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INTEGRAL SOLUTION OF A CONFORMABLE FRACTIONAL INTEGRO-DIFFERENTIAL EQUATION WITH NONLOCAL CONDITION

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Abstract. This paper deals with the existence and uniqueness of the integral solution of a nondense integro-differential equation with nonlocal condition in the frame of conformable fractional derivative. The main results are obtained by using some fixed point theorems combined with a integrated semigroup approach.

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1. INTRODUCTION

The so-called conformable fractional derivative has been introduced in [24]. In short time, the better effect of this new fractional derivative attracts many researchers in several areas of applications [1, 4–12, 16, 18–20, 22, 26, 38–41]. For more details about fractional calculus, we refer to works [25, 27, 30, 33, 35]. On the other hand, the nonlocal condition introduced in [14] is crucial in the description of dynamical processes with unknown initial behaviors in various areas of modeling [17, 31]. For more details about the better effects of non local conditions in differential equations theory, we refer to [13, 15, 29, 36] and references therein. In this present work, we study a class of integro-differential equation with conformable fractional derivative and nonlocal condition. Precisely, we consider the following fractional Cauchy problem with nonlocal condition:

$$(1)\qquad \begin{cases} \frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} &= Ax(t) + f(t, x(t)) \\ &+ \displaystyle\int_0^t a(t-\sigma)\varphi(\sigma, x(\sigma))\mathrm{d}\sigma,\ t \in [0, \tau],\ 0 < \alpha \le 1, \\ x(0) &= x_0 + g(x), \end{cases}$$

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where $\frac{\mathrm{d}^\alpha(.)}{\mathrm{d}t^\alpha}$ presents the conformable fractional derivative [24] and $x(0) = x_0 + g(x)$ is the nonlocal condition [14, 17, 31]. Concerning the linear part A, it is well known that many concrete applications prove that the density of its domain $D(A)$ is a very harsh condition to be imposed [2,3,21,23,28,34,37]. So, we assume that the linear part $(A, D(A))$ is closed and satisfying the Hille-Yosida property without assuming the density of $D(A)$ in a Banach space $(X, \|.\|)$. The functions $f : [0,\tau] \times X \to X$, $\varphi : [0,\tau] \times X \to X$, $a : [0,\tau] \to \mathbb{R}$, $g : \mathcal{C} \to \overline{D(A)}$ satisfied some assumptions and x_0 is an element of $\overline{D(A)}$, where $\mathcal{C}$ is the Banach space of continuous functions from $[0,\tau]$ into X with the norm $|x|_c = \sup_{t\in[0,\tau]} \|x(t)\|$. We also denote by $|.|$ the norm in the space $\mathcal{L}(X)$ of bounded operators defined from X into itself.

Our purpose in this paper is to establish the existence and uniqueness of the integral solution of equation (1) based on the following Duhamel formula:

$$\begin{aligned} x(t) &= \dot{S}\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] \\ &+ \lim_{\lambda\to+\infty} \int_0^t s^{\alpha-1} \dot{S}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) \lambda(\lambda - A)^{-1}[f(s, x(s)) + k(s, x(s))]\mathrm{d}s, \end{aligned} \tag{2}$$

where $(S(t))_{t\geq 0}$ is the integrated semigroup generated by the operator A (see [3, 21, 23]) and $k(s, x(s))$ is the convolution operator given as follows:

$$k(s, x(s)) = \int_0^s a(s-\sigma)\varphi(\sigma, x(\sigma))\mathrm{d}\sigma. \tag{3}$$

The rest of this work is organized as follows. In Section 2, we recall some preliminary facts on the conformable fractional calculus and integrated semigroup theory. Section 3 is devoted to prove the main results.

2. PRELIMINARIES

In this section, we introduce some preliminaries concerning the conformable fractional derivative and integrated semigroup theory.

Definition 2.1 ([24]). For $\alpha \in]0, 1]$, the conformable fractional derivative of order α of a function $x(.)$ is defined as

$$\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = \lim_{\varepsilon\to 0} \frac{x(t + \varepsilon t^{1-\alpha}) - x(t)}{\varepsilon}, \quad t > 0,$$

$$\frac{\mathrm{d}^\alpha x(0)}{\mathrm{d}t^\alpha} = \lim_{t\to 0^+} \frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha}.$$

The conformable fractional integral $I^\alpha(.)$ associated with the conformable fractional derivative is defined as follows

$$I^\alpha(x)(t) = \int_0^t s^{\alpha-1} x(s)\mathrm{d}s.$$

THEOREM 2.2 ([24]). *If $x(.)$ is a continuous function in the domain of $I^\alpha(.)$, then we have*

$$\frac{\mathrm{d}^\alpha(I^\alpha(x)(t))}{\mathrm{d}t^\alpha} = x(t).$$

DEFINITION 2.3 ([1]). The conformable fractional Laplace transform of order $\alpha \in]0,1]$ of a function $x(.)$ is defined as follows

$$\mathcal{L}_\alpha(x(t))(\lambda) = \int_0^{+\infty} t^{\alpha-1} e^{-\frac{\lambda t^\alpha}{\alpha}} x(t)\mathrm{d}t, \quad \lambda > 0.$$

The following proposition gives us the actions of the conformable fractional integral and the conformable fractional Laplace transform on the conformable fractional derivative, respectively.

