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A NEW OPERATOR OF PRIMAL TOPOLOGICAL SPACES

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Abstract. Recently, Acharjee, Özkoç and Issaka introduced a new structure in topology named primal. Primal is the dual structure of grill. The main purpose of this paper is to introduce an operator using primal and obtain some of its fundamental properties. Also, we define the notion of topology suitable for a primal. We not only obtain some characterizations of this new notion, but also investigate many properties.

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Key words. Primal, primal topological space, Ψ-operator, topology suitable, grill.

1. INTRODUCTION

Topology is one of the indispensable branches of mathematics. Due to its various generalized applications in both science and social science, several associated structures such as ideal [10], filter [11], grill [7], etc. have been introduced. The notion of ideal is the dual of filter. Applications of filter are highly available in literature [12, 13]. Moreover, ideal and its generalized fuzzy notion have been finding several uses in general topology [4], summability theory [14], fuzzy summability theory [15], and many others. Similar to ideal, one of the classical structures in topology is grill. Definition of the grill was introduced by Chóquet [7] in 1947. Like ideal and filter, grill also has several applications in general topology [3], fuzzy topology [6], and many others. Recently, the notion of primal was introduced by Acharjee et al. [1]. Primal is the dual structure of the grill. In [1], the authors obtained many fundamental properties of this new structure. On the other hand, the set operator Ψ on ideal topological space was introduced and studied by Hamlett and Janković [8]. Recently, primal has been used to study proximity spaces by Al-Omari et al. [2]. This new proximity space is called primal-proximity space. Also, different types of Ψ-operator can be found in [3, 4, 5, 9], and many others.

In this paper, we continue to study the properties of the operator $\diamond$ as defined in [1]. We define an operator Ψ using primal and investigate its various fundamental properties. Also, we introduce the notion of topology suitable

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for a primal and obtain its characterizations along with several properties.

2. PRELIMINARIES

Throughout this paper, (X, τ) and (Y, σ) (briefly, X and Y) represent topological spaces unless otherwise stated. For any subset A of a space X, $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ denote the closure and interior of A, respectively. The powerset of a set X will be denoted by 2^X. The family of all open neighborhoods of a point x of X is denoted by $\tau(x)$. Also, the family of all closed subsets of a space X will be denoted by $C(X)$. Now, we procure the following notions and results, which will be required in the next section.

Definition 2.1 ([7]). A family $\mathcal{G}$ of 2^X is called a grill on X if $\mathcal{G}$ satisfies the following conditions:

(1) $\emptyset \notin \mathcal{G}$,
(2) if $A \in \mathcal{G}$ and $A \subseteq B$, then $B \in \mathcal{G}$,
(3) if $A \cup B \in \mathcal{G}$, then $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

Definition 2.2 ([1]). Let X be a nonempty set. A collection $\mathcal{P} \subseteq 2^X$ is called a primal on X if it satisfies the following conditions:

(1) $X \notin \mathcal{P}$,
(2) if $A \in \mathcal{P}$ and $B \subseteq A$, then $B \in \mathcal{P}$,
(3) if $A \cap B \in \mathcal{P}$, then $A \in \mathcal{P}$ or $B \in \mathcal{P}$.

Corollary 2.3 ([1]). *Let X be a nonempty set. A collection $\mathcal{P} \subseteq 2^X$ is a primal on X if and only if it satisfies the following conditions:*

(1) $X \notin \mathcal{P}$,
(2) *if $B \notin \mathcal{P}$ and $B \subseteq A$, then $A \notin \mathcal{P}$,*
(3) *if $A \notin \mathcal{P}$ and $B \notin \mathcal{P}$, then $A \cap B \notin \mathcal{P}$.*

Definition 2.4. ([1]) A topological space (X, τ) with a primal $\mathcal{P}$ on X is called a primal topological space and denoted by $(X, \tau, \mathcal{P})$.

Definition 2.5. ([1]) Let $(X, \tau, \mathcal{P})$ be a primal topological space. We consider a map $(\cdot)^\diamond : 2^X \to 2^X$ as

$$A^\diamond(X, \tau, \mathcal{P}) = \{x \in X : (\forall U \in \tau(x))(A^c \cup U^c \in \mathcal{P})\}$$

for any subset A of X. We can also write $A^\diamond_{\mathcal{P}}$ as $A^\diamond(X, \tau, \mathcal{P})$ to specify the primal as per our requirements.

