero semigroup right ideal (resp. semigroup left ideal) and x is an element of $\mathcal{N}$ such that $Ux = \{0\}$ (resp. $xU = \{0\}$), then $x = 0$.*
(iii) [3, Lemma 1.5] *If $Z(\mathcal{N})$ contains a nonzero semigroup left ideal or semigroup right ideal, then $\mathcal{N}$ is a commutative ring.*

The following lemma generalizes Lemma 2.1 (iii) (it suffices to take $\alpha = \mathrm{id}$).

Lemma 2.2. *Let $\mathcal{N}$ be 3-prime near-ring and α a nonzero map on $\mathcal{N}$, then we have the following properties:*

(i) *If U is a nonzero semigroup left ideal of $\mathcal{N}$ and α a nonzero left multiplier on $\mathcal{N}$ such that $\alpha(U) \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

(ii) *If U is a nonzero semigroup right ideal of $\mathcal{N}$ and α a nonzero right multiplier on $\mathcal{N}$ such that $\alpha(U) \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

Proof. For (i) and (ii) it suffices to note that if U is a semigroup left ideal of $\mathcal{N}$ (resp. semigroup right ideal of $\mathcal{N}$), then its image by a left multiplier (resp. right ideal) of $\mathcal{N}$ is also is a left ideal semigroup of $\mathcal{N}$ (resp. right ideal of $\mathcal{N}$). $\square$

3. MAIN RESULTS

In [7, Theorem 2.3], B. Dhara et al. proves that a prime ring $\mathcal{R}$ with $\mathrm{char}(\mathcal{R}) \neq 2$ is a commutative ring if $\mathcal{R}$ admits a non-trivial left multiplier $F : \mathcal{R} \to \mathcal{R}$ such that $F([x, y]) - [x, y] \in Z(\mathcal{R})$, for all $x, y \in \mathcal{R}$. In the following result, we will generalize this result by replacing the Lie product with the product $[x, y]_{\alpha,\beta} = \alpha(x)y - y\beta(x)$.

THEOREM 3.1. *Let $\mathcal{N}$ be 3-prime near-ring, H, α and β are nonzero left multipliers on $\mathcal{N}$, then the following assertions are equivalent:*

(i) *$H([x, y]_{\alpha,\beta}) \in Z(\mathcal{N})$, for all $x, y \in \mathcal{N}$;*
(ii) *$H((x \circ y)_{\alpha,\beta}) \in Z(\mathcal{N})$, for all $x, y \in \mathcal{N}$;*
(iii) *$\mathcal{N}$ is a commutative ring.*

Proof. The implications (iii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii) are obvious.

(i) $\Rightarrow$ (iii). Suppose that

$$H([x, y]_{\alpha,\beta}) \in Z(\mathcal{N}), \text{ for all } x, y \in \mathcal{N}. \tag{1}$$

Replace y by $y\beta(x)$, to get $H([x, y]_{\alpha,\beta})\beta(x) \in Z(\mathcal{N})$, for all $x, y \in \mathcal{N}$ and by using Lemma 2.1 (i), we obtain

$$H([x, y]_{\alpha,\beta}) = 0 \quad \text{or} \quad \beta(x) \in Z(\mathcal{N}), \text{ for all } x, y \in \mathcal{N}. \tag{2}$$

Suppose there exists $x_0 \in \mathcal{N}$ such that $\beta(x_0) \in Z(\mathcal{N})$. Then (1) implies that $\big(H(\alpha(x_0)) - H(\beta(x_0))\big)y \in Z(\mathcal{N})$ for all $y \in \mathcal{N}$ and taking yt in place of y and using Lemma 2.1 (i), we get $\big(H(\alpha(x_0)) - H(\beta(x_0))\big)y = 0$ or $t \in Z(\mathcal{N})$, for all $y \in \mathcal{N}$. Hence, the application of Lemma 2.1 (ii) yields

$$H(\alpha(x_0)) = H(\beta(x_0) \quad \text{or} \quad \mathcal{N} \text{ is a commutative ring.} \tag{3}$$

Since $H(\alpha(x_0)) = H(\beta(x_0)$, we have

$$\begin{aligned} H\big([x_0, y]_{\alpha,\beta}\big) &= H\big(\alpha(x_0)y - y\beta(x_0)\big) \\ &= H\big(\alpha(x_0)y - \beta(x_0)y\big) \\ &= H(\alpha(x_0))y - H(\beta(x_0))y \\ &= 0, \text{ for all } y \in \mathcal{N}. \end{aligned}$$

