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SOME GENERAL DISTORTION RESULTS FOR $K(\alpha)$ AND $S^*(\alpha)$

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Abstract. In this paper we present some general distortion results for the classes $K(\alpha)$ and $S^*(\alpha)$ of convex, respectively starlike functions of order α on the unit disc. For this, we start from a classical result for the class S of univalent and normalized functions on the unit disc. Furthermore, since when $\alpha = 0$ these classes reduce to the well-known classes of starlike and convex functions, we obtain also some general distortion results for the classes K and S^* of convex, respectively starlike functions on the unit disc.

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1. INTRODUCTION

In this paper we denote $U = U(0,1)$ the open unit disc in the complex plane $\mathbb{C}$ and S the family of all univalent (holomorphic and injective) normalized $(f(0) = f'(0) - 1 = 0)$ functions on the unit disc. It is well-known (see [2,4,7]) that if $f \in S$, then

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U. \tag{1}$$

and it is easy to prove that

$$f^{(k)}(z) = \sum_{p=0}^{\infty} \frac{(k+p)!}{p!} a_{p+k} z^p, \quad z \in U. \tag{2}$$

The basic result from which we start in this paper is the following estimate which can be found, for example, in (see [2, p. 70, ex. 6]):

$$\forall\ f \in S: \quad |f^{(k)}(z)| \le \frac{k! \cdot (k + |z|)}{(1 - |z|)^{k+2}}, \quad z \in U, \quad k \in \mathbb{N}.$$

Taking into account the Bieberbach's Conjecture (proved in 1984 by L. de Branges), we can prove the previous inequality using the technique of dominant power series. Also, there are similar results for the classes K and S^* of convex, respectively starlike functions on the unit disc (see [4, p. 117, Th. 8 and Th.

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9]). Recall that $f \in K$ if f is univalent on U and $f(U)$ is a convex domain, respectively $f \in S^*$ if f is univalent on U, $f(0) = 0$ and $f(U)$ is a starlike domain with respect to the origin.

In this paper, we extend the previous inequalities for the classes $K(\alpha)$ and $S^*(\alpha)$ of convex, respectively starlike functions of order α on the unit disc, with $\alpha \in [0, 1)$. These classes were introduced by M.S. Robertson in [10]. We denote

$$K(\alpha) = \left\{ f \in S \ : \mathrm{Re}\left[\frac{zf''(z)}{f'(z)} + 1\right] > \alpha, \quad z \in U \right\}$$

the *class of convex (normalized) functions of order* α on U and

$$S^*(\alpha) = \left\{ f \in S \ : \mathrm{Re}\left[\frac{zf'(z)}{f(z)}\right] > \alpha, \quad z \in U \right\}$$

the *class of starlike (normalized) functions of order* α on U.

Remark 1.1. It is obvious that for $\alpha = 0$ we obtain the well-known classes

$$K = \left\{ f \in S \ : \mathrm{Re}\left[\frac{zf''(z)}{f'(z)} + 1\right] > 0, \quad z \in U \right\}$$

and

$$S^* = \left\{ f \in S \ : \mathrm{Re}\left[\frac{zf'(z)}{f(z)}\right] > 0, \quad z \in U \right\}$$

of convex, respectively starlike functions on the unit disc U in the complex plane $\mathbb{C}$.

In the following two sections we present some general distortion results for the classes $K(\alpha)$ and $S^*(\alpha)$ of convex, respectively starlike functions of order α on the unit disc, where $\alpha \in [0, 1)$. For this, we start from a classical result for the class S of univalent and normalized functions on the unit disc. This inequality can be found, for example, in [2, p. 70, ex. 6]. The classes of starlike, respectively convex functions of order α were first introduced by M. S. Robertson (see [10]). More results about the starlike and convex functions of order α can be found in [4–6, 8, 9, 11].

2. PRELIMINARIES

Similar to Bieberbach's Conjecture, we have estimates of coefficients also for the classes $K(\alpha)$ and $S^*(\alpha)$ as we can see in the following proposition. For details and proofs, one may consult [10, p. 386], [6, Lemma 2.1] or [11, p. 65, Th. 2].