Definition 2.6 ([1]). Let $(X, \tau, \mathcal{P})$ be a primal topological space. We consider a map $\mathrm{cl}^\diamond : 2^X \to 2^X$ as $\mathrm{cl}^\diamond(A) = A \cup A^\diamond$, where A is any subset of X.

Definition 2.7 ([1]). Let $(X, \tau, \mathcal{P})$ be a primal topological space. Then, the family $\tau^\diamond = \{A \subseteq X | \mathrm{cl}^\diamond(A^c) = A^c\}$ is a topology on X induced by topology τ and primal $\mathcal{P}$. It is called primal topology on X. We can also write $\tau^\diamond_{\mathcal{P}}$ instead of $\tau^\diamond$ to specify the primal as per our requirements.

THEOREM 2.8 ([1]). *Let $(X, \tau, \mathcal{P})$ be a primal topological space and $A, B \subseteq X$. If A is open in X, then $A \cap B^\diamond \subseteq (A \cap B)^\diamond$.*

COROLLARY 2.9. *Let $(X, \tau, \mathcal{P})$ be a primal topological space and $A, B \subseteq X$. If A is open in X, then $A \cap B^\diamond = A \cap (A \cap B)^\diamond \subseteq (A \cap B)^\diamond$.*

Proof. We have $A \cap (A \cap B)^\diamond \subseteq A \cap B^\diamond$. Let $x \in A \cap B^\diamond$ and $V \in \tau(x)$. Then $A \cap V \in \tau(x)$ and $x \in B^\diamond$, then $(A \cap V)^c \cup B^c \in \mathcal{P}$ i.e. $(A \cap B)^c \cup V^c \in \mathcal{P}$, then $x \in (A \cap B)^\diamond$ and $x \in A \cap (A \cap B)^\diamond$. Hence by Theorem 2.8, we have $A \cap B^\diamond = A \cap (A \cap B)^\diamond \subseteq (A \cap B)^\diamond$. □

THEOREM 2.10 ([1]). *Let $(X, \tau, \mathcal{P})$ be a primal topological space. Then, the following statements hold for any two subsets of A and B of X:*
(1) *if $A^c \in \tau$, then $A^\diamond \subseteq A$,*
(2) $\emptyset^\diamond = \emptyset$,
(3) $\mathrm{cl}(A^\diamond) = A^\diamond$,
(4) $(A^\diamond)^\diamond \subseteq A^\diamond$,
(5) *if $A \subseteq B$, then $A^\diamond \subseteq B^\diamond$,*
(6) $A^\diamond \cup B^\diamond = (A \cup B)^\diamond$,
(7) $(A \cap B)^\diamond \subseteq A^\diamond \cap B^\diamond$.

THEOREM 2.11 ([1]). *Let $(X, \tau, \mathcal{P})$ be a primal topological space. Then, the family $\mathcal{B}_\mathcal{P} = \{T \cap P | T \in \tau$ and $P \notin \mathcal{P}\}$ is a base for the primal topology $\tau^\diamond$ on X.*

3. MAIN RESULTS

In this section, we introduce some results related to Ψ operator using primal on a primal topological space $(X, \tau, \mathcal{P})$.

THEOREM 3.1. *Let $(X, \tau, \mathcal{P})$ be a primal topological space. If $C(X) - \{X\} \subseteq \mathcal{P}$, then $U \subseteq U^\diamond$ for all $U \in \tau$.*

Proof. In case $U = \emptyset$, we obviously have $U^\diamond = \emptyset = U$. Now note that if $C(X) - \{X\} \subseteq \mathcal{P}$, then $X^\diamond = X$. In fact $x \notin X^\diamond$, then there is $V \in \tau(x)$ such that $V^c \cup X^c \notin \mathcal{P}$. Hence, $V^c \notin \mathcal{P}$ is a contradiction. Now by using Theorem 2.8, we have for any $U \in \tau$, $U = U \cap X^\diamond \subseteq (U \cap X)^\diamond = U^\diamond$. Thus $U \subseteq U^\diamond$. □

LEMMA 3.2. *Let $(X, \tau, \mathcal{P})$ be a primal topological space. If $A^c \notin \mathcal{P}$, then $A^\diamond = \emptyset$.*

Proof. Suppose that $x \in A^\diamond$. Then, for any open set U containing x we have $U^c \cup A^c \in \mathcal{P}$. Since $A^c \notin \mathcal{P}$, $U^c \cup A^c \notin \mathcal{P}$ for some open sets U containing x. This is a contradiction. Hence, $A^\diamond = \emptyset$. □