Here, from (2) and (3), we find that

$$H([x, y]_{\alpha,\beta}) = 0, \text{ for all } x, y \in \mathcal{N} \quad \text{or} \quad \mathcal{N} \text{ is a commutative ring.}$$

Now assume the first case. Then $H(\alpha(x))y = H(y)\beta(x)$, for all $x, y \in \mathcal{N}$. By replacing y by $[u, v]_{\alpha,\beta}$, we get $H(\alpha(x))[u, v]_{\alpha,\beta} = 0$, for all $x, u, v \in \mathcal{N}$. Putting xr in place of x, we clearly get $H(\alpha(x))\mathcal{N}[u, v]_{\alpha,\beta} = \{0\}$, for all $x, u, v \in \mathcal{N}$. Since $\mathcal{N}$ is 3-prime, we obtain $H(\alpha(x)) = 0$ or $[u, v]_{\alpha,\beta} = 0$, for all $x, u, v \in \mathcal{N}$.

If $H(\alpha(x)) = 0$, for all $x \in \mathcal{N}$, then $H(y)\beta(x) = 0$, for all $x, y \in \mathcal{N}$. By replacing y by ys, we arrive at $H(y)\mathcal{N}\beta(x) = \{0\}$, for all $x, y \in \mathcal{N}$ and hence the 3-primeness of $\mathcal{N}$. We conclude that $H = \beta = 0$ a contradiction.

If $[u, v]_{\alpha,\beta} = 0$, for all $u, v \in \mathcal{N}$, then $\alpha(u)v = v\beta(u)$, for all $u, v \in \mathcal{N}$. Further, replace vt in place of v, thus having $\alpha(u)vt = vt\beta(u) = v\alpha(u)t$, for all $u, v, t \in \mathcal{N}$, which implies that $[\alpha(u), v]\mathcal{N} = \{0\}$, for all $u, v \in \mathcal{N}$, so $\alpha(\mathcal{N}) \subseteq Z(\mathcal{N})$, which shows the commutativity of $\mathcal{N}$.

(ii) $\Rightarrow$ (iii). Assume that

$$H((x \circ y)_{\alpha,\beta}) \in Z(\mathcal{N}), \text{ for all } x, y \in \mathcal{N}. \tag{4}$$

Taking $y\beta(x)$ instead of y in (4) we get $H((x \circ y)_{\alpha,\beta})\beta(x) \in Z(\mathcal{N})$, for all $x, y \in \mathcal{N}$ and by using Lemma 2.1 (i), we obtain

$$H((x \circ y)_{\alpha,\beta}) = 0 \quad \text{or} \quad \beta(x) \in Z(\mathcal{N}), \text{ for all } x, y \in \mathcal{N}. \tag{5}$$

Suppose there exists $x_0 \in \mathcal{N}$ such that $\beta(x_0) \in Z(\mathcal{N})$, then (4) implies that $\big(H(\alpha(x_0)) + H(\beta(x_0))\big)y \in Z(\mathcal{N})$ for all $y \in \mathcal{N}$. Taking yt in place of y and also by Lemma 2.1 (i), we obtain $\big(H(\alpha(x_0)) + H(\beta(x_0))\big)y = 0$ or $t \in Z(\mathcal{N})$, for all $y \in \mathcal{N}$. By Lemma 2.1 (ii), we conclude that

$$H(\alpha(x_0)) = -H(\beta(x_0)) \quad \text{or} \quad \mathcal{N} \text{ is a commutative ring.} \tag{6}$$

Since $H(\alpha(x_0)) = -H(\beta(x_0))$, we have

$$\begin{aligned} H\big((x_0 \circ y)_{\alpha,\beta}\big) &= H\big(\alpha(x_0)y + y\beta(x_0)\big) \\ &= H\big(\alpha(x_0)y + \beta(x_0)y\big) \\ &= H(\alpha(x_0))y - H(\alpha(x_0))y \\ &= 0, \text{ for all } y \in \mathcal{N}. \end{aligned}$$

Here, (5) and (6) force

$$H((x \circ y)_{\alpha,\beta} = 0, \text{ for all } x, y \in \mathcal{N} \quad \text{or} \quad \mathcal{N} \text{ is a commutative ring .}$$

If $H((x \circ y)_{\alpha,\beta} = 0$, for all $x, y \in \mathcal{N}$, then $H(\alpha(x))y = -H(y)\beta(x)$, for all $x, y \in \mathcal{N}$. For $y = (u \circ v)_{\alpha,\beta}$, we get $H(\alpha(x))(u \circ v)_{\alpha,\beta} = 0$ for all $x, u, v \in \mathcal{N}$ and putting xr in place of x, we can easily arrive at $H(\alpha(x))\mathcal{N}(u \circ v)_{\alpha,\beta} = \{0\}$, for all $x, u, v \in \mathcal{N}$. Since $\mathcal{N}$ is 3-prime, the last relation reduces to $H(\alpha(x)) = 0$ or $(u \circ v)_{\alpha,\beta} = 0$, for all $x, u, v \in \mathcal{N}$.