PROPOSITION 2.4 ([1]). *If $x(.)$ is a differentiable function, then we have the following results*

$$I^\alpha\left(\frac{\mathrm{d}^\alpha x(.)}{\mathrm{d}t^\alpha}\right)(t) = x(t) - x(0),$$
$$\mathcal{L}_\alpha\left(\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha}\right)(\lambda) = \lambda\mathcal{L}_\alpha(x(t))(\lambda) - x(0).$$

According to [6], we have the following remark.

REMARK 2.5. For two functions $x(.)$ and $y(.)$, we have

$$\mathcal{L}_\alpha\left(x\left(\frac{t^\alpha}{\alpha}\right)\right)(\lambda) = \mathcal{L}_1(x(t))(\lambda),$$
$$\mathcal{L}_\alpha\left(\int_0^t s^{\alpha-1} x\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) y(s)\mathrm{d}s\right)(\lambda) = \mathcal{L}_1(x(t))(\lambda)\mathcal{L}_\alpha(y(t))(\lambda),$$

provided that the both terms of each equality exist.

Now, we recall some definitions and results on the integrated semigroup theory.

DEFINITION 2.6 ([3, 23]). An integrated semigroup is a family $(S(t))_{t\geq 0}$ of bounded linear operators $S(t)$ on a Banach space X with the following properties:

(1) $S(0) = 0$,
(2) $t \longmapsto S(t)$ is strongly continuous,
(3) $S(s)S(t) = \int_0^s (S(t+\tau) - S(\tau))\mathrm{d}\tau$ for all $t, s \geq 0$.

DEFINITION 2.7 ([23]). Let $(S(t))_{t\geq 0}$ be an integrated semigroup.

(1) $(S(t))_{t\geq 0}$ is called exponentially bounded, if there exist constants $M \geq 0$ and $\omega \in \mathbb{R}$ such that $|S(t)| \leq Me^{\omega t}$ for all $t \geq 0$.
(2) $(S(t))_{t\geq 0}$ is called non-degenerate for all $t \geq 0$, if $S(t)x = 0$ implies $x = 0$.

(3) $(S(t))_{t\geq 0}$ is called locally Lipschitz continuous if for all $b > 0$ there exists a constant L such that $|S(t) - S(s)| \leq L\,|t-s|$, for all $s, t \in [0, b]$.

DEFINITION 2.8 ([23]). An operator A is called an infinitesimal generator of an integrated semigroup, if there exists $\omega \in \mathbb{R}$ such that $]\omega, +\infty[\subset \rho(A)$ and there exists a strongly continuous exponentially bounded family $(S(t))_{t\geq 0}$ of bounded operators such that

(1) $S(0) = 0$,

(2) $(\lambda I - A)^{-1} = \lambda \int_0^{+\infty} e^{-\lambda t} S(t)\mathrm{d}t$, for all $\lambda > \omega$.

PROPOSITION 2.9 ([3]). *Let A be the infinitesimal generator of an integrated semigroup $(S(t))_{t\geq 0}$. Then the following properties hold.*

(1) $\int_0^t S(s)x\mathrm{d}s \in D(A)$, *for all* $x \in X$ *and* $t > 0$.

(2) $S(t)y \in D(A)$, *for all* $y \in D(A)$ *and* $t > 0$.

(3) $S(t)x = A\int_0^t S(s)x\mathrm{d}s + tx$, *for all* $x \in X$ *and* $t > 0$.

(4) $AS(t)y = S(t)Ay$, *for all* $y \in D(A)$ *and* $t > 0$.

(5) $S(t)y = \int_0^t S(s)Ay\mathrm{d}s + ty$, *for all* $y \in D(A)$ *and* $t > 0$.

If $x \in \overline{D(A)}$ then the function $t \longmapsto S(t)x$ is continuously differentiable and $\dot{S}\,(t)$ becomes a semigroup on $\overline{D(A)}$.

DEFINITION 2.10 ([2]). A linear operator A is called a Hille-Yosida operator if there exist constants $M \geq 0$ and $\omega \in \mathbb{R}$ such that

(1) $]\omega, +\infty[\subset \rho(A)$, ($\rho(A)$: is the resolvent set of A),

(2) $(\forall n \in \mathbb{N})$ $(\forall \lambda > \omega)$, $|R(\lambda, A)^n| \leq \frac{M}{(\lambda-\omega)^n}$, $(R(\lambda, A) := (\lambda - A)^{-1})$.