LEMMA 3.3. *Let $(X, \tau, \mathcal{P})$ be a primal topological space and A, B be subsets of X. Then, $A^\diamond - B^\diamond = (A - B)^\diamond - B^\diamond$.*

Proof. We have by Theorem 2.10:

$$A^\diamond = [(A-B)\cup(A\cap B)]^\diamond = (A-B)^\diamond \cup (A\cap B)^\diamond \subseteq (A-B)^\diamond \cup B^\diamond.$$

Thus $A^\diamond - B^\diamond \subseteq (A-B)^\diamond - B^\diamond$. Again by Theorem 2.10, $(A-B)^\diamond \subseteq A^\diamond$ and hence, $(A-B)^\diamond - B^\diamond \subseteq A^\diamond - B^\diamond$. Hence, $A^\diamond - B^\diamond = (A-B)^\diamond - B^\diamond$. $\square$

COROLLARY 3.4. *Let $(X,\tau,\mathcal{P})$ be a primal topological space and A, B be subsets of X with $B^c \notin \mathcal{P}$. Then $(A\cup B)^\diamond = A^\diamond = (A-B)^\diamond$.*

Proof. Since $B^c \notin \mathcal{P}$, thus $B^\diamond = \emptyset$. Again by Lemma 3.3, $A^\diamond = (A-B)^\diamond$ and by Theorem 2.10, $(A\cup B)^\diamond = A^\diamond \cup B^\diamond = A^\diamond$ $\square$

DEFINITION 3.5. Let $(X,\tau,\mathcal{P})$ be a primal topological space. An operator $\Psi : 2^X \to 2^X$ is defined as $\Psi(A) = \{x \in X : (\exists U \in \tau(x))((U-A)^c \notin \mathcal{P})\}$ for every $A \subseteq X$.

Several basic facts concerning the behavior of the operator Ψ are included in the following theorem.

THEOREM 3.6. *Let $(X,\tau,\mathcal{P})$ be a primal topological space. Then, the following properties hold:*

(1) *if $A \subseteq X$, then $\Psi(A) = X - (X-A)^\diamond$,*
(2) *if $A \subseteq X$, then $\Psi(A)$ is open,*
(3) *if $A \subseteq B$, then $\Psi(A) \subseteq \Psi(B)$,*
(4) *if $A, B \subseteq X$, then $\Psi(A\cap B) = \Psi(A)\cap\Psi(B)$,*
(5) *if $U \in \tau^\diamond$, then $U \subseteq \Psi(U)$,*
(6) *if $A \subseteq X$, then $\Psi(A) \subseteq \Psi(\Psi(A))$,*
(7) *if $A \subseteq X$, then $\Psi(A) = \Psi(\Psi(A))$ if and only if $(X-A)^\diamond = ((X-A)^\diamond)^\diamond$,*
(8) *if $A^c \notin \mathcal{P}$, then $\Psi(A) = X - X^\diamond$,*
(9) *if $A \subseteq X$, then $A \cap \Psi(A) = \mathrm{int}^\diamond(A)$,*
(10) *if $A \subseteq X$ and $I^c \notin \mathcal{P}$, then $\Psi(A-I) = \Psi(A)$,*
(11) *if $A \subseteq X$ and $I^c \notin \mathcal{P}$, then $\Psi(A\cup I) = \Psi(A)$,*
(12) *if $[(A-B)\cup(B-A)]^c \notin \mathcal{P}$, then $\Psi(A) = \Psi(B)$.*

Proof. (1) Let $x \in \Psi(A)$, then there exists $U \in \tau(x)$ such that $U^c \cup A = (U\cap(X-A))^c = (U-A)^c \notin \mathcal{P}$, then $x \notin (X-A)^\diamond$ and $x \in X-(X-A)^\diamond$. Conversely, let $x \in X-(X-A)^\diamond$, then $x \notin (X-A)^\diamond$, then there exists $U \in \tau(x)$ such that $U^c \cup (X-A)^c = (U-A)^c \notin \mathcal{P}$. Hence, $x \in \Psi(A)$ and $\Psi(A) = X-(X-A)^\diamond$.

(2) This follows from (3) of Theorem 2.10.

(3) This follows from (5) of Theorem 2.10.