If $H(\alpha(x)) = 0$, for all $x \in \mathcal{N}$, using our assumption we have $H(y)\beta(x) = 0$ for all $x, y \in \mathcal{N}$. Replacing y by ys, the above equation can be rewritten as $H(y)\mathcal{N}\beta(x) = \{0\}$, for all $x, y \in \mathcal{N}$. By the 3-primeness of $\mathcal{N}$, we conclude that $H = \beta = 0$ is a contradiction.

If $(u \circ v)_{\alpha,\beta} = 0$, for all $u, v \in \mathcal{N}$, then $\alpha(u)v = -v\beta(u)$, for all $u, v \in \mathcal{N}$. By substituting vt in place of v, we obtain $\alpha(u)vt = -vt\beta(u) = -v(-\alpha(u)t) = (-v)(-\alpha(u))t$, for all $u, v, t \in \mathcal{N}$. It follows that $[-\alpha(u), v]\mathcal{N} = \{0\}$, for all $u, v \in \mathcal{N}$, so $-\alpha(\mathcal{N}) \subseteq Z(\mathcal{N})$. Since $-\alpha$ is also a left multiplier of $\mathcal{N}$, then $\mathcal{N}$ is a commutative ring, by Lemma 2.2 (i). $\square$

It is easy to see that in a ring $\mathcal{R}$, if H is a left multiplier of $\mathcal{R}$, $H \pm \mathrm{id}_{\mathcal{R}}$ is also is left multiplier of $\mathcal{R}$. To find the following theorem replace H by $H - \mathrm{id}_{\mathcal{R}}$ in Theorem 3.1. The following theorem extends [7, Theorem 2.3] to its full generalization.

Theorem 3.2. *Let $\mathcal{R}$ be a prime ring, and H, α and β nonzero left multipliers on $\mathcal{R}$ such that $H \neq \mathrm{id}_{\mathcal{R}}$. Then the following assertions are equivalent:*

(i) $H([x,y]_{\alpha,\beta}) - [x,y]_{\alpha,\beta} \in Z(\mathcal{R})$, *for all* $x, y \in \mathcal{R}$;
(ii) $H((x \circ y)_{\alpha,\beta}) - (x \circ y)_{\alpha,\beta} \in Z(\mathcal{N})$, *for all* $x, y \in \mathcal{R}$;
(iii) $\mathcal{R}$ *is commutative.*

If we replace H with $H + \mathrm{id}_{\mathcal{R}}$, we find the following result:

Theorem 3.3. *Let $\mathcal{N}$ be a 3-prime near-ring, and H, α and β nonzero left multipliers on $\mathcal{N}$ such that $H \neq -\mathrm{id}_{\mathcal{R}}$. Then the following assertions are equivalent:*

(i) $H([x,y]_{\alpha,\beta}) + [x,y]_{\alpha,\beta} \in Z(\mathcal{R})$, *for all* $x, y \in \mathcal{R}$;
(ii) $H((x \circ y)_{\alpha,\beta}) + (x \circ y)_{\alpha,\beta} \in Z(\mathcal{R})$, *for all* $x, y \in \mathcal{R}$;
(iii) $\mathcal{N}$ *is commutative.*

The above theorems have the following interesting consequences:

Corollary 3.4 ([7, Theorem 2.3]). *Let $\mathcal{R}$ be a prime ring with $\mathrm{char}(\mathcal{R}) \neq 2$. If $\mathcal{R}$ admits a non trivial left multiplier $F : \mathcal{R} \to \mathcal{R}$ such that $F([x,y]) - [x,y] \in Z(\mathcal{R})$, for all $x, y \in \mathcal{R}$, then $\mathcal{R}$ is a commutative ring.*

Corollary 3.5. *Let $\mathcal{N}$ be a 3-prime near-ring, and α and β nonzero left multipliers on $\mathcal{N}$, then the following assertions are equivalent:*

