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WELL-POSEDNESS AND GENERAL ENERGY DECAY OF SOLUTIONS FOR A PETROVSKY EQUATION WITH A NONLINEAR STRONG DISSIPATION

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Abstract. In this paper we consider a nonlinear Petrovsky equation in a bounded domain with a strong dissipation

$$u'' + \Delta^2 u - g(\Delta u') = 0.$$

and prove the existence and the uniqueness of the solution using the energy method combined with the Faedo-Galerkin procedure under certain assumptions for g. Furthermore, we study the asymptotic behaviour of the solutions using a perturbed energy method.

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1. INTRODUCTION

In this article, we consider the initial-boundary value problem for the nonlinear Petrovsky equation

$$\begin{cases} u'' + \Delta^2 u - g(\Delta u') = 0, & x \in \Omega \times [0, +\infty[, \\ u(0,t) = \Delta u(0,t) = 0, & x \in \Gamma \times [0, \infty[, \\ u(x,0) = u_0(x), \,, \; u_t(x,0) = u_1(x) & x \in \Omega \times [0, +\infty[, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, Γ is a smooth boundary, (u_0, u_1) are the initial data in a suitable function space and g is real function satisfying some conditions to be specied later. In [6], Guesmia considered the following problem

$$\begin{cases} u'' + \Delta^2 u + q(x)u(x,t) + g(u'(x,t)) = 0 & x \in \Omega \times (0, +\infty) \\ u(0,t) = \partial_\nu u = 0 \text{ in} & x \in \Gamma \times (0, +\infty) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{2}$$

where g is continuous, increasing, satisfying $g(0) = 0$ and $q : \Omega \to \mathbb{R}_+$ is a bounded under suitable growth conditions on g, decay results for weak,

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as well as strong, solutions. Precisely, he showed that the solution decays exponentially if g behaves like a linear function, whereas the decay is of a polynomial order otherwise. Also the system composed of the equation (2), with $u'|u'|^{m-2} - u|u|^{p-2}$ in the place of $q(x)u(x,t) + g(u'(x,t))$ has been treated by Messaoudi [9], he established an existence result and showed that the solution continues to exist globally if $m \geq p$, however, it blows up in finite time if $m < p$. Moreover, Komornik [4] treated the problem (1) for g having a polynomial growth near the origin, used semigroup to prove the existence and uniqueness of solutions and established energy decay results depending on g.

In this paper, we prove the global existence of weak solutions of the problem (1) by using the Galerkin method (see Lions [12]) we use some technique from [10] to establish an explicit and general decay result, depending on g. The proof is based on the multiplier method and makes use of some properties of convex functions, the general Young inequality and Jensen's inequality. These convexity arguments were introduced and developed by Lasiecka and co-workers ([8],[11]) and used, with appropriate modifications, by Liu and Zuazua [13], Alabau-Boussouira [2] and others.

The paper is organized as follows. In section 2 we present some assumptions and technical lemmas. In section 3 we prove the existence and the uniqueness of a global solution. In section 4 we prove the energy estimates.

2. NOTATION AND PRELIMINARIES

We begin by introducing some notation that will be used throughout this work.

Let us introduce three real Hilbert spaces $\mathcal{H}$, V and W by setting

$$\mathcal{H} = H_0^1(\Omega), \quad \|v\|_{\mathcal{H}}^2 = \int_\Omega |\nabla v|^2 \mathrm{d}x$$

and

$$V = \{v \in H^3(\Omega) | v = \Delta v = 0 \text{ on } \Gamma\}, \quad \|v\|_V^2 = \int_\Omega |\nabla \Delta v|^2 \mathrm{d}x$$

$$W = \{v \in H^5(\Omega) | v = \Delta v = \Delta^2 v = 0 \text{ on } \Gamma\}, \quad \|v\|_W^2 = \int_\Omega |\nabla \Delta^2 v|^2 \, \mathrm{d}x.$$

Identifying H with its dual H' we have

$$W \subset V \subset \mathcal{H} \subset V' \subset W',$$

with dense and compact imbedings. If $v \in L^2(\Omega)$, we denote $\|v\|_{L^2(\Omega)}^2 = \|v\|^2$.