THEOREM 2.11 ([23]). *The following assertions are equivalent.*

(1) *A is a Hille-Yosida operator.*

(2) *A is the generator of a locally Lipschitz continuous integrated semigroup $(S(t))_{t\geq 0}$.*

We end these preliminaries by the following remark.

REMARK 2.12 ([32]). We have $\lim\limits_{\lambda\to+\infty} \lambda(\lambda - A)^{-1}x = x$, for all $x \in \overline{D(A)}$

3. MAIN RESULTS

Motivated by the works [21, 28], we introduce the following definition.

DEFINITION 3.1. A function $x \in \mathcal{C}$ is called an integral solution of equation (1) if

$$\int_0^t s^{\alpha-1}x(s)\mathrm{d}s \in D(A),\ t \in [0, \tau], \tag{4}$$

(5)
$$x(t) = x_0 + g(x) + A\int_0^t s^{\alpha-1}x(s)\mathrm{d}s + \int_0^t s^{\alpha-1}[f(s,x(s)) + k(s,x(s))]\mathrm{d}s,\ t \in [0,\tau].$$

REMARK 3.2 ([6]). We have the following equality

$$x(t) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon t^{1-\alpha}} s^{\alpha-1}x(s)\mathrm{d}s \in \overline{D(A)}.$$

LEMMA 3.3. *Each integral solution of equation* (1) *satisfies the following Duhamel formula*

$$x(t) = \dot{S}\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] + \lim_{\lambda\to+\infty}\int_0^t s^{\alpha-1}\dot{S}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\lambda(\lambda - A)^{-1}[f(s,x(s)) + k(s,x(s))]\mathrm{d}s,$$

where $(S(t))_{t\geq 0}$ *is the integrated semigroup generated by the operator* A.

Proof. Applying the conformable fractional Laplace transform in equation (5) and using Remark 2.5, we get

$$\begin{aligned}&\mathcal{L}_\alpha(x(t))(\lambda)\\ &= \frac{1}{\lambda}(x_0 + g(x)) + \frac{1}{\lambda}A\mathcal{L}_\alpha(x(t))(\lambda) + \frac{1}{\lambda}\mathcal{L}_\alpha([f(t,x(t)) + k(t,x(t))])(\lambda)\\ &= \frac{1}{\lambda}[x_0 + g(x) + A\mathcal{L}_\alpha(x(t))(\lambda) + \mathcal{L}_\alpha([f(t,x(t)) + k(t,x(t))])(\lambda)].\end{aligned}$$

Then one has

$$(\lambda - A)\mathcal{L}_\alpha(x(t))(\lambda) = x_0 + g(x) + \mathcal{L}_\alpha([f(t,x(t)) + k(t,x(t))])(\lambda).$$

Consequently, we get

$$\mathcal{L}_\alpha(x(t))(\lambda) = (\lambda - A)^{-1}[x_0 + g(x)] + (\lambda - A)^{-1}\mathcal{L}_\alpha([f(t,x(t)) + k(t,x(t))])(\lambda).$$

Since $[x_0 + g(x)] \in \overline{D(A)}$, the expression $(\lambda - A)^{-1}[x_0 + g(x)]$ can be rewritten as

$$(\lambda - A)^{-1}[x_0 + g(x)] = \mathcal{L}_\alpha\left(\dot{S}\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)]\right)(\lambda).$$

Also, by using a simple calculus and Remark 2.5, the expression

$$(\lambda - A)^{-1}\mathcal{L}_\alpha([f(t,x(t)) + k(t,x(t))])(\lambda)$$

can be rewritten as

$$\begin{aligned}&(\lambda - A)^{-1}\mathcal{L}_\alpha([f(t,x(t)) + k(t,x(t))])(\lambda)\\ &= \lambda\left[\frac{1}{\lambda}(\lambda - A)^{-1}\mathcal{L}_\alpha([f(t,x(t)) + k(t,x(t))])(\lambda)\right]\\ &= \lambda\mathcal{L}_1(S(t))(\lambda)\mathcal{L}_\alpha([f(t,x(t)) + k(t,x(t))])(\lambda)\end{aligned}$$

$$= \lambda \mathcal{L}_\alpha \left(\int_0^t s^{\alpha-1} S\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) [f(s,x(s)) + k(s,x(s))] \mathrm{d}s \right)(\lambda)$$
$$= \mathcal{L}_\alpha \left(\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} \int_0^t s^{\alpha-1} S\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) [f(s,x(s)) + k(s,x(s))] \mathrm{d}s \right)(\lambda).$$

Then one has

$$\begin{aligned}\mathcal{L}_\alpha(x(t))(\lambda) &= (\lambda - A)^{-1}[x_0 + g(x)] \\ &\quad + (\lambda - A)^{-1} \mathcal{L}_\alpha([f(t,x(t)) + k(t,x(t))])(\lambda) \\ &= \mathcal{L}_\alpha \left(\dot{S}\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] \right)(\lambda) \\ &\quad + \mathcal{L}_\alpha \left(\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} \int_0^t s^{\alpha-1} S\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) [f(s,x(s)) + k(s,x(s))] \mathrm{d}s \right)(\lambda).\end{aligned}$$