(4) It follows from (3) that $\Psi(A\cap B) \subseteq \Psi(A)$ and $\Psi(A\cap B) \subseteq \Psi(B)$. Hence, $\Psi(A\cap B) \subseteq \Psi(A)\cap\Psi(B)$. Now, let $x \in \Psi(A)\cap\Psi(B)$. Then, there exist $U, V \in \tau(x)$ such that $(U-A)^c \notin \mathcal{P}$ and $(V-B)^c \notin \mathcal{P}$. Let $G = U\cap V \in \tau(x)$ and we have $(G-A)^c \notin \mathcal{P}$ and $(G-B)^c \notin \mathcal{P}$ by heredity. Thus $[G-(A\cap B)]^c = (G-A)^c \cap (G-B)^c \notin \mathcal{P}$ by Corollary 2.3, and hence, $x \in \Psi(A\cap B)$. We have shown $\Psi(A)\cap\Psi(B) \subseteq \Psi(A\cap B)$ and the proof is completed.

(5) If $U \in \tau^{\diamond}$, then $(X-U)^{\diamond} \subseteq X-U$. Hence, $U \subseteq X-(X-U)^{\diamond} = \Psi(U)$.
(6) It follows from (2) and (5).
(7) It follows from the facts:

(a) $\Psi(A) = X-(X-A)^{\diamond}$.
(b) $\Psi(\Psi(A)) = X-[X-(X-(X-A)^{\diamond})]^{\diamond} = X-((X-A)^{\diamond})^{\diamond}$.

(8) By Corollary 3.4, we obtain that $(X-A)^{\diamond} = X^{\diamond}$ if $A^c \notin \mathcal{P}$. Then, $\Psi(A) = X-(X-A)^{\diamond} = X-X^{\diamond}$.

(9) If $x \in A\cap\Psi(A)$, then $x \in A$ and there exists $U_x \in \tau(x)$ such that $(U_x - A)^c \notin \mathcal{P}$. Then by Theorem 2.11, $U_x \cap (U_x-A)^c$ is a $\tau^{\diamond}$-open neighborhood of x and $x \in \operatorname{int}^{\diamond}(A)$. On the other hand, if $x \in \operatorname{int}^{\diamond}(A)$, there exists a basic $\tau^{\diamond}$-open neighborhood $V_x \cap I$ of x, where $V_x \in \tau$ and $I \notin \mathcal{P}$, such that $x \in V_x \cap I \subseteq A$ which implies $I \subseteq (V_x-A)^c$ and hence, $(V_x-A)^c \notin \mathcal{P}$. Hence, $x \in A\cap\Psi(A)$.

(10) This follows from Corollary 3.4 and

$$\Psi(A-I) = X-[X-(A-I)]^{\diamond} = X-[(X-A)\cup I]^{\diamond} = X-(X-A)^{\diamond} = \Psi(A).$$

(11) This follows from Corollary 3.4 and

$$\Psi(A\cup I) = X-[X-(A\cup I)]^{\diamond} = X-[(X-A)-I]^{\diamond} = X-(X-A)^{\diamond} = \Psi(A).$$

(12) Assume $[(A-B)\cup(B-A)]^c \notin \mathcal{P}$. Let $A-B = I$ and $B-A = J$. Observe that $I^c, J^c \notin \mathcal{P}$ by heredity. Also, observe that $B = (A-I)\cup J$. Thus, $\Psi(A) = \Psi(A-I) = \Psi[(A-I)\cup J] = \Psi(B)$ by (10) and (11). □

COROLLARY 3.7. *Let $(X,\tau,\mathcal{P})$ be a primal topological space. Then $U \subseteq \Psi(U)$ for every open set $U \in \tau$.*

Proof. We know that $\Psi(U) = X-(X-U)^{\diamond}$. Now $(X-U)^{\diamond} \subseteq \operatorname{cl}(X-U) = X-U$, since $X-U$ is closed. Therefore, $U = X-(X-U) \subseteq X-(X-U)^{\diamond} = \Psi(U)$. □

THEOREM 3.8. *Let $(X,\tau,\mathcal{P})$ be a primal topological space and $A \subseteq X$. Then, the following properties hold:*

(1) $\Psi(A) = \bigcup\{U \in \tau : (U-A)^c \notin \mathcal{P}\}$,
(2) $\Psi(A) \supseteq \bigcup\{U \in \tau : (U-A)^c \cup (A-U)^c \notin \mathcal{P}\}$.