We impose the following assumptions on g: $g : \mathbb{R} \to \mathbb{R}$ is a non-decreasing continuous function such that there exist constants $\varepsilon, c_1, c_2, \tau > 0$ and a convex increasing function $G : \mathbb{R}_+ \to \mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+^*)$ satisfying G linear on $[0, \varepsilon]$ or ($G'(0) = 0$ and $G'' > 0$ on $]0, \varepsilon]$, such that

$$c_1 |s| \leqslant |g(s)| \leqslant c_2 |s|, \text{ if } |s| > \varepsilon, \tag{3}$$

$$|s|^2 + |g(s)|^2 \leqslant G^{-1}(sg(s)), \text{ if } |s| \leqslant \varepsilon, \tag{4}$$

$$|g'(s)| \leq \tau. \tag{5}$$

Lemma 2.1. *For all $u \in H_0^1(\Omega) \cap H^2(\Omega)$, we have*

$$\|\nabla u\| \leqslant c\, \|\Delta u\|_{H^{-1}(\Omega)} \leqslant c\|\Delta u\|, \tag{6}$$

where $H^{-1}(\Omega) = \left(H_0^1(\Omega)\right)^{'}$.

Remark 2.2. Let us denote by ϕ^* the conjugate function of the differentiable convex function ϕ, i.e.,

$$\phi^*(s) = \sup_{t\in\mathbb{R}_+} (st - \phi(t)).$$

Then ϕ^* is the Legendre transform of ϕ, which is given by (see Arnold [3, p. 61-62])

$$\phi^*(s) = s(\phi^{'})^{-1}(s) - \phi\left((\phi^{'})^{-1}(s)\right), \text{ if } s \in \left]0, \phi^{'}(r)\right],$$

and ϕ^* satisfies the generalized Young inequality

$$ST \leq \phi^*(S) + \phi(T), \text{ if } S \in \left]0, \phi^{'}(r)\right], T \in \left]0, r\right]. \tag{7}$$

3. WELL POSEDENESS AND REGULARITY

Theorem 3.1. *Assume that $(u_0, u_1) \in W \times V$. Then the solution of the problem (1) satisfies $u' \in L^\infty(0,T;V)$, $u'' \in L^\infty(0,T;\mathcal{H})$, $u \in L^\infty(0,T;H^4(\Omega)\cap V)$, such that for any $T > 0$*

$$u''(x,t) + \Delta^2 u(x,t) - g(\Delta u'(x,t)) = 0, \quad in \quad L^\infty(0,T;L^2(\Omega))$$
$$u(0) = u_0, \quad u'(0) = u_1, \quad in \quad \Omega.$$

Proof. **Step 1. Approximate solutions**

We will use the Faedo-Galerkin method to prove the existence of a global solution. Let $T > 0$ be fixed and let $\{w_j\}, j \in \mathbb{N}$ be a basis of $\mathcal{H}$, V and W, i.e. the space generated by $\mathcal{B}_k = \{w_1, w_2, \ldots, w_k\}$ is dense in $\mathcal{H}$, V and W. We construct approximate solutions u_k, $k = 1, 2, 3, \ldots$, in the form

$$u_k(t) = \sum_{j=1}^{k} c_{jk}(t) w_j(x),$$

where c_{jk} is determined by the ordinary differential equations.

For any v in $\mathcal{B}_k$, $u_k(t)$ satisfies the approximate equation

$$\int_\Omega (u_k''(t) + \Delta^2 u_k - g(\Delta u_k'))v \,\mathrm{d}x = 0, \tag{8}$$

with initial conditions

$$u_k(0) = u_k^0 = \sum_{j=1}^{k} \langle u_0, w_j\rangle w_j \to u_0, \quad \text{in } W \text{ as } k \to +\infty, \tag{9}$$

and

$$u_k'(0) = u_k^1 = \sum_{j=1}^{k} \langle u_1, w_j \rangle w_j \to u_1, \quad \text{in } V \text{ as } k \to +\infty. \tag{10}$$

The standard theory of ODE guarantees that the system (8)-(10) has an unique solution in $[0, t_k)$, with $0 < t_k < T$, by Zorn lemma since the nonlinear terms in (8) are locally Lipschitz continuous.

Note that $u_k(t)$ is of class $\mathcal{C}^2$.

In the next step, we obtain a priori estimates for the solution of the system (8)-(10), so that it can be extended outside $[0, t_k)$ to obtain one solution defined for all $T > 0$, using a standard compactness argument for the limiting procedure.