Thus

$$\begin{aligned}\mathcal{L}_\alpha(x(t))(\lambda) = \mathcal{L}_\alpha \Bigg(& \dot{S}\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] \\ & + \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} \int_0^t s^{\alpha-1} S\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) [f(s,x(s)) + k(s,x(s))] \mathrm{d}s \Bigg)(\lambda).\end{aligned}$$

Now taking the inverse conformable fractional Laplace transform in the last equation, we get

$$\begin{aligned} x(t) &= \dot{S}\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] \\ &+ \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} \int_0^t s^{\alpha-1} S\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) [f(s,x(s)) + k(s,x(s))] \mathrm{d}s.\end{aligned}$$

According to first point of Proposition 2.9, we have

$$\int_0^t s^{\alpha-1} S\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) [f(s,x(s)) + k(s,x(s))] \mathrm{d}s \in D(A).$$

Then, by using Remark 2.9, we get

$$\begin{aligned}&\lim_{\lambda \to +\infty} \lambda(\lambda - A)^{-1} \int_0^t s^{\alpha-1} S\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) [f(s,x(s)) + k(s,x(s))] \\ &= \int_0^t s^{\alpha-1} S\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) [f(s,x(s)) + k(s,x(s))].\end{aligned}$$

Therefore we obtain

$$\begin{aligned} x(t) &= \dot{S}\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] \\ &\quad + \lim_{\lambda \to +\infty} \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} \int_0^t s^{\alpha-1} S\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) \lambda(\lambda - A)^{-1} [f(s,x(s)) + k(s,x(s))] \mathrm{d}s\end{aligned}$$

$$= \dot{S}\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)]$$
$$+ \lim_{\lambda \to +\infty} \int_0^t s^{\alpha-1} \dot{S}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) \lambda(\lambda - A)^{-1}[f(s, x(s)) + k(s, x(s))]\mathrm{d}s.$$

Finally, we have the following Duhamel formula

$$x(t) = \dot{S}\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)]$$
$$+ \lim_{\lambda \to +\infty} \int_0^t s^{\alpha-1} \dot{S}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) \lambda(\lambda - A)^{-1}[f(s, x(s)) + k(s, x(s))]\mathrm{d}s.$$

□

Before proving the existence results of the integral solution, we introduce the following assumptions.

- (H_1) There exists a constant $L_1 > 0$ such that $\|f(t, y) - f(t, x)\| \leq L_1 \|y - x\|$, for all $x, y \in X$ and $t \in [0, \tau]$.
- (H_2) The function $f(., x) : [0, \tau] \longrightarrow X$ is continuous for all $x \in X$.
- (H_3) The function $a(.) : [0, \tau] \longrightarrow \mathbb{R}$ is continuous and there exists a constant $a_0 > 0$ such that $\sup_{t \in [0,\tau]} |a(t)| \leq a_0$.
- (H_4) There exists a constant $L_2 > 0$ such that $\|\varphi(t, y) - \varphi(t, x)\| \leq L_2 \|y - x\|$, for all $x, y \in X$ and $t \in [0, \tau]$.
- (H_5) The function $\varphi(., x) : [0, \tau] \longrightarrow X$ is continuous for all $x \in X$.
- (H_6) There exists a constant $L_3 > 0$ such that $\|g(y) - g(x)\| \leq L_3 |y - x|_c$, for all $x, y \in \mathcal{C}$.

LEMMA 3.4. *If the assumptions* (H_3) *and* (H_4) *hold, then, for the convolution operator* k *defined in* (3), *we have the following inequalities, for all* $x, y \in \mathcal{C}$.

$$(1)\quad \|k(s, y(s)) - k(s, x(s))\| \leq a_0 L_2 \left[s \sup_{\sigma \in [0,s]} \|y(\sigma) - x(\sigma)\| \right]$$
$$(2)\quad \int_0^t s^{\alpha-1} \|k(s, y(s)) - k(s, x(s))\| \mathrm{d}s \leq a_0 L_2 \frac{\tau^{\alpha+1}}{\alpha+1} |y - x|_c$$
$$(3)\quad \int_0^t s^{\alpha-1} \|k(s, x(s))\| \mathrm{d}s \leq a_0 \frac{\tau^{\alpha+1}}{\alpha+1} \left[L_2 |x|_c + \sup_{t \in [0,\tau]} \|\varphi(t, 0)\| \right]$$

THEOREM 3.5. *Assume that* $(H_1) - (H_6)$ *hold, then equation* (1) *has an unique integral solution, provided that*

$$\left(L_3 + M \frac{\tau^\alpha}{\alpha} L_1 + a_0 M \frac{\tau^{\alpha+1}}{\alpha+1} L_2 \right) \sup_{t \in [0,\tau]} \left| \dot{S}\left(\frac{t^\alpha}{\alpha}\right) \right| < 1.$$