Proof. (1) This follows immediately from the definition of Ψ-operator.
(2) Since $\mathcal{P}$ is heredity, it is obvious that

$$\bigcup\{U \in \tau : (U-A)^c \cup (A-U)^c \notin \mathcal{P}\} \subseteq \bigcup\{U \in \tau : (U-A)^c \notin \mathcal{P}\} = \Psi(A),$$

for every $A \subseteq X$. □

THEOREM 3.9. *Let $(X,\tau,\mathcal{P})$ be a primal topological space. If $\sigma = \{A \subseteq X : A \subseteq \Psi(A)\}$. Then, σ is a topology on X and $\sigma = \tau^{\diamond}$.*

Proof. Let $\sigma = \{A \subseteq X : A \subseteq \Psi(A)\}$. First, we show that σ is a topology. Observe that $\emptyset \subseteq \Psi(\emptyset)$ and $X \subseteq \Psi(X) = X$, and thus $\emptyset$ and $X \in \sigma$. Now, if $A, B \in \sigma$, then $A \cap B \subseteq \Psi(A) \cap \Psi(B) = \Psi(A \cap B)$. It implies that $A \cap B \in \sigma$. If $\{A_\alpha : \alpha \in \Delta\} \subseteq \sigma$, then $A_\alpha \subseteq \Psi(A_\alpha) \subseteq \Psi(\bigcup_{\alpha\in\Delta} A_\alpha)$, for every $\alpha \in \Delta$, and hence $\bigcup A_\alpha \subseteq \Psi(\bigcup_{\alpha\in\Delta} A_\alpha)$. This shows that σ is a topology on X.

Now, if $U \in \tau^\diamond$ and $x \in U$, then by Theorem 2.11, there exists $V \in \tau(x)$ and $I \notin \mathcal{P}$ such that $x \in V \cap I \subseteq U$. Clearly $I \subseteq (V - U)^c$, so that $(V - U)^c \notin \mathcal{P}$ by heredity, and hence $x \in \Psi(U)$. Thus, $U \subseteq \Psi(U)$ and $\tau^\diamond \subseteq \sigma$. Now, let $A \in \sigma$, then we have $A \subseteq \Psi(A)$, i.e. $A \subseteq X - (X - A)^\diamond$ and $(X - A)^\diamond \subseteq X - A$. This shows that $X - A$ is $\tau^\diamond$-closed and hence $A \in \tau^\diamond$. Thus, $\sigma \subseteq \tau^\diamond$, and hence $\sigma = \tau^\diamond$. □

4. TOPOLOGY SUITABLE FOR A PRIMAL

In this section, we introduce topology suitable for a primal on a primal topological space and investigate some properties.

Definition 4.1. Let $(X, \tau, \mathcal{P})$ be a primal topological space. Then, τ is said to be suitable for the primal $\mathcal{P}$ if $A^c \cup A^\diamond \notin \mathcal{P}$, for all $A \subseteq X$.

We now give some equivalent descriptions of this definition.

Theorem 4.2. *For a primal topological space $(X, \tau, \mathcal{P})$, the following are equivalent:*

(1) *τ is suitable for the primal $\mathcal{P}$,*
(2) *for any $\tau^\diamond$-closed subset A of X, $A^c \cup A^\diamond \notin \mathcal{P}$,*
(3) *whenever for any $A \subseteq X$ and each $x \in A$ there corresponds some $U_x \in \tau(x)$ with $U_x^c \cup A^c \notin \mathcal{P}$, it follows that $A^c \notin \mathcal{P}$,*
(4) *for $A \subseteq X$ and $A \cap A^\diamond = \emptyset$, it follows that $A^c \notin \mathcal{P}$.*

Proof. (1) $\Rightarrow$ (2) It is trivial.

(2) $\Rightarrow$ (3) Let $A \subseteq X$ and assume that for every $x \in A$ there exists $U \in \tau(x)$ such that $U^c \cup A^c \notin \mathcal{P}$. Then, $x \notin A^\diamond$ so that $A \cap A^\diamond = \emptyset$. Since $A \cup A^\diamond$ is $\tau^\diamond$-closed, thus by (2) we have $(A \cup A^\diamond)^c \cup (A \cup A^\diamond)^\diamond \notin \mathcal{P}$. i.e. $(A \cup A^\diamond)^c \cup (A^\diamond \cup (A^\diamond)^\diamond) \notin \mathcal{P}$ by Theorem 2.10 (6) i.e. $(A \cup A^\diamond)^c \cup A^\diamond \notin \mathcal{P}$ by Theorem 2.10 (4) i.e., $A^c \notin \mathcal{P}$ (as $A \cap A^\diamond = \emptyset$).