Step 2. The first estimate

Setting $v = -2\Delta u_k'$ in (8), we have

$$\frac{d}{\mathrm{d}t}\Big[\|\nabla u_k'\|^2 + \|\nabla \Delta u_k\|^2\Big] + 2\int_\Omega \Delta u_k' g(\Delta u_k')\,\mathrm{d}x = 0.$$

Integrating in $[0, t]$, $t < t_k$ and using (9) and (10), we obtain

$$\begin{aligned} &\|\nabla u_k'(t)\|^2 + \|\nabla \Delta u_k(t)\|^2 + 2\int_0^t \int_\Omega \Delta u_k'(s) g(\Delta u_k'(s))\,\mathrm{d}x\,\mathrm{d}s \\ &\qquad \le c\Big(\|\nabla u_k^1\|^2 + \|\nabla \Delta u_k^0\|^2\Big) \le C_1 \end{aligned} \tag{11}$$

and C_1 is a positive constant depending only on $\|u_1\|_V$ and $\|u_0\|_W$.

This estimate implies that the solution u_k exists globally in $[0, +\infty)$. Estimate (42) yields that

$$u_k \text{ is bounded in } L^\infty(0, T; V), \tag{12}$$

$$u_k' \text{ is bounded in } L^\infty(0, T; \mathcal{H}), \tag{13}$$

$$\Delta u_k' g(\Delta u_k') \text{ is bounded in } L^1(\Omega \times (0, T)). \tag{14}$$

From (3), (4) and (14), it follows that

$$g(\Delta u_k') \text{ is bounded in } L^2(\Omega \times (0, T)). \tag{15}$$

Step 3. The second estimate

First, we estimate $u_k''(0)$. Differentiating (8) with respect to x, taking $v = \nabla u_k''(t)$ and choosing $t = 0$, we obtain that

$$\|\nabla u_k''(0)\|^2 + \Big(\nabla u_k''(0), \nabla \Delta^2 u_k^0 - \nabla\Big(g(\Delta u_k^1)\Big)\Big) = 0.$$

Using Cauchy-Schwarz inequality and (5), we have

$$\begin{aligned}\|\nabla u_k''(0)\| &\le \|\nabla\Delta^2 u_k^0\| + \|\nabla\Delta u_k^1 g'(\Delta u_k^1)\| \\ &\le \|\nabla\Delta^2 u_k^0\| + \tau\|\nabla\Delta u_k^1\|.\end{aligned} \tag{16}$$

By (9) and (10) yields

$$u_k''(0) \quad \text{is bounded in} \quad \mathcal{H}. \tag{17}$$

Step 4. The third estimate

Differentiating (8) with respect to t get

$$\int_\Omega \Big(u_k'''(t) + \Delta^2 u_k'\Big) v \,\mathrm{d}x - \int_\Omega \Delta u_k'' g'(\Delta u_k') v \,\mathrm{d}x = 0.$$

Taking $v = 2\Delta u_k''$, applying the Green formula, we obtain

$$\frac{d}{\mathrm{d}t}\Big[\|\nabla u_k''\|^2 + \|\nabla\Delta u_k'\|^2\Big] + 2\int_\Omega (\Delta u_k'')^2 g'(\Delta u_k') \,\mathrm{d}x = 0.$$

By integrating it over $(0,t)$, we get

$$\begin{aligned}\|\nabla u_k''(t)\|^2 + \|\nabla\Delta u_k'(t)\|^2 + 2\int_0^t \int_\Omega (\Delta u_k''(s))^2 g'(\Delta u_k'(s) \,\mathrm{d}x \,\mathrm{d}s \\ = \|\nabla u_k''(0)\|^2 + \|\nabla\Delta u_k^1\|^2.\end{aligned} \tag{18}$$

By (9) and (17), we deduce that

$$u_k' \text{ is bounded in } L^\infty(0,T;V) \tag{19}$$

and

$$u_k'' \text{ is bounded in } L^\infty(0,T;\mathcal{H}) \tag{20}$$

By (19) we deduce that

$$u_k' \text{ is bounded in } L^2(0,T;V).$$

Applying Rellich compactenes theorem given in [12], we deduce that

$$u_k' \quad \text{is precompact in} \quad L^2(0,T;L^2(\Omega)). \tag{21}$$

Step 5. The fourth estimate

Setting $v = 2\Delta^2 u_k'$ in (8), we have

$$2\int_\Omega u_k''\Delta^2 u_k' \,\mathrm{d}x + \frac{d}{\mathrm{d}t}\Big[\|\Delta^2 u_k\|^2\Big] - 2\int_\Omega g(\Delta u_k').\Delta^2 u_k' \,\mathrm{d}x = 0.$$

