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ALMOST EVERYWHERE CONVERGENCE OF VARYING-PARAMETER SETTING CESÁRO MEANS OF FOURIER SERIES ON THE GROUP OF 2-ADIC INTEGERS

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Abstract. In this paper we prove that the maximal operator of Cesáro-means for one-dimensional Fourier series on the group of 2-adic integers is of weak type (L^1, L^1). Moreover, we prove the almost everywhere convergence of Cesáro means of integrable functions, i.e. $\sigma_{2^n}^{\alpha_n} f \longrightarrow f$, for every $f \in L^1(I)$ and for every sequence $\alpha = (\alpha_n)$ with $0 < \alpha_n < 1$.

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1. INTRODUCTION

The Fejér-Lebesgue theorem, i.e. that the almost everywhere convergence $\sigma_n^1 f \to f$ holds for all integrable function f, was a question of Taibleson [14] open for a long time.

In 1997 Gát [2] proved the almost everywhere convergence $\sigma_n^1 f \to f$ for every integrable function f. Zheng [16] and Gát [3] generalized this result for more general orthonormal systems.

The idea of Cesáro means with variable parameters of numerical sequences is due to Kaplan [10]. In 2007, Akhobadez [1] introduced the notion of Cesáro means of Trigonometric Fourier series with variable parameter setting. Anas and Gát [8] proved for the varying parameter settings that (C, α) means for one-dimensional Walsh-Paley system converges almost everywhere For two-dimensional Walsh-Paley system this was proved by Anas and Gát [9] for the case of $(2^n, 2^n)(C, \alpha)$ means. Maximal operators of Cesáro means with varying parameters of Walsh-Fourier series were investigated by Gát and Goginava [5]. Cesáro and Riesz summability with varying parameters of multidimensional Walsh-Fourier series was proved by Weisz [15]. The one-dimensional case for this varying-parameter settings with respect to one-dimensional Vilenkin system was proved by Gát and Anteneh [6]. Gát [4] proved the almost everywhere

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convergence of Cesáro means of Fourier series on the group of 2-adic integers for a constant parameter. However, for the case of varying parameter, nothing has been done yet. In this paper, we prove it for the case of Fourier series on the group of 2-adic integers in the case of varying parameter setting.

Now, we introduce the basics by following the standard notions of dyadic analysis introduced by F. Schipp, P. Simon, W. R. Wade (see e.g. [12, 13]) and others.

Denote by $\mathbb{N} := \{0, 1, ...\}, \mathbb{P} := \mathbb{N} \setminus \{0\}$, and $I := [0, 1)$, the set of natural numbers, the set of positive integers and the unit interval, respectively. Denote by $(B) = |B|$ the Lebesgue measure of the set B $(B \subset I)$. Denote by $L^p(I)$ the usual Lebesgue spaces and $\|.\|_p$ the corresponding norms $(1 \leq p \leq \infty)$. Set

$$\tau := \left\{ \left[\frac{p}{2^n}, \frac{p+1}{2^n} \right) : p,\, n \in \mathbb{N} \right\},$$

the set of 2-adic intervals and for a given $x \in I$ let $I_n(x)$ denote the interval $I_n(x) \in \tau$ of length 2^{-n} which contains x $(n \in \mathbb{N})$. Also use the notation $I_n := I_n(0) \quad (n \in \mathbb{N})$. Let

$$x = \sum_{n=0}^{\infty} x_n 2^{-(n+1)}$$

the 2-adic expansion of $x \in I$, where $x_n = 0$ or 1 and if x is a dyadic rational number $(x \in \frac{p}{2^n} : p, n \in \mathbb{N})$ we choose the expansion which terminates in 0's. The notion of the Hardy space $H(I)$ is introduced in the following way [12]. A function $a \in L^\infty(I)$ is called an atom, if either $a = 1$ or a has the following properties:

(i) $\operatorname{supp} a \subset I_a$
(ii) $\|a\|_\infty \leq |I_a|^{-1}$
(iii) $\int_I a = 0$,

for some $I_a \in I$. We say that the function f belongs to H, if f can be represented as $f = \sum_{i=0}^{\infty} \lambda_i a_i$, where a_i's are atoms and for the coefficients (λ_i) the inequality $\sum_{i=0}^{\infty} |\lambda_i| < \infty$ is true. It is known that H is a Banach space with respect to the norm

$$\|f\|_H := \inf \sum_{i=0}^{\infty} |\lambda_i|,$$

where the infimum is taken over all decomposition $f = \sum_{i=0}^{\infty} \lambda_i a_i \in H$. The 2-adic (or arithmetic) sum $a + b := \sum_{n=0}^{\infty} r_n 2^{-(n+1)} (a, b \in I)$, where bits $q_n, r_n \in \{0, 1\}$ $(n \in \mathbb{N})$ are defined recursively as follows: $q_{-1} := 0, a_n + b_n + q_{n-1} = 2q_n + r_n$ for $n \in \mathbb{N}$. Since q_n, r_n take only the values 0 or 1, these equations uniquely determine the coefficients q_n and r_n. The group $(I, +)$ is called the group of 2-adic integers.