Proposition 2.1. *Let $\alpha \in [0,1)$. The following estimates hold:*

(1) *If $f \in K(\alpha)$, then*

$$|a_n| \le \frac{1}{n!}\prod_{p=2}^{n}(p-2\alpha), \quad n \ge 2. \tag{3}$$

(2) *If $f \in S^*(\alpha)$, then*

$$|a_n| \le \frac{1}{(n-1)!}\prod_{p=2}^{n}(p-2\alpha), \quad n \ge 2. \tag{4}$$

These estimates for $|a_n|$ are sharp. Notice also that for $\alpha = 0$, we obtain the well-known result $|a_n| \le n$.

Next, we present two distortion theorems for $K(\alpha)$, respectively $S^*(\alpha)$. These theorems are essentially due to Robertson (see [10]). For details and proofs, one may consult also [8, p. 86, Th. 4.4.5 and Th. 4.4.6] , [5, p. 56, Th. 2.3.6 and Th. 2.3.7] or [9, p. 727, Th. 3 and Th. 4].

Theorem 2.2 (Growth and distortion theorem for $K(\alpha)$). *Let $\alpha \in [0,1)$ and $f \in K(\alpha)$. Then*

$$\frac{1}{(1+|z|)^{2(1-\alpha)}} \le |f'(z)| \le \frac{1}{(1-|z|)^{2(1-\alpha)}}.$$

If $\alpha = \frac{1}{2}$, then

$$\log(1+|z|) \le |f(z)| \le -\log(1-|z|).$$

If $\alpha \ne \frac{1}{2}$, then

$$\frac{(1+|z|)^{2\alpha-1}-1}{2\alpha-1} \le |f(z)| \le \frac{1-(1-|z|)^{2\alpha-1}}{2\alpha-1},$$

for all $z \in U$. These bounds are sharp. Equality holds in each of the above relations for

$$f(z) = \begin{cases} \dfrac{1-(1-z)^{2\alpha-1}}{2\alpha-1}, & \alpha \ne \frac{1}{2} \\ \\ -\log(1-z), & \alpha = \frac{1}{2}, \end{cases} \tag{5}$$

where $(1-z)^{2\alpha-1}\big|_{z=0} = 1$ and $\log(1-z)\big|_{z=0} = 0$.

Theorem 2.3 (Growth and distortion theorem for $S^*(\alpha)$). *Let $\alpha \in [0,1)$ and $f \in S^*(\alpha)$. Then*

$$\frac{|z|}{(1+|z|)^{2(1-\alpha)}} \le |f(z)| \le \frac{|z|}{(1-|z|)^{2(1-\alpha)}}$$

and

$$\frac{1-(1-2\alpha)|z|}{(1+|z|)^{3-2\alpha}} \le |f'(z)| \le \frac{1+(1-2\alpha)|z|}{(1-|z|)^{3-2\alpha}},$$

for all $z \in U$. These bounds are sharp. Equality holds for

$$f(z) = \frac{z}{(1-z)^{2(1-\alpha)}}, \quad z \in U. \tag{6}$$

REMARK 2.4. When $\alpha = 0$, we obtain the growth and distortion results for the classes K and S^* of convex, respectively starlike functions on the unit disc.

REMARK 2.5. Let $r = |z| < 1$. Then, for every $k \in \mathbb{N}^*$, the following relation holds

$$T_k = \frac{1}{(1-r)^k} = \sum_{p=0}^{\infty} \frac{(k+p-1)! \cdot r^p}{p! \cdot (k-1)!}. \tag{7}$$

This remark will be used in the next section as part of the proofs of the main results.