Therefore by using the Green's formula, we have

$$\begin{aligned}\frac{d}{\mathrm{d}t}\Big[\|\Delta^2 u_k\|^2\Big] &= -2\int_\Omega \Delta u_k''\Delta u_k' \,\mathrm{d}x - 2\int_\Omega g'(\Delta u_k').(\nabla\Delta u_k')^2 \,\mathrm{d}x \\ &= -\frac{d}{\mathrm{d}t}\Big[\|\Delta u_k'\|^2\Big] - 2\int_\Omega g'(\Delta u_k').(\nabla\Delta u_k')^2 \,\mathrm{d}x.\end{aligned}$$

Integrating it over $(0,t)$ we arrive at

$$\|\Delta^2 u_k(t)\|^2 + \|\Delta u_k'(t)\|^2 + 2\int_0^t \int_\Omega g'(\Delta u_k').(\nabla \Delta u_k')^2 \,\mathrm{d}x\,\mathrm{d}s = \|\Delta^2 u_k^0\|^2 + \|\Delta u_k^1\|^2.$$

By using (5), (9) and (10), we deduce that

$$\|\Delta^2 u_k(t)\|^2 + \|\Delta u_k'(t)\|^2 + 2\tau \int_0^t \int_\Omega (\nabla \Delta u_k')^2 \,\mathrm{d}x\,\mathrm{d}s \le \|\Delta^2 u_0\|^2 + \|\Delta u_1\|^2,$$

then

(22) $$\Delta^2 u_k \text{ is bounded in } L^\infty(0,T;L^2(\Omega)).$$

Step 6. Passage to the limit.

Applying Dunford-Petit theorem we conclude from (12), (15), (19) and (20), replacing the sequence u_k, with a subsequence if needed, that

(23) $$u_k \rightharpoonup u, \text{ weak-star in } L^\infty(0,T;V\cap H^4(\Omega))$$

(24) $$u_k' \rightharpoonup u', \text{ weak-star in } L^\infty(0,T;V)$$

(25) $$u_k'' \rightharpoonup u'', \text{ weak-star in } L^\infty(0,T;\mathcal{H})$$

(26) $$u_k' \to u', \text{ almost everywhere in } \mathcal{A},$$

(27) $$g(\Delta u_k') \rightharpoonup \phi, \text{ weak-star in } L^2(\mathcal{A})$$

where $\mathcal{A} = \Omega \times [0,T]$. It follows at once from (23) and (25) that for each fixed $v \in L^2([0,T]\times L^2(\Omega))$

(28) $$\int_0^T \int_\Omega \Big(u_k''(x,t) + \Delta^2 u_k(x,t)\Big) v \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega \Big(u''(x,t) + \Delta^2 u(x,t)\Big) v \,\mathrm{d}x\,\mathrm{d}t.$$

As (u_k') is bounded in $L^\infty(0,T;V)$ and the injection of V in $\mathcal{H}$ is compact, we have

(29) $$u_k' \to u', \text{ strong in } L^2(\mathcal{A}).$$

It remains to show that

$$\int_0^T \int_\Omega g(\Delta u_k')\ v \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega g(\Delta u')\ v \,\mathrm{d}x\,\mathrm{d}t.$$

Lemma 3.2. *For each $T>0$, $g(\Delta u') \in L^1(\mathcal{A})$, $\|g(\Delta u')\|_{L^1(\mathcal{A})} \le K$, where K is a constant independent of t and $g(\Delta u_k') \to g(\Delta u')$ in $L^1(\mathcal{A})$.*

Proof. We claim that

$$g(\Delta u') \in L^1(\mathcal{A}).$$

Indeed, since g is continuous, we deduce from (26)

$$g(\Delta u'_k) \to g(\Delta u') \quad \text{almost everywhere in} \quad \mathcal{A}. \tag{30}$$

$$\Delta u'_k g(\Delta u'_k) \to \Delta u' g(\Delta u') \quad \text{almost everywhere in} \quad \mathcal{A}.$$

Hence, by (14) and Fatou's Lemma, we have

$$\int_0^T \int_\Omega \Delta u'(x,t) g(\Delta u'(x,t))\,\mathrm{d}x\,\mathrm{d}t \le K_1, \quad \text{for } T > 0 \tag{31}$$