(i) $[x,y]_{\alpha,\beta} \in Z(\mathcal{N})$, *for all* $x, y \in \mathcal{N}$;
(ii) $-[x,y]_{\alpha,\beta} \in Z(\mathcal{N})$, *for all* $x, y \in \mathcal{N}$;
(iii) $\alpha([x,y]_{\alpha,\beta}) \in Z(\mathcal{N})$, *for all* $x, y \in \mathcal{N}$;
(iv) $(x \circ y)_{\alpha,\beta} \in Z(\mathcal{N})$, *for all* $x, y \in \mathcal{N}$;
(v) $\alpha((x \circ y)_{\alpha,\beta}) \in Z(\mathcal{N})$, *for all* $x, y \in \mathcal{N}$;
(vi) $-(x \circ y)_{\alpha,\beta} \in Z(\mathcal{N})$, *for all* $x, y \in \mathcal{N}$;
(vii) $\mathcal{N}$ *is a commutative ring.*

Corollary 3.6. *Let $\mathcal{N}$ be a 3-prime near-ring, then the following assertions are equivalent:*

(i) $[x,y] \in Z(\mathcal{N})$, *for all* $x, y \in \mathcal{N}$;
(ii) $-([x,y]) \in Z(\mathcal{N})$, *for all* $x, y \in \mathcal{N}$;
(iii) $x \circ y \in Z(\mathcal{N})$, *for all* $x, y \in \mathcal{N}$;

(iv) $-(x \circ y) \in Z(\mathcal{N})$, *for all* $x, y \in \mathcal{N}$;
(v) $\mathcal{N}$ *is a commutative ring.*

If we replace the product $[x, y]_{\alpha,\beta}$ by $[x, [y, z]\alpha, \beta]_{\alpha,\beta}$, our result remains valid by adding the condition $[u, \alpha(y)]_\alpha = 0$, for all $y, u \in \mathcal{N}$.

THEOREM 3.7. *Let $\mathcal{N}$ be a 3-prime near-ring, and H, α and β nonzero left multipliers on $\mathcal{N}$. Then the following assertions are equivalent:*
(i) $H([x, [y, z]_{\alpha,\beta}]_{\alpha,\beta}) \in Z(\mathcal{N})$, *for all* $x, y, z \in \mathcal{N}$;
(ii) $\mathcal{N}$ *is a commutative ring and* $[u, \alpha(y)]_\alpha = 0$, *for all* $y, u \in \mathcal{N}$.

Proof. It can be easily shown that (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). Suppose that

$$H([x, [y, z]_{\alpha,\beta}]_{\alpha,\beta}) \in Z(\mathcal{N}), \text{ for all } x, y, z \in \mathcal{N}. \tag{7}$$

Taking $x[y, z]_{\alpha,\beta}$ in place of x, we get $H([x, [y, z]_{\alpha,\beta}]_{\alpha,\beta})[y, z]_{\alpha,\beta} \in Z(\mathcal{N})$, for all $x, y, z \in \mathcal{N}$. Using Lemma 2.1 (i), we obtain

$$H([x, [y, z]_{\alpha,\beta}]_{\alpha,\beta}) = 0 \quad \text{or} \quad [y, z]_{\alpha,\beta} \in Z(\mathcal{N}), \text{ for all } x, y \in \mathcal{N}.$$

Suppose there exists $(y_0, z_0) \in \mathcal{N} \times \mathcal{N}$ such that $[y_0, z_0]_{\alpha,\beta} \in Z(\mathcal{N})$. We have

$$\begin{aligned} H\big([x, [y_0, z_0]_{\alpha,\beta}]_{\alpha,\beta}\big) &= H\big(\alpha(x)[y_0, z_0]_{\alpha,\beta} - [y_0, z_0]_{\alpha,\beta}\beta(x)\big) \\ &= H(\alpha(x) - \beta(x))[y_0, z_0]_{\alpha,\beta} \in Z(\mathcal{N}). \end{aligned}$$

By Lemma 2.1 (i), we find that either $[y_0, z_0]_{\alpha,\beta} = 0$ or $H(\alpha(x)) - H(\beta(x)) \in Z(\mathcal{N})$. Writing xt instead of x and using Lemma 2.1(i), we conclude that

$$[y_0, z_0]_{\alpha,\beta} = 0 \quad \text{or} \quad H(\alpha(x)) = H(\beta(x)), \text{ for all } x \in \mathcal{N}.$$