Proof. Let us consider the following Taylor series expansion

$$\frac{1}{1-r} = 1 + r + r^2 + ... + r^n + ..., \quad -1 < r < 1.$$

Then

$$\frac{1}{(1-r)^2} = \frac{\partial}{\partial r}\left[\frac{1}{1-r}\right] = 1 + 2r + 3r^2 + ... + nr^{n-1} + ...$$

It is easy to prove relation (7) using mathematical induction. For this, let us consider

$$P(k) : \frac{1}{(1-r)^k} = \sum_{p=0}^{\infty} \frac{(k+p-1)! \cdot r^p}{p! \cdot (k-1)!}, \quad k \geq 1.$$

Assume that $P(k)$ is true and let us prove that $P(k+1)$ is also true, where

$$P(k+1) : \frac{1}{(1-r)^{k+1}} = \sum_{p=0}^{\infty} \frac{(k+p)! \cdot r^p}{p! \cdot k!}.$$

Indeed,

$$\frac{k}{(1-r)^{k+1}} = \frac{\partial}{\partial r}\left[\frac{1}{(1-r)^k}\right] = \frac{\partial}{\partial r}\left[\sum_{p=0}^{\infty} \frac{(k+p-1)! \cdot r^p}{p! \cdot (k-1)!}\right]$$

$$= \sum_{p=1}^{\infty} \frac{(k+p-1)! \cdot p \cdot r^{p-1}}{p! \cdot (k-1)!} = \sum_{p=0}^{\infty} \frac{(k+p)! \cdot r^p}{p! \cdot (k-1)!}$$

and then

$$\frac{1}{(1-r)^{k+1}} = \sum_{p=0}^{\infty} \frac{(k+p)! \cdot r^p}{p! \cdot k!}$$

Hence, $P(k)$ is true for all $k \geq 1$ and the relation (7) holds. $\square$

3. THE MAIN RESULTS

3.1. CONVEXITY OF ORDER α

In this section we present a general distortion result for convex functions of order α.

THEOREM 3.1. *Let $\alpha \in [0,1)$ and $f \in K(\alpha)$. Then the following estimate holds:*

$$|f^{(k)}(z)| \leq \frac{M(k,\alpha)}{(1-|z|)^{k+1-2\alpha}}, \quad \forall\ z \in U, \quad k \geq 1,$$

where

$$M(k,\alpha) = \begin{cases} \dfrac{1}{1-2\alpha} \displaystyle\prod_{p=1}^{k}(p-2\alpha), & \alpha \neq \dfrac{1}{2} \\ \\ (k-1)!, & \alpha = \dfrac{1}{2} \end{cases}$$

These bounds are sharp. Equality holds for the function given by (5).

Proof. Let $f \in K(\alpha)$. Then f is of the form (1) and f has the following form of the k-th derivative:

$$f^{(k)}(z) = \sum_{p=0}^{\infty} \frac{(k+p)!}{p!} a_{p+k} z^p, \quad z \in U. \tag{8}$$

Let $|z| \leq r < 1$. In view of relations (3) and (8) we obtain

$$\begin{aligned} |f^{(k)}(z)| &= \left| \sum_{p=0}^{\infty} \frac{(k+p)!}{p!} a_{p+k} z^p \right| \\ &\leq \sum_{p=0}^{\infty} \frac{(k+p)!}{p!} |a_{p+k}| \cdot |z|^p \\ &\leq \sum_{p=0}^{\infty} \left(\frac{(k+p)!}{p!} \cdot \frac{1}{(p+k)!} \cdot \prod_{p=2}^{p+k}(p-2\alpha) r^p \right) \\ &= \sum_{p=0}^{\infty} \frac{1}{p!} \prod_{p=2}^{p+k}(p-2\alpha) r^p \end{aligned}$$

We have two cases.