Now, we can estimate $\int_0^T \int_\Omega |\Delta g(u'(x,t))|\,\mathrm{d}x\,\mathrm{d}t$. By Cauchy-Schwarz inequality, we have

$$\int_0^T \int_\Omega |\Delta g(u'(x,t))|\,\mathrm{d}x\,\mathrm{d}t \le c|\mathcal{A}|^{1/2}\Big(\int_0^T \int_\Omega |\Delta g(u'(x,t))|^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2}.$$

Using (3), (4) and (31), we obtain

$$\begin{aligned}
&\int_0^T \int_\Omega |\Delta g(u'(x,t))|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\le \int_0^T \int_{|\Delta u'|>\varepsilon} \Delta u' g(\Delta u')\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{|\Delta u'|\le\varepsilon} G^{-1}(\Delta u' g(\Delta u'))\,\mathrm{d}x\,\mathrm{d}t \\
&\le c\int_0^T \int_\Omega \Delta u' g(\Delta u')\,\mathrm{d}x\,\mathrm{d}t + cG^{-1}\Big(\int_{\mathcal{A}} \Delta u' g(\Delta u')\,\mathrm{d}x\,\mathrm{d}t\Big) \\
&\le c\int_0^T \int_\Omega \Delta u' g(\Delta u')\,\mathrm{d}x\,\mathrm{d}t + c'G^*(1) + c''\int_\Omega \Delta u' g(\Delta u')\,\mathrm{d}x\,\mathrm{d}t \\
&\le cK_1 + c'G^*(1), \ \text{for } \ T > 0.
\end{aligned}$$

Then $\int_0^T \int_{\mathcal{A}} |\Delta g(u'(x,t))|\,\mathrm{d}x\,d \le K$, for $T > 0$.

Let $E \subset \Omega \times [0,T]$ and set $E_1 = \Big\{(x,t) \in E : |g(\Delta u'_k(x,t))| \le \frac{1}{\sqrt{|E|}}\Big\}$, $E_2 = E \backslash E_1$, where $|E|$ is the measure of E. If $M(r) = \inf\{|s| : s \in \mathbb{R} \text{ and } |g(s)| \ge r\}$, then

$$\int_E |g(\Delta u'_k)|\,\mathrm{d}x\,\mathrm{d}t \le c\sqrt{|E|} + \Big(M\Big(\frac{1}{\sqrt{|E|}}\Big)\Big)^{-1} \int_{E_2} |\Delta u'_k g(\Delta u'_k)|\,\mathrm{d}x\,\mathrm{d}t.$$

By applying (14) we deduce that

$$\sup_k \int_E g(\Delta u'_k)\ \mathrm{d}x\ \mathrm{d}t \to 0, \ \text{when } |E| \to 0.$$

From Vitali's convergence theorem we deduce that

$$g(\Delta u'_k) \to g(\Delta u') \quad \text{in } L^1(\mathcal{A}).$$

This completes the proof. □

Then (27) implies that

$$g(\Delta u_k') \rightharpoonup g(\Delta u'), \text{ weak-star in } L^2([0,T]\times\Omega).$$

We deduce, for all $v \in L^2([0,T]\times L^2(\Omega)$, that

$$\int_0^T \int_\Omega g(\Delta u_k')v \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega g(\Delta u')v \,\mathrm{d}x\,\mathrm{d}t.$$

Finally we have shown that, for all $v \in L^2([0,T]\times L^2(\Omega))$:

$$\int_0^T \int_\Omega \Big(u''(x,t) + \Delta^2 u(x,t) - g(\Delta u')\Big)v \,\mathrm{d}x\,\mathrm{d}t = 0.$$

Therefore, u is a solution for problem (1).