The last two cases give easily $H\big([x, [y_0, z_0]_{\alpha,\beta}]_{\alpha,\beta}\big) = 0$, for all $x \in \mathcal{N}$. In this case, (7) becomes $H\big([x, [y, z]_{\alpha,\beta}]_{\alpha,\beta}\big) = 0$, for all $x \in \mathcal{N}$ and so

$$H(\alpha(x))[y, z]_{\alpha,\beta} = H([y, z]_{\alpha,\beta})\beta(x), \text{ for all } x, y, z \in \mathcal{N}. \tag{8}$$

Replacing z by $[u, t]_{\alpha,\beta}$ and x by xr, we obtain $H(\alpha(x))\mathcal{N}[y, [u, t]_{\alpha,\beta}]_{\alpha,\beta} = \{0\}$, for all $t, u, x, y \in \mathcal{N}$. Since $\mathcal{N}$ is 3-prime, we get

$$H(\alpha(x)) = 0 \quad \text{or} \quad [y, [u, t]_{\alpha,\beta}]_{\alpha,\beta} = 0, \text{ for all } t, u, x, y \in \mathcal{N}. \tag{9}$$

If $H(\alpha(x)) = 0$ for all $x \in \mathcal{N}$, then by using (8) we get $H([y, z]_{\alpha,\beta})\beta(x) = 0$ for all $x, y, z \in \mathcal{N}$, which implies that $H(z)\beta(y)\beta(x) = 0$ for all $x, y, z \in \mathcal{N}$. Replacing z by zr and y by ys and by using the 3-primeness of $\mathcal{N}$, we conclude that $H = \beta = 0$; is a contradiction. So we must have $[y, [u, t]_{\alpha,\beta}]_{\alpha,\beta} = 0$, for all $t, u, y \in \mathcal{N}$. It follows that

$$\alpha(y)[u, t]_{\alpha,\beta} = [u, t]_{\alpha,\beta}\beta(y), \text{ for all } t, u, y \in \mathcal{N}. \tag{10}$$

Replacing y by $y\beta(r)$ in (10), we obtain

$$\begin{aligned} \alpha(y)\beta(r)[u, t]_{\alpha,\beta} &= [u, t]_{\alpha,\beta}\beta(y)\beta(r) \\ &= \alpha(y)[u, t]_{\alpha,\beta}\beta(r) \\ &= \alpha(y)\alpha(r)[u, t]_{\alpha,\beta}, \text{ for all } r, t, x, y \in \mathcal{N}. \end{aligned}$$

Which leads to $\big(\alpha(y)\beta(r)-\alpha(y)\alpha(r)\big)[u,t]_{\alpha,\beta}=0$, for all $r,t,u,y\in\mathcal{N}$, so

$$\big(\alpha(y)\beta(r)-\alpha(y)\alpha(r)\big)\mathcal{N}[u,t]_{\alpha,\beta}=\{0\},\ \text{for all } r,t,u,y\in\mathcal{N}.$$

Since $\mathcal{N}$ is 3-prime, we obtain

$$\alpha(y)\beta(r)=\alpha(y)\alpha(r)\quad\text{or}\quad [u,t]_{\alpha,\beta}=0,\ \text{for all } r,t,u,y\in\mathcal{N}.$$

Suppose that $[u,t]_{\alpha,\beta}=0$, for all $t,u\in\mathcal{N}$. Replacing t by tr in the last expression and using it again, we have $\alpha(u)tr=tr\beta(u)=t\alpha(u)r$, for all $r,t,u\in\mathcal{N}$, which gives $[\alpha(u),t]\mathcal{N}=\{0\}$. By the 3-primeness of $\mathcal{N}$, we conclude that $\alpha(\mathcal{N})\subseteq Z(\mathcal{N})$ and that Lemma 2.2(i) forces $\mathcal{N}$ to be a commutative ring. Returning to our hypothesis, we conclude that $\alpha=\beta$.

