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SOME NEW INEQUALITIES FOR CONVEX FUNCTIONS VIA GENERALIZED INTEGRAL OPERATORS AND THEIR APPLICATIONS

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Abstract. The authors discover an identity for a generalized integral operator via differentiable function. By using this integral equation, we derive some new bounds on Hermite–Hadamard type integral inequality for differentiable mappings that are in absolute value at certain powers convex. Our results include several new and known results as particular cases. At the end, some applications of presented results for special means and error estimates for the mixed trapezium and midpoint formula have been analyzed. The ideas and techniques of this paper may stimulate further research in the field of integral inequalities.

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1. INTRODUCTION AND PRELIMINARIES

Convex functions and their generalizations have various applications in the fields of pure and applied sciences. Due to these applications, it is the most attractive area for researchers now a days. The class of convex functions is well known in the literature and is usually defined in the following way:

Definition 1.1. Let J be an interval in $\Re$. A function $f : J \to \Re$, is said to be convex on J, if the inequality

$$f(ta_1 + (1-t)a_2) \leq tf(a_1) + (1-t)f(a_2) \tag{1}$$

holds for all $a_1, a_2 \in J$ and $t \in [0,1]$. Also, we say that f is concave, if the inequality in (1) holds in the reverse direction.

The following inequality, named Hermite–Hadamard inequality (or H–H inequality), is one of the most famous inequalities in the literature for convex functions.

Theorem 1.2. *Let $f : J \subset \Re \to \Re$ be a convex function and $a_1, a_2 \in J$ with $a_1 < a_2$. Then the following inequality holds:*

$$f\left(\frac{a_1+a_2}{2}\right) \leq \frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(x)\mathrm{d}x \leq \frac{f(a_1)+f(a_2)}{2}. \tag{2}$$

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This inequality (2) *is also known as trapezium inequality.*

The trapezium inequality has remained an area of great interest due to its wide applications in the field of mathematical analysis. Authors of recent decades have studied (2) in the premises of newly invented definitions due to motivation of convex function. Interested readers see the references [1, 2, 3, 4, 6, 7, 9, 10, 12, 14, 16, 17].

The aim of this paper is to establish some new trapezium type generalized integral inequalities for convex functions. Interestingly, the special cases of presented results, are fractional integral inequalities, see also the references [4, 12]. Therefore, it is important to summarize the study of fractional integrals.

Let us recall some special functions and evoke some basic definitions as follows:

Definition 1.3. For $k \in \Re^+$ and $x \in \mathbb{C}$, the k–gamma function is defined by

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}. \tag{3}$$

Its integral representation is given by

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} \mathrm{e}^{-\frac{t^k}{k}} \mathrm{d}t. \tag{4}$$

One can note that

$$\Gamma_k(\alpha + k) = \alpha\Gamma_k(\alpha). \tag{5}$$

For $k = 1$, (4) gives integral representation of gamma function.

Definition 1.4. Let $f \in L[a_1, a_2]$. Then k–fractional integrals of order $\alpha, k > 0$ with $a_1 \geq 0$ are defined by

$$I^{\alpha,k}_{a_1^+} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_{a_1}^x (x-t)^{\frac{\alpha}{k}-1} f(t)\mathrm{d}t, \quad x > a_1$$

and

$$I^{\alpha,k}_{a_2^-} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^{a_2} (t-x)^{\frac{\alpha}{k}-1} f(t)\mathrm{d}t, \quad a_2 > x. \tag{6}$$

For $k = 1$, k–fractional integrals give Riemann–Liouville integrals. For $\alpha = k = 1$, k–fractional integrals give classical integrals.

Also, lets define a function $\phi : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

$$\int_0^1 \frac{\phi(t)}{t} \mathrm{d}t < +\infty, \tag{7}$$

$$\frac{1}{A} \leq \frac{\phi(s)}{\phi(r)} \leq \mathcal{A} \text{ for } \frac{1}{2} \leq \frac{s}{r} \leq 2 \tag{8}$$

$$\frac{\phi(r)}{r^2} \le \mathcal{B}\frac{\phi(s)}{s^2} \quad \text{for} \quad s \le r \tag{9}$$