(3) $\Rightarrow$ (4) If $A \subseteq X$ and $A \cap A^\diamond = \emptyset$, then $A \subseteq X \setminus A^\diamond$. Let $x \in A$. Then $x \notin A^\diamond$. So there exists $U \in \tau(x)$ such that $U^c \cup A^c \notin \mathcal{P}$. Then by (3), $A^c \notin \mathcal{P}$.

(4) $\Rightarrow$ (1) Let $A \subseteq X$. We first claim that $(A \setminus A^\diamond) \cap (A \setminus A^\diamond)^\diamond = \emptyset$. In fact, if $x \in (A \setminus A^\diamond) \cap (A \setminus A^\diamond)^\diamond$, then $x \in A \setminus A^\diamond$. Thus $x \in A$ and $x \notin A^\diamond$. Then, there exists $U \in \tau(x)$ such that $U^c \cup A^c \notin \mathcal{P}$. Now, $U^c \cup A^c \subseteq U^c \cup (A \setminus A^\diamond)^c$ by Corollary 2.3 (2), $U^c \cup (A \setminus A^\diamond)^c \notin \mathcal{P}$. Hence, $x \notin (A \setminus A^\diamond)^\diamond$, which is a contradiction. Hence, by (4), $(A \setminus A^\diamond)^c = A^c \cup A^\diamond \notin \mathcal{P}$ and τ is suitable for the primal $\mathcal{P}$. □

THEOREM 4.3. *For a primal topological space $(X, \tau, \mathcal{P})$, the following conditions are equivalent and any of these three conditions is necessary for τ to be suitable for the primal $\mathcal{P}$:*

(1) *for any $A \subseteq X$, $A \cap A^\diamond = \emptyset$, then $A^\diamond = \emptyset$,*
(2) *for any $A \subseteq X$, $(A \setminus A^\diamond)^\diamond = \emptyset$,*
(3) *for any $A \subseteq X$, $(A \cap A^\diamond)^\diamond = A^\diamond$.*

Proof. (1) $\Rightarrow$ (2) It follows because $(A \setminus A^\diamond) \cap (A \setminus A^\diamond)^\diamond = \emptyset$, for all $A \subseteq X$.

(2) $\Rightarrow$ (3) Since $A = (A \setminus (A \cap A^\diamond)) \cup (A \cap A^\diamond)$, we have $A^\diamond = (A \setminus (A \cap A^\diamond))^\diamond \cup (A \cap A^\diamond)^\diamond = (A \setminus A^\diamond)^\diamond \cup (A \cap A^\diamond)^\diamond = (A \cap A^\diamond)^\diamond$ by (2).

(3) $\Rightarrow$ (1) Let $A \subseteq X$ and $A \cap A^\diamond = \emptyset$. Then $A^\diamond = (A \cap A^\diamond)^\diamond = \emptyset^\diamond = \emptyset$. □

COROLLARY 4.4. *If $(X, \tau, \mathcal{P})$ be a primal topological space such that τ is suitable for $\mathcal{P}$., then the operator $\diamond$ is an idempotent operator i.e., $A^\diamond = (A^\diamond)^\diamond$ for any $A \subseteq X$.*

Proof. By Theorem 2.10 (4), we have $(A^\diamond)^\diamond \subseteq A^\diamond$. By Theorem 4.3 and Theorem 2.10 (5), we get $A^\diamond = (A \cap A^\diamond)^\diamond \subseteq (A^\diamond)^\diamond$. □

THEOREM 4.5. *Let $(X, \tau, \mathcal{P})$ be a primal topological space such that τ is suitable for $\mathcal{P}$. Then, a subset A of X is $\tau^\diamond$-closed if and only if it can be expressed as a union of a closed set in (X, τ) and a complement not in $\mathcal{P}$.*

Proof. Let A be a $\tau^\diamond$-closed subset of X. Then, $A^\diamond \subseteq A$. Now, $A = A^\diamond \cup (A \setminus A^\diamond)$. Since τ is suitable for $\mathcal{P}$, then by Theorem 4.2 $(A \setminus A^\diamond)^c \notin \mathcal{P}$ and by Theorem 2.10 (3), $A^\diamond$ is closed.