Proof. Define the operator $\Gamma : \mathcal{C} \longrightarrow \mathcal{C}$ by:

$$\Gamma(x)(t) = \dot{S}\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] + \lim_{\lambda\to+\infty}\int_0^t s^{\alpha-1}\dot{S}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\lambda(\lambda - A)^{-1}[f(s,x(s)) + k(s,x(s))]\mathrm{d}s.$$

For $x, y \in \mathcal{C}$, we have

$$\Gamma(y)(t) - \Gamma(x)(t) = \dot{S}\left(\frac{t^\alpha}{\alpha}\right)[g(y) - g(x)] + \lim_{\lambda\to+\infty}\int_0^t s^{\alpha-1}\dot{S}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\lambda(\lambda - A)^{-1}[f(s,y(s)) - f(s,x(s))]\mathrm{d}s + \lim_{\lambda\to+\infty}\int_0^t s^{\alpha-1}\dot{S}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\lambda(\lambda - A)^{-1}[k(s,y(s)) - k(s,x(s))]\mathrm{d}s.$$

The second point of Definition 2.10, for $n = 1$, proves that

$$\lim_{\lambda\to+\infty}\left|\lambda(\lambda - A)^{-1}\right| \leq M.$$

Accordingly, we obtain

$$\|\Gamma(y)(t) - \Gamma(x)(t)\| \leq \sup_{t\in[0,\tau]}\left|\dot{S}\left(\frac{t^\alpha}{\alpha}\right)\right|\|g(y) - g(x)\| + \sup_{t\in[0,\tau]}\left|\dot{S}\left(\frac{t^\alpha}{\alpha}\right)\right| M\int_0^t s^{\alpha-1}\|f(s,y(s)) - f(s,x(s))\|\mathrm{d}s + \sup_{t\in[0,\tau]}\left|\dot{S}\left(\frac{t^\alpha}{\alpha}\right)\right| M\int_0^t s^{\alpha-1}\|k(s,y(s)) - k(s,x(s))\|\mathrm{d}s.$$

According to assumptions (H_1), (H_6) and the second point of Lemma 3.4, we conclude that

$$\|\Gamma(y)(t) - \Gamma(x)(t)\| \leq \left(L_3 + M\frac{\tau^\alpha}{\alpha}L_1 + Ma_0\frac{\tau^{\alpha+1}}{\alpha+1}L_2\right)\sup_{t\in[0,\tau]}\left|\dot{S}\left(\frac{t^\alpha}{\alpha}\right)\right||y - x|_c.$$

Taking the supremum, we get

$$|\Gamma(y) - \Gamma(x)|_c \leq \left(L_3 + M\frac{\tau^\alpha}{\alpha}L_1 + Ma_0\frac{\tau^{\alpha+1}}{\alpha+1}L_2\right)\sup_{t\in[0,\tau]}\left|\dot{S}\left(\frac{t^\alpha}{\alpha}\right)\right||y - x|_c.$$

Since $\left(L_3 + M\frac{\tau^\alpha}{\alpha}L_1 + Ma_0\frac{\tau^{\alpha+1}}{\alpha+1}L_2\right)\sup_{t\in[0,\tau]}\left|\dot{S}\left(\frac{t^\alpha}{\alpha}\right)\right| < 1$, the operator Γ has an unique fixed point in $\mathcal{C}$, which is the integral solution of equation (1). □

Notice that in several concrete applications the semigroup $(\dot{S}(t))_{t>0}$ is compact and in this case we can dispense with the strong Lipschitz condition imposed in hypothesis (H_1). Precisely, we can replace hypothesis (H_1) by the following weak assumption:

(H_7) The function $f(t,.) : X \longrightarrow X$ is continuous and there exists a function $\mu \in L^\infty([0,\tau],\mathbb{R}^+)$ such that $\|f(t,x)\| \leq \mu(t)$, for all $t \in [0,\tau]$.

THEOREM 3.6. *If the semigroup $(\dot{S}(t))_{t>0}$ is compact and $(H_2)-(H_7)$ are satisfied, then equation* (1) *has at least one integral solution, provided only that*

$$\left(L_3 + Ma_0\frac{\tau^{\alpha+1}}{\alpha+1}L_2\right)\sup_{t\in[0,\tau]}\left|\dot{S}\left(\frac{t^\alpha}{\alpha}\right)\right| < 1.$$

Proof. Let $B_r = \{x \in \mathcal{C},\ |x|_c \leq r\}$, where r is bigger or equal than

$$\frac{\sup_{t\in[0,\tau]}\left|\dot{S}\left(\frac{t^\alpha}{\alpha}\right)\right|\left[\|x_0\| + \|g(0)\| + Ma_0\frac{\tau^{\alpha+1}}{\alpha+1}\sup_{t\in[0,\tau]}\|\varphi(t,0)\| + M\frac{\tau^\alpha}{\alpha}|\mu|_{L^\infty([0,\tau],\mathbb{R}^+)}\right]}{1-\left(L_3 + Ma_0\frac{\tau^{\alpha+1}}{\alpha+1}L_2\right)\sup_{t\in[0,\tau]}\left|\dot{S}\left(\frac{t^\alpha}{\alpha}\right)\right|}$$