Step 7. Proof of uniqueness

Let u_1, u_2 be two solutions of (1) with the same initial data. It is straightforward to see that $z = u_1 - u_2$ satisfies

$$\|\nabla z'\|^2 + \|\nabla\Delta z\|^2 + 2\int_0^t \int_\Omega \Delta z'(s) g(\Delta z'(s))\,\mathrm{d}x\,\mathrm{d}s = 0.$$

Using the monotonicity of g, we conclude that

$$\|\nabla z'\|^2 + \|\nabla\Delta z\|^2 \le 0,$$

which implies $z = 0$. This finishes the proof of Theorem 3.1. □

4. ASYMPTOTIC BEHAVIOR

Now we define the energy associated with the solution of the problem (1) by the following formula

$$E(t) = \frac{1}{2}\left\|\nabla u'\right\|^2 + \frac{1}{2}\left\|\nabla\Delta u\right\|^2.$$

LEMMA 4.1. *Let u be a solution to the problem* (1). *Then $E(t)$ is a nonincreasing function for all $t \in \mathbb{R}_+$.*

Proof. Multiplying the first equation in (1) by $-\Delta u'$ and integrating over Ω, we get

$$\frac{d}{\mathrm{d}t}\left(\frac{1}{2}\left\|\nabla u'\right\|^2 + \frac{1}{2}\left\|\nabla\Delta u\right\|^2\right) = -\int_\Omega \Delta u' g(\Delta u')\mathrm{d}x,$$

$$E^{'}(t) = -\int_\Omega \Delta u' g(\Delta u')\mathrm{d}x \le 0. \tag{32}$$

□

Lemma 4.2. *We define the following functional F by*

$$F(t) = ME(t) - \int_\Omega \Delta u u' \,\mathrm{d}x, \tag{33}$$

where $M > 0$ will be determined later. Then there exist positive constants λ_1, λ_2 such that

$$\lambda_1 E(t) \leq F(t) \leq \lambda_2 E(t), \quad \forall t \in \mathbb{R}_+. \tag{34}$$

Proof. Using the obvious estimates

$$\|u'\| \leq c_3 \|\nabla u'\| \tag{35}$$

and

$$\|\Delta u\| \leq c_4 \|\nabla \Delta u\|, \tag{36}$$

by Cauchy-Schwarz's inequality, we obtain

$$-\int_\Omega \Delta u.u' \mathrm{d}x \leq \frac{c_3^2}{2} \|\nabla \Delta u\|^2 + \frac{c_4^2}{2} \|\nabla u'\|^2 \leq \max(c_3^2, c_4^2) E(t),$$

hence

$$(M - \max(c_3^2, c_4^2))E(t) \leq F(t) \leq (M + \max(c_3^2, c_4^2))E(t),$$

choosing $M > \max(c_3^2, c_4^2)$, and we obtain (34), where $\lambda_1 = M - \max(c_3^2, c_4^2)$ and $\lambda_2 = M + \max(c_3^2, c_4^2)$. □

Lemma 4.3. *We define the following functional L by*

$$L(t) = F(t) + \lambda E(t), \tag{37}$$

where λ will be chosen later. Then there exist positive constants μ_1, μ_2 such that

$$\mu_1 E(t) \leq L(t) \leq \mu_2 E(t), \quad \forall t \in \mathbb{R}_+. \tag{38}$$

It is easy to see (38) holds, from Lemma 4.2 with $\mu_1 = \lambda_1 + \lambda$ and $\mu_2 = \lambda_2 + \lambda$.

Theorem 4.4. *Assume that (3) and (4) hold. Then there exist positive constants k_1, k_2, k_3 and ε_0 such that the solution of problem (1) satisfies*

$$E(t) \leq k_3 G_1^{-1}(k_1 t + k_2), \quad \forall\, t \in \mathbb{R}_+, \tag{39}$$

where

$$G_1(t) = \int_t^1 \frac{1}{s G_2'(\varepsilon_0 s)} \mathrm{d}s, \;\; G_2(t) = tG'(\varepsilon_0 t). \tag{40}$$

Here G_1 is strictly decreasing and convex on $]0,1]$, with $\lim_{t\to 0} G_1(t) = +\infty$.

Proof. Let $\varepsilon_1 \in (0, \varepsilon)$, we define two sets such that

$$\Omega_1 = \left\{x \in \Omega : |\Delta u'| \leq \varepsilon_1\right\}, \; \Omega_2 = \left\{x \in \Omega : |\Delta u'| > \varepsilon_1\right\}.$$