Now assume that $\alpha(y)\beta(r)=\alpha(y)\alpha(r)$, for all $r,y\in\mathcal{N}$, then $\alpha(y\beta(r)-y\alpha(r))=0$ for all $r,y\in\mathcal{N}$. Replacing y by yt, we get

$$\alpha(y)t\beta(r)=\alpha(y)t\alpha(r),\ \text{for all } r,t,y\in\mathcal{N}.$$

Using the previous assumption, we have $\alpha(s)\beta(y\beta(r)-y\alpha(r))=\alpha(s)\alpha(y\beta(r)-y\alpha(r))=0$ for all $r,s,y\in\mathcal{N}$. Substituting st with s and using the 3-primeness of $\mathcal{N}$ with $\alpha\neq 0$, we obtain

$$\beta(y)\beta(r)=\beta(y)\alpha(r),\ \text{for all } r,y\in\mathcal{N}. \tag{11}$$

From (10) and (11), we obtain

$$\begin{aligned}\alpha(y)[u,t]_{\alpha,\beta} &= [u,t]_{\alpha,\beta}\beta(y)\\ &= \alpha(u)t\beta(y)-t\beta(u)\beta(y)\\ &= \alpha(u)t\beta(y)-t\beta(u)\alpha(y)\\ &= [u,t]_{\alpha,\beta}\alpha(y),\ \text{for all } r,y,u,t\in\mathcal{N}.\end{aligned} \tag{12}$$

The above computation shows that $\alpha(y)[u,t]_{\alpha,\beta}=[u,t]_{\alpha,\beta}\alpha(y)$ for all $r,y,u,t\in\mathcal{N}$. By replacing t with $t\beta(u)$ and by using (10), we get $[\alpha(y),\alpha(u)][u,t]_{\alpha,\beta}=0$, for all $r,y,u,t\in\mathcal{N}$. It follows that $[\alpha(y),\alpha(u)]\alpha(s)[u,t]_{\alpha,\beta}=0$, for all $r,y,u,t\in\mathcal{N}$. From the above we can see that $[\alpha(y),\alpha(u)]\alpha(s)\mathcal{N}[u,t]_{\alpha,\beta}=\{0\}$, for all $s,r,y,u,t\in\mathcal{N}$ and by the 3-primeness of $\mathcal{N}$, we can conclude that

$$[\alpha(y),\alpha(u)]\alpha(s)=0\quad\text{or}\quad [u,t]_{\alpha,\beta}=0,\ \text{for all } s,r,y,u,t\in\mathcal{N}.$$

Using expression (12) we get $[u,\alpha(y)]_{\alpha,\beta}=0$ or $[u,t]_{\alpha,\beta}=0$, for all $y,u,t\in\mathcal{N}$. Both results give $[u,\alpha(y)]_\alpha=0$, for all $y,u\in\mathcal{N}$. □

Corollary 3.8. *Let $\mathcal{N}$ be a 3-prime near-ring and H a nonzero left multiplier on $\mathcal{N}$, then the following assertions are equivalent:*

(i) $[x,[y,z]]\in Z(\mathcal{N})$, *for all* $x,y,z\in\mathcal{N}$;
(ii) $-([x,[y,z]])\in Z(\mathcal{N})$, *for all* $x,y,z\in\mathcal{N}$;
(iii) $H([x,[y,z]])\in Z(\mathcal{N})$, *for all* $x,y,z\in\mathcal{N}$;
(iv) $\mathcal{N}$ *is a commutative ring.*

Theorem 3.9. *Let $\mathcal{N}$ be a3-prime near-ring, and H, α and β nonzero left multipliers on $\mathcal{N}$ such that $[u,\alpha(y)]_\alpha\neq 0$, for all $y,u\in\mathcal{N}$. Then the following assertions are equivalent:*

(i) $H\big((x \circ (y \circ z)_{\alpha,\beta})_{\alpha,\beta}\big) \in Z(\mathcal{N})$, *for all* $x, y, z \in \mathcal{N}$;
(ii) $\mathcal{N}$ *is a commutative ring and* $[u, \alpha(y)]_\alpha = 0$, *for all* $y, u \in \mathcal{N}$.

Proof. The implication (ii) $\Rightarrow$ (i) is clear.

(i) $\Rightarrow$ (ii). Suppose that

$$H\big((x \circ (y \circ z)_{\alpha,\beta})_{\alpha,\beta}\big) \in Z(\mathcal{N}), \text{ for all } x, y, z \in \mathcal{N}.$$

Replacing x by $x(y \circ z)_{\alpha,\beta}$, we get $H\big((x \circ (y \circ z)_{\alpha,\beta})_{\alpha,\beta}\big)(y \circ z)_{\alpha,\beta} \in Z(\mathcal{N})$, for all $x, y, z \in \mathcal{N}$. Further, by using Lemma 2.1(i), we obtain

$$H\big((x \circ (y \circ z)_{\alpha,\beta})_{\alpha,\beta}\big) = 0 \quad \text{or} \quad (y \circ z)_{\alpha,\beta}, \text{ for all } x, y \in \mathcal{N}.$$