Set

$$\epsilon(t) := \exp(2\pi i t),\, t \in \mathbb{R},$$

where $i = (1)^{-\frac{1}{2}}$. Set

$$v_{2^n} := \epsilon\left(\frac{x_n}{2} + ... + \frac{x_o}{2^{n+1}}\right) \qquad (x \in I,\, n \in \mathbb{N})$$

and

$$v_n := \prod_{i=0}^{\infty} v_{2^i}^{n_i},$$

where $n = \sum_{i=0}^{\infty} n_i 2^i$ ($n_i \in \{0,1\}$ for $i \in \mathbb{N}$), $n \in \mathbb{N}$. It is known [7] that the system (v_n, $n \in \mathbb{N}$) is the character system of $(I, +)$. For more details on the Fourier theory with respect to the character system of the group of 2-adic integers see for instance [11]. Denote by

$$\hat{f}(n) := \int_I f\bar{v}_n \mathrm{d}\lambda, \qquad D_n := \sum_{k=0}^{n-1} v_k, \qquad K_n^1 := \frac{1}{n+1}\sum_{k=0}^{n} D_k,$$

the Fourier coefficients, the Dirichlet and the Fejér or $(C, 1)$ kernels, respectively. It is also known that the Fejér or $(C, 1)$ means of f is

$$\begin{aligned}\sigma_n f(y) &:= \frac{1}{n+1}\sum_{k=0}^{n} S_k f(y) = \int_I f(x) K_n^1(y-x)\mathrm{d}\lambda(x)\\ &= \frac{1}{n+1}\int_I f(x) D_k^1(y-x)\mathrm{d}\lambda(x) \qquad (n \in \mathbb{N},\, y \in I)\end{aligned}$$

It is known [2, 11] that for $n \in \mathbb{N}$, $x \in I$

$$D_{2^n}(x) = \begin{cases} 0, & \text{if } x \in I_n, \\ 2^n, & \text{if } x \notin I_n \end{cases}$$

and also that

$$D_n(x) = v_n(x)\sum_{k=0}^{\infty} D_{2^k}(x) n_k (-1)^{x_k}.$$

Denote by K_n^α the kernel of the summability method (C, α), and call it the (C, α) kernel, or the Cesáro kernel for $\alpha \in \mathbb{R} \setminus \{..., -3-2, -1\}$

$$K_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=0}^{n} A_{n-v}^{\alpha-1} D_v,$$

where

$$A_k^\alpha = \frac{(\alpha+1)(\alpha+2)...(\alpha+k)}{k!}.$$

The (C,α) means of the integrable function f are

$$\sigma_n^\alpha f(y) = \frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_k f(y) = \int_I f(x) K_n^\alpha (y-x)\mathrm{d}\lambda(x),$$

$$\sigma_n^\alpha f(y) = \frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1} \int_I f(x) D_k (y-x)\mathrm{d}\lambda(x).$$

It is well-known [17] that

$$A_n^\alpha = \sum_{k=0}^{n} A_{n-k}^{\alpha-1}, \qquad A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1}, \qquad A_n^\alpha \approx n^\alpha$$

Basically, in order to prove Theorem 3.6 we verify that the maximal operator $\sigma_{*,q}^\alpha$ is of weak type (L^1, L^1). In order to do this, we investigate kernel functions, and its maximal function on the unit interval I by making a hole around zero. To obtain the proof of Theorem 3.7, we follow the standard way of using the fact that $\sigma_{*,q}^\alpha$ is of weak type (L^1, L^1). We need several lemmas.

The following notations as well as definitions of functions and operators are used through the proofs of this paper.

For a, s, $n \in \mathbb{N}$, let $n_{(s)} := \sum_{j=0}^{s-1} n_i 2^i$, that is, $n_{(0)} = 0$, $n_{(1)} = n_0$ and for $2^B \le n < 2^{B+1}$, $|n| := B$, $n_{(B+1)} = n$.