In order to use the Krasnoselskii fixed-point theorem, we define the operators Γ_1 and Γ_2, for $x \in B_r$ and $t \in [0,\tau]$, as follows

$$\Gamma_1(x)(t) = \dot{S}\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] + \lim_{\lambda\to+\infty}\int_0^t s^{\alpha-1}\dot{S}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\lambda(\lambda-A)^{-1}k(s,x(s))\mathrm{d}s,$$

$$\Gamma_2(x)(t) = \lim_{\lambda\to+\infty}\int_0^t s^{\alpha-1}\dot{S}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\lambda(\lambda-A)^{-1}f(s,x(s))\mathrm{d}s.$$

The proof will be given in four steps.

Step 1: Prove that $\Gamma_1(x) + \Gamma_2(y) \in B_r$ whenever $x,\ y \in B_r$.

Let $x,\ y \in B_r$, we have

$$\begin{aligned}\Gamma_1(x)(t) + \Gamma_2(y)(t) = {} & \dot{S}\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] \\ & + \lim_{\lambda\to+\infty}\int_0^t s^{\alpha-1}\dot{S}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\lambda(\lambda-A)^{-1}k(s,x(s))\mathrm{d}s \\ & + \lim_{\lambda\to+\infty}\int_0^t s^{\alpha-1}\dot{S}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\lambda(\lambda-A)^{-1}f(s,y(s))\mathrm{d}s.\end{aligned}$$

Then, we obtain

$$\|\Gamma_1(x)(t) + \Gamma_2(y)(t)\| \leq \sup_{t\in[0,\tau]}\left|\dot{S}\left(\frac{t^\alpha}{\alpha}\right)\right|[\|x_0\| + \|g(0)\| + \|g(x) - g(0)\|]$$

$$+ \sup_{t\in[0,\tau]} \left| \dot{S}\left(\frac{t^\alpha}{\alpha}\right) \right| M \int_0^t s^{\alpha-1} \|k(s,x(s))\| \mathrm{d}s$$
$$+ \sup_{t\in[0,\tau]} \left| \dot{S}\left(\frac{t^\alpha}{\alpha}\right) \right| M \int_0^t s^{\alpha-1} \|f(s,y(s))\| \mathrm{d}s.$$

According to assumptions (H_6), (H_7) and the third point of Lemma 3.4, we conclude that

$$\begin{aligned}
\|\Gamma_1(x)(t)+\Gamma_2(y)(t)\| \leq & \sup_{t\in[0,\tau]} \left| \dot{S}\left(\frac{t^\alpha}{\alpha}\right) \right| [\|x_0\| + \|g(0)\| + L_3 |x|_c] \\
& + \sup_{t\in[0,\tau]} \left| \dot{S}\left(\frac{t^\alpha}{\alpha}\right) \right| M a_0 \frac{\tau^{\alpha+1}}{\alpha+1} [L_2 |x|_c + \sup_{t\in[0,\tau]} \|\varphi(t,0)\|] \\
& + \sup_{t\in[0,\tau]} \left| \dot{S}\left(\frac{t^\alpha}{\alpha}\right) \right| |\mu|_{L^\infty([0,\tau],\mathbb{R}^+)} M \frac{\tau^\alpha}{\alpha}.
\end{aligned}$$

Using the fact that $x,\ y \in B_r$, we conclude that

$$\begin{aligned}
\|\Gamma_1(x)(t)+\Gamma_2(y)(t)\| \leq & \sup_{t\in[0,\tau]} \left| \dot{S}\left(\frac{t^\alpha}{\alpha}\right) \right| [\|x_0\| + \|g(0)\| + L_3 r] \\
& + \sup_{t\in[0,\tau]} \left| \dot{S}\left(\frac{t^\alpha}{\alpha}\right) \right| M a_0 \frac{\tau^{\alpha+1}}{\alpha+1} \left[L_2 r + \sup_{t\in[0,\tau]} \|\varphi(t,0)\|\right] \\
& + \sup_{t\in[0,\tau]} \left| \dot{S}\left(\frac{t^\alpha}{\alpha}\right) \right| |\mu|_{L^\infty([0,\tau],\mathbb{R}^+)} M \frac{\tau^\alpha}{\alpha}.
\end{aligned}$$

Taking the supremum, we get

$$\begin{aligned}
|\Gamma_1(x)+\Gamma_2(y)|_c \leq & \sup_{t\in[0,\tau]} \left| \dot{S}\left(\frac{t^\alpha}{\alpha}\right) \right| [\|x_0\| + \|g(0)\| + L_3 r] \\
& + \sup_{t\in[0,\tau]} \left| \dot{S}\left(\frac{t^\alpha}{\alpha}\right) \right| M a_0 \frac{\tau^{\alpha+1}}{\alpha+1} \left[L_2 r + \sup_{t\in[0,\tau]} \|\varphi(t,0)\|\right] \\
& + \sup_{t\in[0,\tau]} \left| \dot{S}\left(\frac{t^\alpha}{\alpha}\right) \right| |\mu|_{L^\infty([0,\tau],\mathbb{R}^+)} M \frac{\tau^\alpha}{\alpha} \\
\leq & \ r.
\end{aligned}$$

Hence, we conclude that $\Gamma_1(x) + \Gamma_2(y) \in B_r$, for all $x,\ y \in B_r$.