Differentiating (33) with respect to t, using (6), (32), and the first equation of the problem (1), we get

$$
\begin{aligned}
F^{'}(t) &= ME^{'}(t) + \int_{\Omega} |\nabla u_t|^2 \,\mathrm{d}x - \int_{\Omega} |\nabla \Delta u|^2 \,\mathrm{d}x - \int_{\Omega} \Delta u g(\Delta u')\,\mathrm{d}x \\
&\leq -2E(t) + \int_{\Omega} \Big[2\left|\nabla u'\right|^2 + \left|\Delta u g(\Delta u')\right| \Big] \mathrm{d}x \\
&\leq -2E(t) + 2\int_{\Omega} \Big[\left|\nabla u'\right|^2 + \left|\Delta u g(\Delta u')\right| \Big] \mathrm{d}x \\
&\leq -2E(t) + C\int_{\Omega} \Big[\left|\Delta u'\right|^2 + \left|\Delta u g(\Delta u')\right| \Big] \mathrm{d}x.
\end{aligned} \tag{41}
$$

Using Young's inequality, (36), we obtain

$$
\begin{aligned}
\int_{\Omega_1} \left|\Delta u g(\Delta u')\right| \,\mathrm{d}x &\leq c_4\delta \left\|\nabla \Delta u\right\|^2 + C_\delta \int_{\Omega_1} \left|g(\Delta u')\right|^2 \,\mathrm{d}x \\
&\leq c\delta E(t) + C_\delta \int_{\Omega_1} \left|g(\Delta u')\right|^2 \,\mathrm{d}x.
\end{aligned} \tag{42}
$$

By Hölder's inequality and (36), we obtain

$$
\begin{aligned}
\int_{\Omega_2} \left|\Delta u g(\Delta u')\right| \,\mathrm{d}x \;&\leq\; \Big(\int_{\Omega_2} |\Delta u|^2 \,\mathrm{d}x\Big)^{\frac{1}{2}} \Big(\int_{\Omega_2} \left|g(\Delta u')\right|^2 \,\mathrm{d}x\Big)^{\frac{1}{2}} \\
&\leq\; c_4 \left\|\nabla \Delta u\right\| \Big(\int_{\Omega_2} \left|g(\Delta u')\right|^2 \,\mathrm{d}x\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Then, we use Young's inequality and (3), for any $\delta > 0$ and we have

$$
\begin{aligned}
&\int_{\Omega_2} \Big[\left|\Delta u'\right|^2 + \left|\Delta u g(\Delta u')\right| \Big] \mathrm{d}x \\
&\leq \int_{\Omega_2} \left|\Delta u' g(\Delta u')\right| \,\mathrm{d}x + c_4 \left\|\nabla \Delta u\right\| \Big(\int_{\Omega_2} \left|\Delta u' g(\Delta u')\right| \,\mathrm{d}x\Big)^{\frac{1}{2}} \\
&\leq -cE'(t) + c_4 E^{\frac{1}{2}}(t)(-E'(t))^{\frac{1}{2}} \\
&\leq -cE'(t) + c_4\delta E(t) + C_\delta(-E'(t)) \\
&\leq c_4\delta E(t) - C_\delta E'(t).
\end{aligned} \tag{43}
$$

By (41), (42) and (44), for δ small enough, the function $L = F + C_\delta E$ satisfies

$$
\begin{aligned}
L'(t) &\leq (-2 + c\delta + c_4\delta)E(t) + C\int_{\Omega_1} \Big[|\Delta u_t|^2 \,\mathrm{d}x + C_\delta \int_{\Omega} \left|g(\Delta u')\right|^2 \Big] \mathrm{d}x \\
&\leq -dE(t) + c\int_{\Omega_1} \Big[\left|\Delta u'\right|^2 + \int_{\Omega} \left|g(\Delta u')\right|^2 \Big] \mathrm{d}x
\end{aligned} \tag{44}
$$

and $L(t) \sim E(t)$.

Case 1 : G is linear on $[0,\varepsilon]$, using (4), we deduce that

$$\begin{aligned} L^{'}(t) &\leq -dE(t) + c\int_{\Omega_1} G^{-1}(\Delta u' g(\Delta u'))\mathrm{d}x \\ &\leq -dE(t) + c\int_{\Omega_1} \Delta u' g(\Delta u')\,\mathrm{d}x \\ &\leq -dE(t) - cE'(t), \end{aligned} \tag{45}$$

we deduce that $(L(t)+cE(t))^{'} \leq -dE(t)$. Recalling that $L(t)+cE(t) \sim E(t)$, we obtain $E(t) \leq E(0)e^{-\mathrm{d}t}$. Thus, we have

$$E(t) \leq E(0)e^{-\mathrm{d}t} = E(0)G_1^{-1}(\mathrm{d}t).$$

Case 2 : G is nonlinear, we define the following functional I by

$$I(t) = \frac{1}{|\Omega_1|}\int_{\Omega_1} \Delta u' g(\Delta u')\mathrm{d}x.$$