Define a two variable function $P(n,\alpha) := \sum_{i=0}^{\infty} n_i (2^i)^\alpha$, for $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$ [8]. For example $P(n,1) = n$. Besides, set for a sequence $\alpha = (\alpha_n)$ and a positive real number q, the subset of natural numbers

$$\mathbb{N}_{\alpha,q} := \left\{ n \in \mathbb{N} : \frac{P(n,\alpha_n)}{n^{\alpha_n}} \le q \right\}.$$

For a sequence α such that $0 < \alpha_0 \le \alpha_n < 1$ we have $\mathbb{N}_{\alpha,q} = \mathbb{N}$ for some q depending only on α_0. We remark that $2^n \in \mathbb{N}_{\alpha,q}$ for every $\alpha = (\alpha_n)$, $0 < \alpha_n < 1$ and $q \ge 1$. In this paper, C denotes an absolute constant and C_q another one which may depend only on q. Define the following kernel function and operator.

$$T_n^\alpha := \frac{1}{A_n^\alpha} \sum_{k=0}^{2^B-1} A_{n-k}^{\alpha_a - 1} D_k,$$

$$t_n^\alpha f(y) := \int_I f(x) T_n^\alpha (y-x) \mathrm{d}\mu(x).$$

For $f \in L^1(I)$ and for all real numbers $\alpha_n \ne -1, -2, -3, ...$, define the kernel of (C,α_n) summability method as follows

$$K_n^{\alpha_n} := \frac{1}{A_n^{\alpha_n}} \sum_{t=0}^{n} A_{n-t}^{\alpha_n - 1} D_t, \tag{1}$$

where $A_k^{\alpha_n}$ is defined in [1] for the case where $\alpha = (\alpha_n)$. Besides, introduce the following kernel functions and operators where $0 < \alpha_n < 1$:

$$\tilde{K}_n^{\alpha_n} := \left|T_n^{\alpha_n}\right| + \sum_{l=0}^{B} \frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}} D_{2^l} + \sum_{l=0}^{B} \frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}} |T_{n_{(l-1)}}^{\alpha_n}|,$$

$$\tilde{\sigma}_n^{\alpha_n} f(y) := \int_I f(x)\tilde{K}_n^{\alpha_n}(y-x)\mathrm{d}\mu(x).$$

2. PRELIMINARY LEMMAS

LEMMA 2.1 ([4]). *For $j, n \in \mathbb{N}, j < 2^n$ we have $D_{2^n - j} = D_{2^n} - v_{2^n-1}\bar{D}_j$.*

LEMMA 2.2 ([4]). *$\int_{I \setminus I_k(u)} \sup_{n \geq 2^k} |T_n^\alpha| \leq C$, where C depends only on α.*

LEMMA 2.3 ([4]). *$\int_{I \setminus I_k(u)} \sup_{n \geq 2^A} |T_n^\alpha| \leq C(k-A)$, where C depends only on α.*

3. MAIN RESULT

The following lemma plays a central role in the proof of next lemmas and the main theorem too.

LEMMA 3.1. *Let $0 < \alpha_n < 1$, $n \in \mathbb{N}$, $2^B \leq n < 2^{B+1}$, $|n| = B$. Then,*

$$|K_n^{\alpha_n}| \leq \tilde{K}_n^{\alpha_n}.$$

Proof. By definition, we have

$$\begin{aligned} K_n^{\alpha_n} &= \frac{1}{A_n^{\alpha_n}} \sum_{j=0}^{n-1} A_{n-j}^{\alpha_n - 1} D_j \\ &= \frac{1}{A_n^{\alpha_n}} \sum_{j=0}^{2^B-1} A_{n-j}^{\alpha_n - 1} D_j + \frac{1}{A_n^{\alpha_n}} \sum_{j=2^B}^{n-1} A_{n-j}^{\alpha_n - 1} D_j \\ &= T_n^{\alpha_n} + \frac{1}{A_n^{\alpha_n}} \sum_{j=2^B}^{2^B + n_{(B)} - 1} A_{n_{(B)} + 2^B - j}^{\alpha_n - 1} D_j. \end{aligned}$$

By Lemma 2.1 we have

$$\begin{aligned} &\frac{1}{A_n^{\alpha_n}} \sum_{j=2^B}^{2^B + n_{(B)} - 1} A_{n_{(B)} + 2^B - j}^{\alpha_n - 1} D_j \\ &= \frac{1}{A_n^{\alpha_n}} \sum_{t=0}^{n-1} A_{n-t}^{\alpha_n - 1} D_{t+2^B} \end{aligned}$$

$$= \frac{1}{A_n^{\alpha_n}} \sum_{t=0}^{n_{(B)}-1} A_{n-t}^{\alpha_n-1}\Bigg(D_{2^B} + v_{2^B-1}D_t\Bigg)$$

$$= \frac{D_{2^B}}{A_n^{\alpha_n}} \sum_{t=0}^{n_{(B)}-1} A_{n-t}^{\alpha_n-1} + \frac{v_{2^B-1}}{A_n^{\alpha_n}} \sum_{t=0}^{n_{(B)}-1} A_{n-t}^{\alpha_n-1}D_t$$

$$= \frac{A_{n_{(B)}}^{\alpha_n}}{A_n^{\alpha_n}}\Bigg(D_{2^B} + v_{2^B-1}K_{n_{(B)}}^{\alpha_n}\Bigg).$$

Then,

$$K_n^{\alpha_n} = T_n^{\alpha_n} + \frac{A_{n_{(B)}}^{\alpha_n}}{A_n^{\alpha_n}}\Bigg(D_{2^B} + v_{2^B-1}K_{n_{(B)}}^{\alpha_n}\Bigg).$$