$$\left|\frac{\phi(r)}{r^2} - \frac{\phi(s)}{s^2}\right| \le \mathcal{C}|r-s|\frac{\phi(r)}{r^2} \quad \text{for} \quad \frac{1}{2} \le \frac{s}{r} \le 2, \tag{10}$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C} > 0$ are independent of $r, s > 0$. If $\phi(r)r^{\alpha}$ is increasing for some $\alpha \ge 0$ and $\frac{\phi(r)}{r^{\beta}}$ is decreasing for some $\beta \ge 0$, then ϕ satisfies (7)-(10), see [11]. Therefore, the left–sided and right–sided generalized integral operators are defined as follows:

$${}_{a_1^+}\mathcal{I}_{\phi}f(x) = \int_{a_1}^{x} \frac{\phi(x-t)}{x-t} f(t)\mathrm{d}t, \quad x > a_1, \tag{11}$$

$${}_{a_2^-}\mathcal{I}_{\phi}f(x) = \int_{x}^{a_2} \frac{\phi(t-x)}{t-x} f(t)\mathrm{d}t, \quad x < a_2. \tag{12}$$

The most important feature of generalized integrals is that; they produce Riemann–Liouville fractional integrals, k–Riemann–Liouville fractional integrals, Katugampola fractional integrals, conformable fractional integrals, Hadamard fractional integrals, etc.

Motivated by the above literature, the main objective of this paper is to discover in Section 2 an identity for a generalized integral operator via differentiable function. By using the established identity as an auxiliary result, some new estimates on Hermite–Hadamard type integral inequality for differentiable mappings that are in absolute value at certain powers convex are obtained. Moreover, our results include several new and known results as particular cases. In Section 3, some applications of presented results for special means and error estimates for the mixed trapezium and midpoint formula are given. In Section 4, a brief conclusion is given as well.

2. MAIN RESULTS

Throughout this study, let $P = [ma_1, a_2]$, $P^{\circ} = (ma_1, a_2)$ with $a_1 < a_2$, $L(P)$ is the set of all integrable functions on P and λ, $m \in (0,1]$. Finally, for all $t \in [0,1]$, we define

$$\Lambda_m(t) = \int_0^t \frac{\phi\left(\left(\lambda\left(\frac{ma_1+a_2}{2}\right) - ma_1\right)\frac{u}{\lambda}\right)}{u}\mathrm{d}u < +\infty \tag{13}$$

and

$$\Delta_m(t) = \int_0^t \frac{\phi\left(\left(a_2 - \lambda\left(\frac{ma_1+a_2}{2}\right)\right)\frac{u}{\lambda}\right)}{u}\mathrm{d}u < +\infty. \tag{14}$$

For establishing some new results regarding general fractional integrals we need to prove Lemma 2.1.

LEMMA 2.1. *Let $f : P \to \Re$ be a differentiable mapping on P°. If $f' \in L(P)$ and α, $\beta \in \Re$ then the following identity for generalized fractional integrals holds:*

$$\begin{aligned}
&\frac{\alpha f(ma_1)+\beta f(a_2)}{2}+\frac{\lambda}{2}\left[\frac{\Lambda_m(\lambda)}{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}+\frac{\Delta_m(\lambda)}{a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)}\right.\\
&\left.-\frac{\alpha+\beta}{\lambda}\right]f\left(\lambda\left(\frac{ma_1+a_2}{2}\right)\right)\\
&-\frac{\lambda}{2}\left[\frac{1}{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}\times\ {}_{\left(\lambda\left(\frac{ma_1+a_2}{2}\right)\right)^-}\mathcal{I}_\phi f(ma_1)\right.\\
&\left.+\frac{1}{a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)}\times\ {}_{\left(\lambda\left(\frac{ma_1+a_2}{2}\right)\right)^+}\mathcal{I}_\phi f(a_2)\right]\\
&=\left(\frac{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}{2}\right)\\
&\times\int_0^1\left[\frac{\lambda\Lambda_m((1-t)\lambda)}{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}-\alpha\right]f'\left(ma_1t+\lambda(1-t)\left(\frac{ma_1+a_2}{2}\right)\right)\mathrm{d}t\\
&+\left(\frac{a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)}{2}\right)\\
&\times\int_0^1\left[\beta-\frac{\lambda\Delta_m(\lambda t)}{a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)}\right]f'\left(\lambda\left(\frac{ma_1+a_2}{2}\right)t+(1-t)a_2\right)\mathrm{d}t.
\end{aligned}\tag{15}$$