Conversely, let $A = F \cup B$, where F is closed and $B^c \notin \mathcal{P}$. Then, $A^\diamond = (F \cup B)^\diamond = F^\diamond$ by Corollary 3.4, and hence by Theorem 2.10(3) $A^\diamond = (F \cup B)^\diamond = F^\diamond = \mathrm{cl}(F) = F \subseteq A$. Hence, A is $\tau^\diamond$-closed. □

COROLLARY 4.6. *Let the topology τ on a space X be suitable for a primal $\mathcal{P}$ on X. Then, $\mathcal{B}_\mathcal{P} = \{U \cap P : (U \in \tau)(P \notin \mathcal{P})\}$ is a topology on X and hence, $\mathcal{B}_\mathcal{P} = \tau^\diamond$.*

Proof. Let $U \in \tau^\diamond$. Then by Theorem 4.5, $X \setminus U = F \cup B$, where F is closed and $B^c \notin \mathcal{P}$. Then, $U = X \setminus (F \cup B) = (X \setminus F) \cap (X \setminus B) = V \cap P$, where $V = F^c \in \tau$ and $P = B^c \notin \mathcal{P}$. Thus, every $\tau^\diamond$-open set is of the form $V \cap P$, where $V \in \tau$ and $P \notin \mathcal{P}$. The rest follows from Theorem 2.11. □

THEOREM 4.7. *Let $(X, \tau, \mathcal{P})$ be a primal topological space and A be any subset of X such that $A \subseteq A^\diamond$. Then $\mathrm{cl}(A) = \mathrm{cl}^\diamond(A) = \mathrm{cl}(A^\diamond) = A^\diamond$.*

Proof. Since $\tau^\diamond$ is finer than τ, then $\mathrm{cl}^\diamond(A) \subseteq \mathrm{cl}(A)$ for any subset A of X. Now $x \notin \mathrm{cl}^\diamond(A)$, there exist $V \in \tau$ and $B \in \mathcal{P}$ such that $x \in V \cap B$ and $(V \cap B) \cap A = \emptyset$, then $[(V \cap B) \cap A]^\diamond = \emptyset$. Thus $[(V \cap A) \setminus B^c]^\diamond = \emptyset$, hence by Corollary 3.4 we have $(V \cap A)^\diamond = \emptyset$. By Theorem 2.8, we get $V \cap (A)^\diamond = \emptyset$ and $V \cap A = \emptyset$ (as $A \subseteq A^\diamond$), then $x \notin \mathrm{cl}(A)$. Thus, $\mathrm{cl}(A) = \mathrm{cl}^\diamond(A)$. Now, by Theorem 2.10 (3), $A^\diamond = \mathrm{cl}(A^\diamond)$. Now, let $x \notin \mathrm{cl}(A)$. Then, there exists $U \in \tau(x)$ such that $U \cap A = \emptyset$. Thus, $(U \cap A)^c = U^c \cup A^c = X \notin \mathcal{P}$. So, $x \notin A^\diamond$ and hence $A^\diamond \subseteq \mathrm{cl}(A)$. Again as $A^\diamond \subseteq \mathrm{cl}(A)$, so we have $\mathrm{cl}(A^\diamond) \subseteq \mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$. Also, $A \subseteq A^\diamond$, then $\mathrm{cl}(A) \subseteq \mathrm{cl}(A^\diamond)$. Thus, $\mathrm{cl}(A) = \mathrm{cl}(A^\diamond) = A^\diamond$. □

THEOREM 4.8. *Let* $(X, \tau, \mathcal{P})$ *be a primal topological space such that* τ *is suitable for* $\mathcal{P}$ *with* $C(X) - \{X\} \subseteq \mathcal{P}$. *Let* G *be* $\tau^\diamond$*-open set such that* $G = U \cap A$, *where* $U \in \tau$ *and* $A \notin \mathcal{P}$. *Then:* $\mathrm{cl}(G) = \mathrm{cl}^\diamond(G) = G^\diamond = U^\diamond = \mathrm{cl}(U) = \mathrm{cl}^\diamond(U)$.

Proof. Let $G = U \cap A$, where $U \in \tau$ and $A \notin \mathcal{P}$ (by Corollary 4.6, every $\tau^\diamond$-open set G is of this form). Since $C(X) - \{X\} \subseteq \mathcal{P}$, by Theorem 3.1 we have $U \subseteq U^\diamond$. Hence, by Theorem 4.7, we get $U^\diamond = \mathrm{cl}(U) = \mathrm{cl}^\diamond(U)$.