In general, for $j = 1, ..., B+1,$ we get

$$K_{n_{(j)}}^{\alpha_n} = T_{n_{(j)}}^{\alpha_n} + \frac{A_{n_{(j-1)}}^{\alpha_n}}{A_{n_{(j)}}^{\alpha_n}}\Bigg(D_{2^{(j-1)}} + v_{2^{(j-1)}-1}K_{n_{(j-1)}}^{\alpha_n}\Bigg).$$

Recursively applying this formula and considering that

$$n_{(-1)} = 0, \qquad T_0^{\alpha_n} = K_0^{\alpha_n} = 0, \qquad A_0^{\alpha_n} = 1,$$

we get

$$|K_n^{\alpha_n}| \le |T_n^{\alpha_n}| + \sum_{l=0}^{B}\Bigg(\prod_{j=l}^{B}\frac{A_{n_{(j-1)}}^{\alpha_n}}{A_{n_{(j)}}^{\alpha_n}}D_{2^l} + \prod_{j=l}^{B}\frac{A_{n_{(j-1)}}^{\alpha_n}}{A_{n_{(j)}}^{\alpha_n}}|T_{n_{(l-1)}}^{\alpha_n}|\Bigg) = \tilde{K}_n^{\alpha_n}.$$

Hence, the lemma follows. □

We prove Lemma 3.2 below, which means that the maximal operator $t_*^{\alpha} := \sup_{n,\, a\in\mathbb{N}} |t_n^{\alpha}|$ is quasi-local.

Lemma 3.2. *The maximal operator t_*^{α} is quasi-local.*

Proof. By the definition of quasi-locality, let $1 > \alpha > 0$, $f \in L^1(I)$ such that $\operatorname{supp} f \subset I_k(u)$, $\int_{I_k(u)} f \mathrm{d}\mu(x) = 0$ for some 2-adic interval $I_k(u)$. We can easily show that for $n < 2^k$ and $x \in I_k(u)$, $y \in \bar{I}_k(u)$ we have $T_n^{\alpha}(y-x) = T_n^{\alpha}(y-u)$. Then,

$$\int_{I_k(u)} f(x)T_n^{\alpha}(y-x)\mathrm{d}\mu(x) = T_n^{\alpha}(y-u)\int_{I_k(u)} f(x)\mathrm{d}\mu(x) = 0.$$

Consequently,

$$\int_{\bar{I}_k(u)} \sup_{n\in\mathbb{N}} \left|t_n^{\alpha} f\right| \mathrm{d}\mu = \int_{\bar{I}_k(u)} \sup_{n\ge 2^k} \left|t_n^{\alpha} f\right| \mathrm{d}\mu.$$

By the shift invariance of the Haar measure it can be assumed that $u = 0$. That is, $I_k(u) = I_k$. Thus,

$$\int_{\bar{I}_k(u)} \sup_{n\geq 2^k} |t_n^\alpha f| \mathrm{d}\mu = \int_{\bar{I}_k} \sup_{n\geq 2^k} \left| \int_{I_k} T_n^\alpha(y-x) f(x) \mathrm{d}\mu(x) \right| \mathrm{d}\mu(y).$$

By Lemma 2.2 we have,

$$\begin{aligned}
&\int_{\bar{I}_k} \sup_{n\geq 2^k} \left| \int_{I_k} T_n^\alpha(y-x) f(x) \mathrm{d}\mu(x) \right| \mathrm{d}\mu(y) \\
&= \int_{I_k} |f(x)| \left| \int_{\bar{I}_k} \sup_{n\geq 2^k} T_n^\alpha(y-x) \mathrm{d}\mu(x) \right| \mathrm{d}\mu(y) \\
&\leq C \|f\|_1
\end{aligned}$$

□

In the following corollary, it is also proved that operators t_n^α are of type $(L^1,\, L^1)$ and $(L^\infty,\, L^\infty)$ uniformly in n.

Corollary 3.3. *Let $1 > \alpha > 0$, $a \in \mathbb{N}$. Then, we have*

$$\|T_n^\alpha\|_1 \leq C, \qquad \|t_n^\alpha f\|_1 \leq C\|f\|_1 \quad \textit{and} \quad \|t_n^{\alpha_a} g\|_\infty \leq C\|g\|_\infty,$$

for all natural numbers n and where C is some absolute constant and $f \in L^1(I)$, $g \in L^\infty(I)$. That is, operator t_n^α is of type $(L^1,\, L^1)$ and $(L^\infty,\, L^\infty)$ uniformly in n.