Step 2: Prove that Γ_1 is a contraction operator on B_r.

For $x, y \in \mathcal{C}$, we have

$$\Gamma_1(y)(t) - \Gamma_1(x)(t) = \dot{S}\left(\frac{t^\alpha}{\alpha}\right) [g(y) - g(x)]$$

$$+\lim_{\lambda\to+\infty}\int_0^t s^{\alpha-1}\dot{S}\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)\lambda(\lambda-A)^{-1}[k(s,y(s))-k(s,x(s))]\mathrm{d}s.$$

Using the fact that $\lim_{\lambda\to+\infty}\left|\lambda(\lambda-A)^{-1}\right|\leq M$, we get

$$\|\Gamma_1(y)(t)-\Gamma_1(x)(t)\|\leq\sup_{t\in[0,\tau]}\left|\dot{S}\left(\frac{t^\alpha}{\alpha}\right)\right|\|g(y)-g(x)\|$$

$$+\sup_{t\in[0,\tau]}\left|\dot{S}\left(\frac{t^\alpha}{\alpha}\right)\right|M\int_0^t s^{\alpha-1}\|k(s,y(s))-k(s,x(s))\|\mathrm{d}s.$$

According to assumptions (H_3), (H_4), (H_6) and Lemma 3.4, we obtain

$$\|\Gamma_1(y)(t)-\Gamma_1(x)(t)\|\leq(L_3+Ma_0\frac{\tau^{\alpha+1}}{\alpha+1}L_2)\sup_{t\in[0,\tau]}\left|\dot{S}\left(\frac{t^\alpha}{\alpha}\right)\right||y-x|_c.$$

Taking the supremum, we get

$$|\Gamma_1(y)-\Gamma_1(x)|_c\leq\left(L_3+Ma_0\frac{\tau^{\alpha+1}}{\alpha+1}L_2\right)\sup_{t\in[0,\tau]}\left|\dot{S}\left(\frac{t^\alpha}{\alpha}\right)\right||y-x|_c.$$

Since $(L_3+Ma_0\frac{\tau^{\alpha+1}}{\alpha+1}L_2)\sup_{t\in[0,\tau]}\left|\dot{S}\left(\frac{t^\alpha}{\alpha}\right)\right|<1$, the operator Γ_1 is a contraction operator on B_r.

Step 3: Prove that Γ_2 is continuous on B_r.

Let $(x_n)\subset B_r$ such that $x_n\longrightarrow x$ in B_r. We have

$$\Gamma_2(x_n)(t)-\Gamma_2(x)(t)$$
$$=\lim_{\lambda\to+\infty}\int_0^t s^{\alpha-1}\dot{S}\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)\lambda(\lambda-A)^{-1}[f(s,x_n(s))-f(s,x(s))]\mathrm{d}s.$$

By using a simple computation, we obtain

$$|\Gamma_2(x_n)-\Gamma_2(x)|_c\leq M\sup_{t\in[0,\tau]}\left|\dot{S}\left(\frac{t^\alpha}{\alpha}\right)\right|\int_0^\tau s^{\alpha-1}\|f(s,x_n(s))-f(s,x(s))\|\mathrm{d}s.$$

According to assumption (H_7), we get

$$\left\|s^{\alpha-1}[f(s,x_n(s))-f(s,x(s))]\right\|\leq 2\mu(s)s^{\alpha-1}$$

and

$$f(s,x_n(s))\longrightarrow f(s,x(s))\ \text{ as }\ n\longrightarrow+\infty.$$

Hence the Lebesgue dominated convergence theorem proves that

$$\lim_{n\to+\infty}|\Gamma_2(x_n)-\Gamma_2(x)|_c=0.$$

Step 4: Prove that Γ_2 is compact.

Claim 1: Prove that $\Gamma_2(B_r)$ is uniformly bounded.

For $x \in B_r$, we have

$$\|\Gamma_2(x)(t)\| \leq \sup_{t\in[0,\tau]} \left| \dot{S}\left(\frac{t^\alpha}{\alpha}\right) \right| |\mu|_{L^\infty([0,\tau],\mathbb{R}^+)} M \frac{\tau^\alpha}{\alpha}.$$

Then $\Gamma_2(B_r)$ is uniformly bounded.