- **Case 1.** If $\alpha = \dfrac{1}{2}$, then

$$
\begin{aligned}
|f^{(k)}(z)| &\leq \sum_{p=0}^{\infty} \frac{1}{p!} \prod_{p=2}^{p+k} (p-1) r^p \\
&= \sum_{p=0}^{\infty} \frac{(p+k-1)! r^p}{p!} \\
&= (k-1)! \cdot \sum_{p=0}^{\infty} \frac{(p+k-1)! r^p}{p!(k-1)!} \\
&= (k-1)! \cdot T_k,
\end{aligned}
$$

where T_k is given by (7). Hence,

$$
|f^{(k)}(z)| \leq \frac{(k-1)!}{(1-|z|)^k}, \quad z \in U.
$$

- **Case 2.** If $\alpha \neq \dfrac{1}{2}$, then

$$
\begin{aligned}
|f^{(k)}(z)| &\leq \sum_{p=0}^{\infty} \frac{1}{p!} \prod_{p=2}^{p+k} (p-2\alpha) r^p \\
&= \sum_{p=0}^{\infty} \frac{(p+k-2\alpha)!}{2\alpha(1-2\alpha)p!} r^p \\
&= \frac{(k-2\alpha)!}{2\alpha(2\alpha-1)} \cdot \sum_{p=0}^{\infty} \frac{(p+k-2\alpha)!}{p!(k-2\alpha)!} r^p \\
&= \frac{(k-2\alpha)!}{2\alpha(2\alpha-1)} \cdot T_{k+1-2\alpha} \\
&= \frac{(k-2\alpha)!}{2\alpha(2\alpha-1)} \cdot \frac{1}{(1-r)^{k+1-2\alpha}} \\
&= \frac{\prod_{p=2}^{k}(p-2\alpha)}{(1-r)^{k+1-2\alpha}},
\end{aligned}
$$

where

$$
(k-2\alpha)! = \prod_{p=0}^{k} (p-2\alpha)
$$

is a factorial notation which depends only on k, and

$$
T_{k+1-2\alpha} = \frac{1}{(1-r)^{k+1-2\alpha}} = \sum_{p=0}^{\infty} \frac{(p+k-2\alpha)!}{p!(k-2\alpha)!} r^p.
$$

This result holds also for $k = 1$, so we obtain that

$$|f^{(k)}(z)| \leq \frac{\prod_{p=2}^{k}(p-2\alpha)}{(1-r)^{k+1-2\alpha}} = \frac{\frac{1}{1-2\alpha}\prod_{p=1}^{k}(p-2\alpha)}{(1-r)^{k+1-2\alpha}} = \frac{M(k,\alpha)}{(1-r)^{k+1-2\alpha}}, \quad \forall\, r < 1,$$

where

$$M(k,\alpha) = \frac{1}{1-2\alpha}\prod_{p=1}^{k}(p-2\alpha), \quad k \geq 1.$$

This completes the proof. $\square$

Remark 3.2. If $\alpha = 0$, then $K(0) = K$ is the class of convex functions on the unit disc and we obtain the upper bounds for the k-th derivative in K:

$$|f^{(k)}(z)| \leq \frac{k!}{(1-|z|)^{k+1}}, \quad z \in U, \quad k \geq 1.$$

The same result can be found in [4, p. 118, Th. 9].

Remark 3.3. Let $\alpha \in [0,1)$ and $f \in K(\alpha)$. If $z = 0$, then we obtain

$$|f^{(k)}(0)| \leq \begin{cases} \dfrac{1}{1-2\alpha}\displaystyle\prod_{p=1}^{k}(p-2\alpha), & \alpha \neq \dfrac{1}{2} \\[2ex] (k-1)!, & \alpha = \dfrac{1}{2}, \end{cases}$$

for all $k \geq 1$. In particular,

$$|f''(0)| \leq 2(1-\alpha). \tag{9}$$

3.2. STARLIKENESS OF ORDER α

In this section we present a general distortion result for starlike functions of order α.

Theorem 3.4. *Let $\alpha \in [0,1)$ and $f \in S^*(\alpha)$. Then the following estimate holds:*

$$|f^{(k)}(z)| \leq \frac{M(k,\alpha)\big[k + |z|\cdot(1-2\alpha)\big]}{(1-|z|)^{k+2-2\alpha}}, \quad \forall\, z \in U, \quad k \geq 1, \tag{10}$$

where

$$M(k,\alpha) = \begin{cases} \dfrac{1}{1-2\alpha}\displaystyle\prod_{p=1}^{k}(p-2\alpha), & \alpha \neq \dfrac{1}{2} \\[2ex] (k-1)!, & \alpha = \dfrac{1}{2} \end{cases}$$

These bounds are sharp. Equality holds for the function given by (6).