Proof. The proof is a direct consequence of Lemma 3.1, Lemma 3.2 and since $\|D_{2^k}\|_1$, $\|K_j\|_1 \leq C$. □

In the next lemma we prove that the maximal operator

$$\tilde{\sigma}_{*,\,q}^\alpha := \sup_{n\in\mathbb{N}_{\alpha,\,q}} |\tilde{\sigma}_n^{\alpha_n}|$$

is quasi-local. We get this by the investigation of kernel functions and its maximal function on the 2-adic group by making a hole around zero.

Lemma 3.4. *Let $0 < \alpha_n < 1$, $f \in L^1(I)$ such that* $\operatorname{supp} f \subset I_k(u)$ *and $\int_{I_k(u)} f \mathrm{d}\mu = 0$ for some 2-adic interval $I_k(u)$. Then we have*

$$\int_{I\setminus I_k(u)} \sigma_{*,\,q}^\alpha f \mathrm{d}\mu \leq C_q \|f\|_1.$$

Where the constants C_q can depend only on q.

Proof. From the formula of kernel function $\tilde{K}_n^{\alpha_n}$ we have

$$\tilde{K}_n^{\alpha_n} = \left|T_n^{\alpha_n}\right| + \sum_{l=0}^{B} \frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}} D_{2^l} + \sum_{l=0}^{B} \frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}} |T_{n_{(l-1)}}^{\alpha_n}| =: N_1 + N_2 + N_3.$$

The integral,

$$\int_{I\setminus I_k(u)} \sup_{n\in N}\left|\int_{I_k(u)} f(x)\Bigg(N_2(y-x)\Bigg)\mathrm{d}\mu(x)\right|\mathrm{d}\mu(y)=0$$

since $f * D_{2^l} = 0$, for $l < s \leq k$ because of the $\mathcal{A}_k$ measurablity of D_{2^l} and $\int f = 0$. Besides, $D_{2^l}(y-x)=0$, for $s>k$, $y-x\notin I_k$.

Since it is known from [1] we have

$$\frac{A^{\alpha_n}_{n_{(l-1)}}}{A^{\alpha_n}_n} \leq \frac{(n_{(l-1)})^{\alpha_n}}{n^{\alpha_n}} \leq C\frac{(2^l)^{\alpha_n}}{n^{\alpha_n}}.$$

Besides, by the help of Lemma 3.2 and by the fact that $n\in\mathbb{N}_{\alpha,\,q}$ implies

$$\sum_{l=0}^{B}\frac{A^{\alpha_n}_{n_{(l-1)}}}{A^{\alpha_n}_n} \leq C\sum_{l=0}^{B}\frac{(2^l)^{\alpha_n}}{n^{\alpha_n}} \leq C_q$$

we get

$$\begin{aligned}
&\int_{I\setminus I_k(u)} \sup_{n\in\mathbb{N}_{\alpha,\,q}}\left|\int_{I_k(u)} f(x)\Bigg(N_1(y-x)+N_3(y-x)\Bigg)\mathrm{d}\mu(x)\right|\mathrm{d}\mu(y)\\
&\leq \int_{I\setminus I_k(u)} \sup_{n\in\mathbb{N}_{\alpha,\,q}}\Bigg|\int_{I_k(u)} f(x)\Bigg(\left|T^{\alpha_n}_n(y-x)\right|\\
&+\sum_{l=0}^{B}\frac{A^{\alpha_n}_{n_{(l-1)}}}{A^{\alpha_n}_n}\left|T^{\alpha_a}_{n_{(l-1)}}(y-x)\right|\Bigg)\mathrm{d}\mu(x)\Bigg|\mathrm{d}\mu(y)\\
&\leq C_q\int_{I\setminus I_k(u)} \sup_{n\in\mathbb{N}_{\alpha,\,q}}\left|\int_{I_k(u)} f(x)\left|T^{\alpha_n}_n(y-x)\right|\mathrm{d}\mu(x)\right|\mathrm{d}\mu(y)\\
&\leq C_q\|f\|_1.
\end{aligned}$$

Hence, the lemma follows. □

For the general case of $n\in\mathbb{N}_{\alpha,q}$. Define (C,α_n) mean as follows

$$\sigma^{\alpha_n}_n f(x) := \frac{1}{A^{\alpha}_n}\sum_{k=0}^{n} A^{\alpha-1}_{n-k}S_k(x) = \int_I f(y)K^{\alpha_n}_n(x-y)\mathrm{d}\mu(y), \tag{2}$$

where $K^{\alpha_n}_n$ is the kernel function defined in (1).