Claim 2: Prove that the set $\{\Gamma_2(x)(t),\ x \in B_r\}$ is relatively compact in X.

To do so, for some fixed $t \in]0,\tau[$ and $x \in B_r$, we define the operator Γ_2^ε by

$$\Gamma_2^\varepsilon(x)(t) = \lim_{\lambda\to+\infty} \int_0^{(t^\alpha-\varepsilon^\alpha)^{\frac{1}{\alpha}}} s^{\alpha-1} \dot{S}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) \lambda(\lambda - A)^{-1} f(s, x(s))\mathrm{d}s,$$

where $\varepsilon \in]0,t[$. We can write Γ_2^ε as follows

$$\Gamma_2^\varepsilon(x)(t)$$
$$= \dot{S}\left(\frac{\varepsilon^\alpha}{\alpha}\right) \lim_{\lambda\to+\infty} \int_0^{(t^\alpha-\varepsilon^\alpha)^{\frac{1}{\alpha}}} s^{\alpha-1} \dot{S}\left(\frac{t^\alpha - s^\alpha - \varepsilon^\alpha}{\alpha}\right) \lambda(\lambda - A)^{-1} f(s, x(s))\mathrm{d}s.$$

Since the compactness of $(\dot{S}(t))_{t>0}$, the set $\{\Gamma_2^\varepsilon(x)(t),\ x \in B_r\}$ is relatively compact in X. By using a simple computation combined with assumption (H_7), we get

$$\|\Gamma_2^\varepsilon(x)(t) - \Gamma_2(x)(t)\| \leq M\, |\mu|_{L^\infty([0,\tau],\mathbb{R}^+)} \sup_{t\in[0,\tau]} \left| \dot{S}\left(\frac{t^\alpha}{\alpha}\right) \right| \frac{\varepsilon^\alpha}{\alpha}.$$

This last inequality proves that the set $\{\Gamma_2(x)(t),\ x \in B_r\}$ is relatively compact in X. For $t = 0$, the set $\{\Gamma_2(x)(0),\ x \in B_r\}$ is compact. Hence, the set $\{\Gamma_2(x)(t),\ x \in B_r\}$ is relatively compact in X for all $t \in [0,\tau]$.

Claim 3: We prove that $\Gamma_2(B_r)$ is equicontinuous.

For $t_1, t_2 \in [0,\tau]$ such that $t_1 < t_2$, we have

$$\Gamma_2(x)(t_2) - \Gamma_2(x)(t_1)$$
$$= \lim_{\lambda\to+\infty} \int_0^{t_1} s^{\alpha-1} \left[\dot{S}\left(\frac{t_2^\alpha - s^\alpha}{\alpha}\right) - \dot{S}\left(\frac{t_1^\alpha - s^\alpha}{\alpha}\right) \right] \lambda(\lambda - A)^{-1} f(s, x(s))\mathrm{d}s$$
$$+ \lim_{\lambda\to+\infty} \int_{t_1}^{t_2} s^{\alpha-1} \dot{S}\left(\frac{t_2^\alpha - s^\alpha}{\alpha}\right) \lambda(\lambda - A)^{-1} f(s, x(s))\mathrm{d}s$$
$$= \left[\dot{S}\left(\frac{t_2^\alpha - t_1^\alpha}{\alpha}\right) - I) \right] \lim_{\lambda\to+\infty} \int_0^{t_1} s^{\alpha-1} \dot{S}\left(\frac{t_1^\alpha - s^\alpha}{\alpha}\right) \lambda(\lambda - A)^{-1} f(s, x(s))\mathrm{d}s$$
$$+ \lim_{\lambda\to+\infty} \int_{t_1}^{t_2} s^{\alpha-1} \dot{S}\left(\frac{t_2^\alpha - s^\alpha}{\alpha}\right) \lambda(\lambda - A)^{-1} f(s, x(s))\mathrm{d}s.$$

By using a simple computation and assumption (H_7), we get

$$\|\Gamma_2(x)(t_2) - \Gamma_2(x)(t_1)\|$$

$$\leq \frac{M \sup\limits_{t\in[0,\tau]} \left|\dot{S}\left(\frac{t^\alpha}{\alpha}\right)\right| |\mu|_{L^\infty([0,\tau],\mathbb{R}^+)}}{\alpha} \left[(t_2^\alpha - t_1^\alpha) + \tau^\alpha \left|\dot{S}\left(\frac{t_2^\alpha - t_1^\alpha}{\alpha}\right) - I\right|\right].$$

This implies that $\Gamma_2(x)$, $x \in B_r$ are equicontinuous at $t \in [0,\tau]$. Hence, the Arzela-Ascoli theorem proves that the operator Γ_2 is compact. Finally, by using the Krasnoselskii fixed-point theorem, we conclude that the operator $\Gamma_1 + \Gamma_2$ has at least one fixed point in $\mathcal{C}$, which is an integral solution of equation (1). □

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