Proof. It is clear that $S^*(\alpha)$ is a subclass of class S. From the first section we know that every function f from class S has the following form of k-th derivative

$$f^{(k)}(z) = \sum_{p=0}^{\infty} \frac{(k+p)!}{p!} a_{p+k} z^p, \quad z \in U. \tag{11}$$

Let $|z| \leq r < 1$. In view of relations (4) and (11) we obtain

$$|f^{(k)}(z)| = \left| \sum_{p=0}^{\infty} \frac{(k+p)!}{p!} a_{p+k} z^p \right| \leq \sum_{p=0}^{\infty} \frac{(k+p)!}{p!} |a_{p+k}| \cdot |z|^p$$

$$\leq \sum_{p=0}^{\infty} \left(\frac{(p+k)!}{p!} \cdot \frac{1}{(p+k-1)!} \cdot \prod_{p=2}^{p+k} (p-2\alpha) r^p \right) = \sum_{p=0}^{\infty} \frac{p+k}{p!} \prod_{p=2}^{p+k} (p-2\alpha) r^p$$

- **Case 1.** If $\alpha = \dfrac{1}{2}$, then

$$|f^{(k)}(z)| \leq \sum_{p=0}^{\infty} \frac{p+k}{p!} \prod_{p=2}^{p+k} (p-1) r^p = \sum_{p=0}^{\infty} \frac{(p+k)(p+k-1)! r^p}{p!}$$

$$= \sum_{p=0}^{\infty} \frac{(p+k)! r^p}{p!} = k! \cdot \sum_{p=0}^{\infty} \frac{(p+k)! r^p}{p! \cdot k!} = k! \cdot T_{k+1},$$

where T_{k+1} is given by (7). Hence,

$$|f^{(k)}(z)| \leq \frac{k!}{(1-|z|)^{k+1}}, \quad z \in U.$$

- **Case 2.** If $\alpha \neq \dfrac{1}{2}$, then

$$|f^{(k)}(z)| \leq \sum_{p=0}^{\infty} \frac{p+k}{p!} \prod_{p=2}^{p+k} (p-2\alpha) r^p = \sum_{p=0}^{\infty} \frac{(p+k)(p+k-2\alpha)! r^p}{2\alpha(2\alpha-1)p!}$$

$$= \frac{1}{2\alpha(2\alpha-1)} \sum_{p=0}^{\infty} \frac{(k+p-2\alpha)!(p+k) r^p}{p!},$$

where

$$(k+p-2\alpha)! = \prod_{p=0}^{k+p} (p-2\alpha)$$

is a factorial notation which depends only on $k+p$.

Next, we determine the sum of the series

$$\sum_{p=0}^{\infty} \frac{(k+p-2\alpha)!(p+k) r^p}{p!}.$$

In view of *the ratio test* (for details see [1]) we obtain that the series converges when $r < 1$.

As in the case of the class S we consider $u_p = a_p + b_p$, where

$$u_p = \frac{(k+p-2\alpha)!(p+k)r^p}{p!},$$

is the general term of the above series,

$$a_p = \frac{(p+k-2\alpha)!(p-2n\alpha)r^p}{p!(k+1-2\alpha)}$$

and

$$b_p = \frac{(p+k-2\alpha)!(k^2+k+pk-2\alpha k)r^p}{p!(k+1-2\alpha)}.$$

It is easy to observe that $a_0 = 0$, so we can consider a new general term h_p for the above series, where

$$h_p = b_p + a_{p+1}$$

$$h_p = \frac{(p+k+1-2\alpha)![k+r(1-2\alpha)]r^p}{p!(k+1-2\alpha)}.$$

Hence,

$$\begin{aligned}
&\sum_{p=0}^{\infty} \frac{(k+p-2\alpha)!(p+k)r^p}{p!} \\
&= \sum_{p=0}^{\infty} \frac{(p+k+1-2\alpha)![k+r(1-2\alpha)]r^p}{p!(k+1-2\alpha)} \\
&= (k-2\alpha)![k+r(1-2\alpha)] \cdot \sum_{p=0}^{\infty} \frac{(p+k+1-2\alpha)!r^p}{p!(k+1-2\alpha)!}.
\end{aligned}$$