Considering the definition in (2), we define maximal operators as follows:

$$\sigma^{\alpha}_{*,\,q}f := \sup_{n\in\mathbb{N}_{\alpha,\,q}}|\sigma^{\alpha_n}_n f|.$$

Lemma 3.5. *The operator $\sigma^{\alpha}_{*,\,q}$ is of type $(L^{\infty},\, L^{\infty})$ and weak type (L^1, L^1).*

Proof. Using Lemma 3.1 and Corollary 3.3 we get

$$\begin{aligned}\|K_n^{\alpha_n}\|_1 &\leq \|T_n^{\alpha_n}\|_1 + \sum_{l=0}^{B} \frac{A^{\alpha_n}_{n_{(l-1)}}}{A^{\alpha_n}_n}\|D_{2^l}\|_1 + \sum_{l=0}^{B} \frac{A^{\alpha_n}_{n_{(l-1)}}}{A^{\alpha_n}_n}\|T^{\alpha_n}_{n_{(l-1)}}\|_1 \\ &\leq C + C\sum_{l=0}^{B} \frac{A^{\alpha_n}_{n_{(l-1)}}}{A^{\alpha_n}_n} \leq C_q\end{aligned}$$

since $n \in \mathbb{N}_{\alpha,\,q}$. Thus, $\sigma^{\alpha}_{*,\,q}$ is of type $(L^{\infty},\, L^{\infty})$.

To prove the weak type $(L^1,\, L^1)$ case we apply the Calderon-Zygmund decomposition lemma [3].

Let $f \in L^1(I)$ and $\|f\|_1 < \delta$. Then there is a decomposition:

$$f = f_0 + \sum_{j=1}^{\infty} f_j$$

such that

$$\|f_0\|_{\infty} \leq C\delta, \qquad \|f_0\|_1 \leq C\|f\|_1$$

and $I^j = I_{k_j}(u^j)$ are disjoint 2-adic intervals, for which

$$\operatorname{supp} f_j \subset I^j, \qquad \int_{I^j} f_j \mathrm{d}\mu = 0,\; |F| \leq \frac{C\|f\|_1}{\delta}$$

$(u^j \in I,\, k_j \in \mathbb{N},\, j \in \mathbb{P}),$ where $F = \bigcup_{i=1}^{\infty} I^j$.

By the σ-sublinearity of the maximal operator with an appropriate constant C_q we have

$$\mu(\sigma^{\alpha}_{*,\,q} f > 2C_q\delta) \leq \mu(\sigma^{\alpha}_{*,\,q} f_0 > C_q\delta) + \mu(\sigma^{\alpha}_{*,\,q}\sum_{j=1}^{\infty} f_j > C_q\delta) =: W + M.$$

Since $\sigma^{\alpha}_{*,\,q}$ is of type $(L^{\infty},\, L^{\infty})$, we have that

$$\|\sigma_{*,\,q} f_0\|_{\infty} \leq C_q\|f_0\|_{\infty} \leq C_q\delta.$$

Then we have $W = 0$. The situation for M becomes,

$$\begin{aligned} M = \mu(\sigma^{\alpha}_{*,\,q}\sum_{j=1}^{\infty} f_j > C_q\delta) &\leq |F| + \mu(\bar{F} \cap [\sigma^{\alpha}_{*,\,q}\sum_{j=1}^{\infty} f_j > C_q\delta]) \\ &\leq \frac{C\|f\|_1}{\delta} + \frac{C_q}{\delta}\sum_{j=1}^{\infty}\int_{I\setminus I^j} \sigma^{\alpha}_{*,\,q} f_j \mathrm{d}\mu =: \frac{C\|f\|_1}{\delta} + \frac{C_q}{\delta}\sum_{j=1}^{\infty} N_j,\end{aligned}$$

in which

$$N_j = \int_{I\setminus I^j} \sigma^{\alpha}_{*,\,q} f_j \mathrm{d}\mu.$$

By Lemma 3.4 we have

$$N_j \le \int_{I\setminus I_{k_j}(u^j)} \sup_{n\in\mathbb{N}_{\alpha,\,q}} \left| \int_{I_{k_j}(u^j)} f_j(x) K_n^{\alpha_n}(y-x)\mathrm{d}\mu(x) \right| \mathrm{d}\mu(y) \le C_q \|f_j\|_1.$$

Thus,

$$\mu(\sigma^{\alpha}_{*,\,q} f > 2C_q\delta) \le C_q \frac{\|f\|_1}{\delta}.$$

We conclude that the maximal operator $\sigma^{\alpha}_{*,\,q}$ is of weak type $(L^1,\, L^1)$. Hence, the lemma follows. □

Following the proofs of the lemmas and corollaries above, the almost everywhere convergence of the (C, α_n) means is proved in the following theorem.

THEOREM 3.6. *Let* $0 < \alpha_n < 1$ *and* $f \in L^1(I)$. *Then* $\sigma_n^{\alpha_n} f \longrightarrow f$ *almost everywhere if* $n \longrightarrow \infty$, $n \in \mathbb{N}_{\alpha,\,q}$.

Proof. Let us consider a polynomial P with respect to the character system of the group of 2-adic integers such that $P(x) = \sum_{i=0}^{2^k-1} c_i v_i(x)$. Then for all natural number $n \ge 2^k$, $n \in \mathbb{N}_{\alpha,\,q}$ we have that $S_n P \equiv P$. Thus, the statement $\sigma_n^{\alpha_n} P \longrightarrow P$ holds everywhere which is not only for $n \in \mathbb{N}_{\alpha,\,q}$, but for arbitrary $n \to \infty$.