Suppose there exists $(y_0, z_0) \in \mathcal{N} \times \mathcal{N}$ such that $(y_0 \circ z_0)_{\alpha,\beta} \in Z(\mathcal{N})$, then

$$\begin{aligned} H\big((x \circ (y_0 \circ z_0)_{\alpha,\beta})_{\alpha,\beta}\big) &= H\big(\alpha(x)(y_0 \circ z_0)_{\alpha,\beta} + (y_0 \circ z_0)_{\alpha,\beta}\beta(x)\big) \\ &= H(\alpha(x) + \beta(x))(y_0 \circ z_0)_{\alpha,\beta} \in Z(\mathcal{N}) \end{aligned}$$

From Lemma 2.1 (i), we get either $(y_0 \circ z_0)_{\alpha,\beta} = 0$ or $H(\alpha(x)) + H(\beta(x)) \in Z(\mathcal{N})$. Substituting xt with x and by using Lemma 2.1 (i), we find that

$$(y_0 \circ z_0)_{\alpha,\beta} = 0 \quad \text{or} \quad H(\alpha(x)) = -H(\beta(x)), \text{ for all } x \in \mathcal{N}.$$

Both expressions give easily that $H\big((x \circ (y_0 \circ z_0)_{\alpha,\beta})_{\alpha,\beta}\big) = 0$ for all $x \in \mathcal{N}$. In this case, (7) becomes $H\big((x \circ (y \circ z)_{\alpha,\beta})_{\alpha,\beta}\big)$, for all $x \in \mathcal{N}$ equivalently,

$$H(\alpha(x))(y \circ z)_{\alpha,\beta} = H((y \circ z)_{\alpha,\beta})\beta(x), \text{ for all } x, y, z \in \mathcal{N}. \tag{13}$$

Replacing z by $(u \circ t)_{\alpha,\beta}$ and x by xr, we obtain $H(\alpha(x))\mathcal{N}(y \circ (u \circ t)_{\alpha,\beta})_{\alpha,\beta} = \{0\}$, for all $t, u, x, y \in \mathcal{N}$. Since $\mathcal{N}$ is 3-prime, we get

$$H(\alpha(x)) = 0 \quad \text{or} \quad (y \circ (u \circ t)_{\alpha,\beta})_{\alpha,\beta} = 0, \text{ for all } t, u, x, y \in \mathcal{N}.$$

Suppose that $H(\alpha(x)) = 0$, for all $x \in \mathcal{N}$. From (13), we see that $H((y \circ z)_{\alpha,\beta})\beta(x) = 0$, for all $x, y, z \in \mathcal{N}$ and so $H(z)\beta(y)\beta(x) = 0$, for all $x, y, z \in \mathcal{N}$. Replacing z with zr and y with ys and by using the 3-primeness of $\mathcal{N}$, we conclude that $H = \beta = 0$; contradiction. Hence, $(y \circ (u \circ t)_{\alpha,\beta})_{\alpha,\beta} = 0$, for all $t, u, y \in \mathcal{N}$, which can be rewritten as

$$\alpha(y)(u \circ t)_{\alpha,\beta} = (u \circ t)_{\alpha,\beta}\beta(y), \text{ for all } t, u, y \in \mathcal{N}. \tag{14}$$

Replacing y by $y\beta(r)$ in (14), we obtain

$$\begin{aligned} \alpha(y)\beta(r)(u \circ t)_{\alpha,\beta} &= (u \circ t)_{\alpha,\beta}\beta(y)\beta(r) \\ &= \alpha(y)(u \circ t)_{\alpha,\beta} \\ &= \alpha(y)\alpha(r)(u \circ t)_{\alpha,\beta}, \text{ for all } r, t, x, y \in \mathcal{N}. \end{aligned}$$

Which leads to $\big(\alpha(y)\beta(r) - \alpha(y)\alpha(r)\big)(u \circ t)_{\alpha,\beta} = 0$, for all $r, t, u, y \in \mathcal{N}$, so

$$\big(\alpha(y)\beta(r) - \alpha(y)\alpha(r)\big)\mathcal{N}(u \circ t)_{\alpha,\beta} = \{0\}, \text{ for all } r, t, u, y \in \mathcal{N}.$$

Since $\mathcal{N}$ is 3-prime, we obtain

$$\alpha(y)\beta(r) = \alpha(y)\alpha(r) \quad \text{or} \quad (u \circ t)_{\alpha,\beta} = 0, \text{ for all } r, t, u, y \in \mathcal{N}.$$