We denote

$$\begin{aligned}
&T_{f,\Lambda_m,\Delta_m}(\lambda,\alpha,\beta,a_1,a_2)\\
&=\left(\frac{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}{2}\right)\\
&\times\int_0^1\left[\frac{\lambda\Lambda_m((1-t)\lambda)}{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}-\alpha\right]f'\left(ma_1t+\lambda(1-t)\left(\frac{ma_1+a_2}{2}\right)\right)\mathrm{d}t\\
&+\left(\frac{a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)}{2}\right)\\
&\int_0^1\left[\beta-\frac{\lambda\Delta_m(\lambda t)}{a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)}\right]f'\left(\lambda\left(\frac{ma_1+a_2}{2}\right)t+(1-t)a_2\right)\mathrm{d}t.
\end{aligned}\tag{16}$$

Proof. Integrating by parts (16) and changing the variables of integration, we have

$$T_{f,\Lambda_m,\Delta_m}(\lambda,\alpha,\beta,a_1,a_2) = \left(\frac{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}{2}\right)\left\{\left(\frac{\lambda}{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}\right)\right.$$
$$\times\int_0^1 \Lambda_m((1-t)\lambda) f'\left(ma_1t+\lambda(1-t)\left(\frac{ma_1+a_2}{2}\right)\right)\mathrm{d}t$$
$$\left.-\alpha\int_0^1 f'\left(ma_1t+\lambda(1-t)\left(\frac{ma_1+a_2}{2}\right)\right)\mathrm{d}t\right\}$$
$$+\left(\frac{a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)}{2}\right)\times\left\{\beta\int_0^1 f'\left(\lambda\left(\frac{ma_1+a_2}{2}\right)t+(1-t)a_2\right)\mathrm{d}t\right.$$
$$\left.-\left(\frac{\lambda}{a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)}\right)\int_0^1 \Delta_m(\lambda t) f'\left(\lambda\left(\frac{ma_1+a_2}{2}\right)t+(1-t)a_2\right)\mathrm{d}t\right\}$$
$$=\left(\frac{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}{2}\right)$$
$$\times\left\{\left(\frac{\lambda}{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}\right)\left[\Lambda_m((1-t)\lambda)\frac{f\left(ma_1t+\lambda(1-t)\left(\frac{ma_1+a_2}{2}\right)\right)}{ma_1-\lambda\left(\frac{ma_1+a_2}{2}\right)}\right|_0^1\right.$$
$$-\frac{1}{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}$$
$$\left.\times\int_0^1\frac{\phi\left(\left(\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1\right)(1-t)\right)}{1-t}f\left(ma_1t+\lambda(1-t)\left(\frac{ma_1+a_2}{2}\right)\right)\mathrm{d}t\right]$$
$$\left.-\frac{\alpha}{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}\left[f\left(\lambda\left(\frac{ma_1+a_2}{2}\right)\right)-f(ma_1)\right]\right\}$$
$$+\left(\frac{a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)}{2}\right)$$
$$\times\left\{\left(\frac{\lambda}{a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)}\right)\left[-\Delta_m(\lambda t)\frac{f\left(\lambda\left(\frac{ma_1+a_2}{2}\right)t+(1-t)a_2\right)}{\lambda\left(\frac{ma_1+a_2}{2}\right)-a_2}\right|_0^1\right.$$
$$+\frac{1}{\lambda\left(\frac{ma_1+a_2}{2}\right)-a_2}\int_0^1\frac{\phi\left(\left(a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)\right)t\right)}{t}$$
$$\left.\times f\left(\lambda\left(\frac{ma_1+a_2}{2}\right)t+(1-t)a_2\right)\mathrm{d}t\right]$$
$$\left.+\frac{\beta}{a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)}\left[f(a_2)-f\left(\lambda\left(\frac{ma_1+a_2}{2}\right)\right)\right]\right\}$$

$$= \frac{\alpha f(ma_1) + \beta f(a_2)}{2} + \frac{\lambda}{2}\left[\frac{\Lambda_m(\lambda)}{\lambda\left(\frac{ma_1+a_2}{2}\right) - ma_1} + \frac{\Delta_m(\lambda)}{a_2 - \lambda\left(\frac{ma_1+a_2}{2}\right)} - \frac{\alpha+\beta}{\lambda}\right]$$