Now, let $\epsilon,\, \delta > 0,\ f \in L^1(I)$. Let P be a polynomial such that $\|f - P\|_1 < \delta$. Then,

$$\begin{aligned}
&\mu\left(\overline{\lim_{n\in\mathbb{N}_{\alpha,\,q}}} |\sigma_n^{\alpha_n} f - f| > \epsilon\right) \\
&\le \mu\left(\overline{\lim_{n\in\mathbb{N}_{\alpha,\,q}}} |\sigma_n^{\alpha_n}(f-P)| > \frac{\epsilon}{3}\right) + \mu\left(\overline{\lim_{n\in\mathbb{N}_{\alpha,\,q}}} |\sigma_n^{\alpha_n} P - P| > \frac{\epsilon}{3}\right) \\
&+ \mu\left(\overline{\lim_{n\in\mathbb{N}_{\alpha,\,q}}} |P - f| > \frac{\epsilon}{3}\right) \\
&\le \mu\left(\overline{\lim_{n\in\mathbb{N}_{\alpha,\,q}}} |\sigma_n^{\alpha_n}(f-P)| > \frac{\epsilon}{3}\right) + 0 + \frac{3}{\epsilon}\|P - f\|_1 \\
&\le C_q \|P - f\|_1 \frac{3}{\epsilon} \le \frac{C_q}{\epsilon}\delta
\end{aligned}$$

since (from Lemma 3.5) $\sigma^{\alpha}_{*,\,q}$ is of weak type (L^1, L^1) with any fixed $q > 0$. This holds for all $\delta > 0$. That is, for an arbitrary $\epsilon > 0$

$$\mu\left(\overline{\lim_{n\in\mathbb{N}_{\alpha,\,q}}} |\sigma_n^{\alpha_n} f - f| > \epsilon\right) = 0$$

and as a result we also have

$$\mu\left(\overline{\lim_{n\in\mathbb{N}_{\alpha,\,q}}} |\sigma_n^{\alpha_n} f - f| > 0\right) = 0.$$

This finally gives

$$\overline{\lim_{n\in\mathbb{N}_{\alpha,\,q}}} \left|\sigma_n^{\alpha_n} f - f\right| = 0 \quad \text{almost everywhere}$$

Consequently,

$$\sigma_n^{\alpha_n} f \longrightarrow f \text{ almost everywhere as } n \longrightarrow \infty,\, n \in \mathbb{N}_{\alpha,\,q}.$$

Hence, the theorem follows. □

THEOREM 3.7. *Let $f \subset H(I)$. Then we have $\left\|\sigma_{*,\,q}^{\alpha} f\right\|_H \leq C_{q,\alpha} \left\|f\right\|_H$. Moreover, the operator $\sigma_{*,\,q}^{\alpha}$ is of type (L^p, L^p), for all $1 < p \leq \infty$. That is, $\left\|\sigma_{*,\,q}^{\alpha} f\right\|_p \leq C_{q,p,\alpha} \left\|f\right\|_p$, for all $1 < p \leq \infty$, where the constants $C_{q,\alpha}$ and $C_{q,p,\alpha}$ depends on the constants q, p, α.*

Proof. By the Marcinkiewicz interpolation theorem of [12] and by using Corollary 3.3 and Lemma 3.5 we have that the operator $\sigma_{*,\,q}^{\alpha}$ is of type (L^p, L^p), for all $1 < p \leq \infty$. Assume $a \neq 1$ is an atom, $I_a := I_k(x)$, $\left\|a\right\|_\infty \leq 2^k$ for some $k \in \mathbb{N}$ and $x \in I$. Then, $n < 2^k$ implies $S_n a = 0$. As a result, we have $\sigma_{*,\,q}^{\alpha} a = \sup_{n\geq 2^k} \left|\sigma_n^{\alpha} a\right|$.