Suppose that $(u \circ t)_{\alpha,\beta} = 0$, for all $t, u \in \mathcal{N}$. Replacing t by tr in the last expression and using it again, we get $\alpha(u)tr = -tr\beta(u) = (-t)(-\alpha(u))r$, for all $r, t, u \in \mathcal{N}$ from this, we have $(-\alpha(u)t+t\alpha(u))\mathcal{N} = \{0\}$. By the 3-primeness of $\mathcal{N}$, we conclude that $\alpha(\mathcal{N}) \subseteq Z(\mathcal{N})$. Further, Lemma 2.2 (i) forces that $\mathcal{N}$ is a commutative ring. In this case, our assumption gives $(\alpha(u) + \beta(u))\mathcal{N} = \{0\}$ so $\alpha = -\beta$, then our hypothesis becomes $[u, \alpha(y)]_\alpha = 0$, for all $u, y \in \mathcal{N}$.

Now, assume that $\alpha(y)\beta(r) = \alpha(y)\alpha(r)$, for all $r, y \in \mathcal{N}$, then $\alpha(y\beta(r) - y\alpha(r)) = 0$ for all $r, y \in \mathcal{N}$. Replacing y by yt, we get

$$\alpha(y)t\beta(r) = \alpha(y)t\alpha(r), \text{ for all } r, t, y \in \mathcal{N}.$$

Using the previous assumption, we have $\alpha(s)\beta(y\beta(r)-y\alpha(r)) = \alpha(s)\alpha(y\beta(r)-y\alpha(r)) = 0$ for all $r, s, y \in \mathcal{N}$. Replacing s by st and using the 3-primeness of $\mathcal{N}$ with $\alpha \neq 0$, we obtain

$$\beta(y)\beta(r) = \beta(y)\alpha(r), \text{ for all } r, y \in \mathcal{N}. \tag{15}$$

By (10) and (15), we get

$$\begin{aligned} \alpha(y)(u \circ t)_{\alpha,\beta} &= (u \circ t)_{\alpha,\beta}\beta(y) \\ &= \alpha(u)t\beta(y) + t\beta(u)\beta(y) \\ &= \alpha(u)t\beta(y) + t\beta(u)\alpha(y) \\ &= (u \circ t)_{\alpha,\beta}\alpha(y), \text{ for all } r, y, u, t \in \mathcal{N} \end{aligned}$$

which implies that $\alpha(y)(u \circ t)_{\alpha,\beta} = (u \circ t)_{\alpha,\beta}\alpha(y)$, for all $r, y, u, t \in \mathcal{N}$. Replacing t by $t\beta(u)$ and using (10), gives us $[\alpha(y), \alpha(u)](u \circ t)_{\alpha,\beta} = 0$, for all $r, y, u, t \in \mathcal{N}$, which in turn further gives $[\alpha(y), \alpha(u)]\alpha(s)(u \circ t)_{\alpha,\beta} = 0$, for all $r, y, u, t \in \mathcal{N}$, so $[\alpha(y), \alpha(u)]\alpha(s)\mathcal{N}(u \circ t)_{\alpha,\beta} = \{0\}$ for all $s, r, y, u, t \in \mathcal{N}$. By the 3-primeness of $\mathcal{N}$, we conclude that

$$[\alpha(y), \alpha(u)]\alpha(s) = 0 \quad \text{or} \quad (u \circ t)_{\alpha,\beta} = 0, \text{ for all } s, r, y, u, t \in \mathcal{N},$$

which forces that $[u, \alpha(y)]_\alpha = 0$, for all $y, u \in \mathcal{N}$. □

COROLLARY 3.10. *Let $\mathcal{N}$ be a 3-prime near-ring, and H a nonzero left multiplier. Then the following assertions are equivalent:*

(i) $(x \circ (y \circ z)) \in Z(\mathcal{N})$, *for all* $x, y, z \in \mathcal{N}$;
(ii) $-(x \circ (y \circ z)) \in Z(\mathcal{N})$, *for all* $x, y, z \in \mathcal{N}$;
(iii) $H\big((x \circ (y \circ z))\big) \in Z(\mathcal{N})$, *for all* $x, y, z \in \mathcal{N}$;
(iv) $\mathcal{N}$ *is a commutative ring.*

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Sidi Mohammed Ben Abdellah University
Polydisciplinary Faculty
LSI, Taza, Morocco
E-mail: `abdelkarimboua@yahoo.fr`

Yazd University
Yazd, Iran
E-mail: `davvaz@yazd.ac.ir`