In view of these computations, we obtain

$$\begin{aligned}
|f^{(k)}(z)| &\le \frac{(k-2\alpha)!}{2\alpha(2\alpha-1)} \cdot [k+r(1-2\alpha)] \cdot \sum_{p=0}^{\infty} \frac{(p+k+1-2\alpha)!r^p}{p!(k+1-2\alpha)!} \\
&= \frac{(k-2\alpha)!}{2\alpha(2\alpha-1)} \cdot [k+r(1-2\alpha)] \cdot T_{k+2-2\alpha} \\
&= \frac{(k-2\alpha)!}{2\alpha(2\alpha-1)} \cdot \frac{[k+r(1-2\alpha)]}{(1-r)^{k+2-2\alpha}} \\
&= \prod_{p=2}^{k} (p-2\alpha) \cdot \frac{[k+r(1-2\alpha)]}{(1-r)^{k+2-2\alpha}}.
\end{aligned}$$

This result holds also for $k = 1$, so we obtain that

$$|f^{(k)}(z)| \le \frac{1}{1-2\alpha} \prod_{p=1}^{k} (p-2\alpha) \cdot \frac{[k+r(1-2\alpha)]}{(1-r)^{k+2-2\alpha}}$$

$$|f^{(k)}(z)| \leq \frac{M(k,\alpha)\big[k + r(1-2\alpha)\big]}{(1-r)^{k+2-2\alpha}}, \quad \forall\, r < 1,$$

where

$$M(k,\alpha) = \frac{1}{1-2\alpha}\prod_{p=1}^{k}(p-2\alpha), \quad k \in \mathbb{N}^*.$$

This completes the proof. $\square$

Remark 3.5. If $\alpha = 0$, then $S^*(0) = S^*$ is the class of starlike functions on the unit disc and we obtain the upper bounds for the k-th derivative in S^*:

$$|f^{(k)}(z)| \leq \frac{k!(k+|z|)}{(1-|z|)^{k+2}}, \quad z \in U.$$

The same result can be found in [4, p. 117, Th. 8].

Remark 3.6. The above result confirm also the inequality from [3, Th. 3.2]:

$$|f''(0)| \leq 2/\beta$$

for $\alpha \in [0,1)$ and $1/\beta = 2(1-\alpha)$. It is enough to take $k = 2$ in (10), and we obtain that

$$|f''(z)| \leq \frac{M(2,\alpha)[2 + |z|\cdot(1-2\alpha)]}{(1-|z|)^{2+2-2\alpha}}, \quad z \in U.$$

Now, for $z = 0$ we obtain

$$|f''(0)| \leq \frac{(2-2\alpha)[2 + 0\cdot(1-2\alpha)]}{(1-0)^{2+2-2\alpha}} = 2\cdot 2(1-\alpha) = 2\cdot\frac{1}{\beta} = \frac{2}{\beta},$$

and hence

$$|f''(0)| \leq 2/\beta.$$

Remark 3.7. Let $\alpha \in [0,1)$ and $f \in S^*(\alpha)$. Then

$$|f^{(k)}(0)| \leq \begin{cases} \dfrac{k}{1-2\alpha}\displaystyle\prod_{p=1}^{k}(p-2\alpha), & \alpha \neq \dfrac{1}{2} \\[2ex] k!, & \alpha = \dfrac{1}{2}, \end{cases}$$

for all $k \geq 1$.

Remark 3.8. If we take $\alpha = 1/2$ in Theorem 3.4, we obtain the general distortion result for the convex functions given also in [4, p. 118, Th. 9] and Remark 3.2.

Remark 3.9. Another proof of Theorem 3.4 can be given using the duality theorem between $K(\alpha)$ and $S^*(\alpha)$ (see [4, 5]).

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