Using Lemma 3.4, we have

$$\begin{aligned}
\int_{I\setminus I_a} \sigma_*^{\alpha} a \mathrm{d}\lambda &= \int_{I\setminus I_k(x)} \sup_{n\geq 2^k} \left|\int_{I_k(x)} a(y) K_n^{\alpha}(z-y) \mathrm{d}\lambda(y)\right| \lambda(z) \\
&\leq \int_{I_k(x)} |a(y)| \int_{I\setminus I_k(x)} \sup_{n\geq 2^k} \left|K_n^{\alpha}(z-y)\right| \mathrm{d}\lambda(z)\mathrm{d}\lambda(y) \\
&\leq \int_{I_k(x)} |a(y)| \int_{I\setminus I_k(x)} \sup_{n\geq 2^k} \Bigg[\left|T_n^{\alpha_n}(z-y)\right| \\
&+ \sum_{l=0}^{B} \frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}} D_{2^l}(z-y) + \sum_{l=0}^{B} \frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}} \left|T_{n_{(l-1)}}^{\alpha_n}(z-y)\right|\Bigg] \\
&\leq \int_{I_k(x)} |a(y)| \Bigg\{\int_{I\setminus I_k(x)} \sup_{n\geq 2^k} \left|T_n^{\alpha_n}(z-y)\right| \\
&+ \int_{I\setminus I_k(x)} \sup_{n\geq 2^k} \sum_{l=0}^{B} \frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}} D_{2^l}(z-y) \\
&+ \int_{I\setminus I_k(x)} \sup_{n\geq 2^k} \sum_{l=0}^{B} \frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}} \left|T_{n_{(l-1)}}^{\alpha_n}(z-y)\right|\Bigg\} \\
&\leq \int_{I_k(x)} |a(y)| \int_{I\setminus I_k(x)} \sup_{n\geq 2^k} \left|T_n^{\alpha_n}(z-y)\right|
\end{aligned}$$

$$
\begin{aligned}
&+\int_{I_k(x)}|a(y)|\int_{I\setminus I_k(x)}\sup_{n\geq 2^k}\sum_{l=0}^{B}\frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}}D_{2^l}(z-y)\\
&+\int_{I_k(x)}|a(y)|\int_{I\setminus I_k(x)}\sup_{n\geq 2^k}\sum_{l=0}^{B}\frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}}\left|T_{n_{(l-1)}}^{\alpha_n}(z-y)\right|\\
&:=\Psi_1+\Psi_2+\Psi_3.
\end{aligned}
$$

By Lemma 2.2, the situation for Ψ_1 becomes

$$
\Psi_1=\int_{I_k(x)}|a(y)|\int_{I\setminus I_k(x)}\sup_{n\geq 2^k}\left|T_n^{\alpha_n}(z-y)\right|\leq C\int_{I_k(x)}|a(y)|\leq C.
$$

For $z-y\in I_k$ and $l\geq k$ we have $D_{2^l}(z-y)=0$. Thus, the situation for Ψ_2 becomes

$$
\begin{aligned}
\Psi_2&=\int_{I_k(x)}|a(y)|\int_{I\setminus I_k(x)}\sup_{n\geq 2^k,\,l\leq k}\sum_{l=0}^{B}\frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}}D_{2^l}(z-y)\\
&=\sum_{t=0}^{k-1}\int_{I_t\setminus I_{t+1}}\sup_{n\geq 2^k,\,l\leq t}\sum_{l=0}^{B}\frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}}2^l\int_{I_k(x)}|a(y)|\leq C_q\int_{I_k(x)}|a(y)|\leq C_q.
\end{aligned}
$$

Similarly, using Lemma 3.1 and Lemma 3.2, we have the situation for

$$
\Psi_3=\int_{I_k(x)}|a(y)|\int_{I\setminus I_k(x)}\sup_{n\geq 2^k}\sum_{l=0}^{B}\frac{A_{n_{(l-1)}}^{\alpha_n}}{A_n^{\alpha_n}}\left|T_{n_{(l-1)}}^{\alpha_n}(z-y)\right|\leq C_q.
$$

We have that the operator $\sigma_{*,q}^{\alpha}$ is of type (L^2,L^2) (i.e. $\left\|\sigma_{*,q}^{\alpha}f\right\|_2\leq C\left\|f\right\|_2$ for all $f\in L^2(I)$), then

$$
\begin{aligned}
\left\|\sigma_{*,q}^{\alpha}a\right\|_1 &= \int_{I\setminus I_a}Ta+\int_{I_a}\sigma_{*,q}^{\alpha}a \;\leq\; C+|I_a|^{\frac{1}{2}}\left\|\sigma_{*,q}a\right\|^{\alpha}\\
&\leq C+C_q2^{\frac{-k}{2}}\left\|a\right\|_2 \;\leq\; C+C_q2^{\frac{-k}{2}}2^{\frac{k}{2}} \;\leq\; C_q.
\end{aligned}
$$

That is, $\left\|\sigma_{*,q}^{\alpha}a\right\|_1\leq C_q$. Hence, by the sublinearity of $\sigma_{*,q}^{\alpha}$, we have

$$
\left\|\sigma_{*,q}^{\alpha}f\right\|_1\leq\sum_{i=0}^{\infty}|\lambda_i|\left\|\sigma_{*,q}^{\alpha}a_i\right\|_1\leq C_q\sum_{i=0}^{\infty}|\lambda_i|\leq C_q\left\|f\right\|_H,
$$

for all $f=\sum_{i=0}^{\infty}\lambda_ia_i\in H$. Therefore, the operator $\sigma_{*,q}^{\alpha}$ is of type (H,L). Hence, the theorem follows. $\square$

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