Let G be $\tau^\diamond$-open. We claim that $G \subseteq G^\diamond$. In fact, $\mathrm{cl}^\diamond(X \setminus G) = X \setminus G$, then $(X \setminus G)^\diamond = X \setminus G$ and $X^\diamond \setminus G^\diamond = X \setminus G$ by Lemma 3.3 and by Theorem 3.1 we have $X \setminus G^\diamond = X \setminus G$, $G \subseteq G^\diamond$. Hence, by Theorem 4.7, $G^\diamond = \mathrm{cl}(G) = \mathrm{cl}^\diamond(G)$.

Again, $G \subseteq U$, so $G^\diamond \subseteq U^\diamond$ and also $G^\diamond = (U \cap A)^\diamond = (U - A^c)^\diamond \supseteq U^\diamond - (A^c)^\diamond = U^\diamond$ by Lemma 3.3 and Lemma 3.2 as $A \notin \mathcal{P}$. Thus, $U^\diamond = G^\diamond$. Therefore, we have $\mathrm{cl}(G) = \mathrm{cl}^\diamond(G) = G^\diamond = U^\diamond = \mathrm{cl}(U) = \mathrm{cl}^\diamond(U)$. □

THEOREM 4.9. *Let* $(X, \tau, \mathcal{P})$ *be a primal topological space such that* τ *is suitable for* $\mathcal{P}$. *Then, for every* $A \in \tau$ *and any* $B \subseteq X$,

$$(A \cap B)^\diamond = (A \cap B^\diamond)^\diamond = \mathrm{cl}(A \cap B^\diamond).$$

Proof. Let $A \in \tau$. Then by Corollary 2.9, $A \cap B^\diamond = A \cap (A \cap B)^\diamond \subseteq (A \cap B)^\diamond$ and hence, $(A \cap B^\diamond)^\diamond \subseteq [(A \cap B)^\diamond]^\diamond = (A \cap B)^\diamond$ by Corollary 4.4.

Now by using Corollary 2.9 and Theorem 4.3, we obtain $[A \cap (B \setminus B^\diamond)]^\diamond = A \cap (B \setminus B^\diamond)^\diamond = A \cap \emptyset = \emptyset$.

Also, $(A \cap B)^\diamond \setminus (A \cap B^\diamond)^\diamond \subseteq [(A \cap B) \setminus (A \cap B^\diamond)]^\diamond = [A \cap (B \setminus B^\diamond)]^\diamond = \emptyset$ by Lemma 3.3. Hence, $(A \cap B)^\diamond \subseteq (A \cap B^\diamond)^\diamond$ and, we get $(A \cap B)^\diamond = (A \cap B^\diamond)^\diamond$.

Again $(A \cap B)^\diamond = (A \cap B^\diamond)^\diamond \subseteq \mathrm{cl}(A \cap B^\diamond)$, since $\tau^\diamond$ is finer than τ. Due to $A \cap B^\diamond \subseteq (A \cap B)^\diamond$, we have $\mathrm{cl}(A \cap B^\diamond) \subseteq \mathrm{cl}((A \cap B)^\diamond) = (A \cap B)^\diamond$. Hence, $(A \cap B)^\diamond = \mathrm{cl}(A \cap B^\diamond)$. □

COROLLARY 4.10. *Let* $(X, \tau, \mathcal{P})$ *be a primal topological space such that* τ *is suitable for* $\mathcal{P}$. *If* $A \in \tau$ *and* $A^c \notin \mathcal{P}$, *then* $A \subseteq X \setminus X^\diamond$.

Proof. Taking $B = X$ in Theorem 4.9, we get $(A \cap X)^\diamond = \mathrm{cl}(A \cap X^\diamond)$. Thus, $A^\diamond = \mathrm{cl}(A \cap X^\diamond)$, for all $A \in \tau$. Now if $A^c \notin \mathcal{P}$, then $A^\diamond = \emptyset$. Thus, $(A \cap X)^\diamond = \mathrm{cl}(A \cap X^\diamond) = \emptyset$. So, $A \cap X^\diamond = \emptyset$ by Theorem 2.8 and hence, $A \subseteq X \setminus X^\diamond$. □

5. CONCLUSION

In this paper, we introduced Ψ operator using primal [1] and studied some fundamental properties. Moreover, we introduced topology suitable for a primal and proved some equivalent conditions. We hope that these results will find these suitable roles in topological research related to primal.

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