From Jensen's inequality and the concavity of G^{-1}, we conclude that

$$G^{-1}(I(t)) \geq c\int_{\Omega_1} G^{-1}(\Delta u' g(\Delta u')\,\mathrm{d}x.$$

By using this inequality and (4), we obtain

$$\int_{\Omega_1} \left|\Delta u'\right|^2 + \left|g(\Delta u')\right|^2\,\mathrm{d}x \leq \int_{\Omega_1} G^{-1}(\Delta u' g(\Delta u'))\,\mathrm{d}x$$

which implies

$$\int_{\Omega_1}\left[|\Delta u_t|^2 + |g(\Delta u_t)|^2\right]\mathrm{d}x \leq cG^{-1}(I(t). \tag{46}$$

$$L'(t) \leq -dE(t) + cG^{-1}(I(t). \tag{47}$$

For $\varepsilon_0 < \varepsilon$ and $c_0 > 0$, we define H_1 by

$$H_1(t) = G^{'}\left(\varepsilon_0\frac{E(t)}{E(0)}\right)L(t) + c_0E(t). \tag{48}$$

Since $L(t)$ is equivalent to $E(t)$ (see (38)), there exist positive constants α_1, α_2, such that

$$\alpha_1 H_1(t) \leq E(t) \leq \alpha_2 H_1(t) \quad , \quad \forall t \in \mathbb{R}_+. \tag{49}$$

By recalling that $E^{'} \leq 0$, $G^{'} > 0$, and $G^{''} > 0$ on $]0,\varepsilon]$, making use of (32) and (47), we obtain

$$\begin{aligned} H_1^{'}(t) &= \varepsilon_0\frac{E'(t)}{E(0)}G''\left(\varepsilon_0\frac{E(t)}{E(0)}\right)L(t) + G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)L'(t) + c_0E^{'}(t) \\ &\leq -dE(t)G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + cG'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)G^{-1}(I(t)) + c_0E^{'}(t). \end{aligned} \tag{50}$$

Using (7) with $S = G^{'}\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)$ and $T = G^{-1}(I(t))$, using the Lemma 4.1, Remark 2.2 and (32), we deduce that

$$\begin{aligned} H_1^{'}(t) &\leq -dE(t)G^{'}\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + cG^{*}\left(G^{'}\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right) + cI(t) + c_0E^{'}(t) \\ &\leq -dE(t)G^{'}\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\varepsilon_0 E(0)\frac{E(t)}{E(0)}G^{'}\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) - cE'(t) + c_0E^{'}(t). \end{aligned}$$

Choosing $c_0 > c$ and ε_0 small enough, we obtain

$$H_1^{'}(t) \leq -k\frac{E(t)}{E(0)}G^{'}\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) = -kG_2\left(\varepsilon_0 \frac{E(t)}{E(0)}\right), \tag{51}$$

where $G_2(t) = tG'(\varepsilon_0 t)$. Since $G_2'(t) = G^{'}(\varepsilon_0 t) + \varepsilon_0 tG^{''}(\varepsilon_0 t)$ and G is convex on $]0,\varepsilon]$, we find that $G_2'(t) > 0$ and $G_2(t) > 0$ on $(0,1]$. By setting

$$H(t) = \frac{\alpha_1}{E(0)}H_1(t), \tag{52}$$

(α_1 is given in (49)) we easily see that, by (49), we have

$$H(t) \sim E(t). \tag{53}$$

Using (51), we arrive at $H^{'}(t) \leq -k_1 G_2(H(t))$. By recalling (40), we deduce $G_2(t) = -1/G_1'(t)$, hence $H^{'}(t) \leq k_1 \frac{1}{G_1'(H(t))}$, which gives $[G_1(H(t))]^{'} = H'(t)G_1'(H(t)) \leq k_1$, by integrating over $[0,t]$, we obtain

$$G_1(H(t)) \geq k_1 t + G_1(H(0)),$$

consequently

$$H(t) \leq G_1^{-1}\left(k_1 t + k_2\right)). \tag{54}$$

Combining (53) and (54), we obtain (39). The proof is complete.

□

Finally, we remark that, with appropriate choices of the defining parameters, our results agree with the existing results.

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