$$\times f\left(\lambda\left(\frac{ma_1+a_2}{2}\right)\right)$$

$$- \frac{\lambda}{2}\left[\frac{1}{\lambda\left(\frac{ma_1+a_2}{2}\right) - ma_1} \times {}_{\left(\lambda\left(\frac{ma_1+a_2}{2}\right)\right)^-}\mathcal{I}_\phi f(ma_1)\right.$$

$$\left. + \frac{1}{a_2 - \lambda\left(\frac{ma_1+a_2}{2}\right)} \times {}_{\left(\lambda\left(\frac{ma_1+a_2}{2}\right)\right)^+}\mathcal{I}_\phi f(a_2)\right].$$

The proof of Lemma 2.1 is completed. □

REMARK 2.2. Taking $\lambda = m = 1$ and $\phi(t) = t$ in Lemma 2.1, we get [16, Lemma 2.1].

THEOREM 2.3. *Let $f : P \to \Re$ be a differentiable mapping on P° and α, $\beta \in [0,1]$. If $|f'|^q$ is convex on P for $q > 1$ and $p^{-1} + q^{-1} = 1$, then the following inequality for generalized fractional integrals holds:*

$$\begin{aligned}
&\left|T_{f,\Lambda_m,\Delta_m}(\lambda,\alpha,\beta,a_1,a_2)\right| \\
&\leq \left(\frac{\lambda\left(\frac{ma_1+a_2}{2}\right) - ma_1}{2\sqrt[q]{2}}\right)\sqrt[p]{B_{\Lambda_m}(\alpha,\lambda;p)} \\
&\times \sqrt[q]{\left|f'(ma_1)\right|^q + \left|f'\left(\lambda\left(\frac{ma_1+a_2}{2}\right)\right)\right|^q} \\
&+ \left(\frac{a_2 - \lambda\left(\frac{ma_1+a_2}{2}\right)}{2\sqrt[q]{2}}\right)\sqrt[p]{C_{\Delta_m}(\beta,\lambda;p)} \\
&\times \sqrt[q]{\left|f'\left(\lambda\left(\frac{ma_1+a_2}{2}\right)\right)\right|^q + \left|f'(a_2)\right|^q},
\end{aligned} \tag{17}$$

where

$$B_{\Lambda_m}(\alpha,\lambda;p) = \int_0^1 \left|\frac{\lambda\Lambda_m((1-t)\lambda)}{\lambda\left(\frac{ma_1+a_2}{2}\right) - ma_1} - \alpha\right|^p \mathrm{d}t \tag{18}$$

and

$$C_{\Delta_m}(\beta,\lambda;p) = \int_0^1 \left|\beta - \frac{\lambda\Delta_m(\lambda t)}{a_2 - \lambda\left(\frac{ma_1+a_2}{2}\right)}\right|^p \mathrm{d}t. \tag{19}$$

Proof. From Lemma 2.1, convexity of $|f'|^q$, Hölder's inequality and properties of the modulus, we have

$$\begin{aligned}
&\left|T_{f,\Lambda_m,\Delta_m}(\lambda,\alpha,\beta,a_1,a_2)\right| \le \left(\frac{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}{2}\right)\\
&\times\int_0^1\left|\frac{\lambda\Lambda_m((1-t)\lambda)}{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}-\alpha\right|\left|f'\left(ma_1t+\lambda(1-t)\left(\frac{ma_1+a_2}{2}\right)\right)\right|\mathrm{d}t\\
&+\left(\frac{a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)}{2}\right)\int_0^1\left|\beta-\frac{\lambda\Delta_m(\lambda t)}{a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)}\right|\\
&\times\left|f'\left(\lambda\left(\frac{ma_1+a_2}{2}\right)t+(1-t)a_2\right)\right|\mathrm{d}t\\
&\le\left(\frac{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}{2}\right)\left(\int_0^1\left|\frac{\lambda\Lambda_m((1-t)\lambda)}{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}-\alpha\right|^p\mathrm{d}t\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1\left|f'\left(ma_1t+\lambda(1-t)\left(\frac{ma_1+a_2}{2}\right)\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\\
&+\left(\frac{a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)}{2}\right)\left(\int_0^1\left|\beta-\frac{\lambda\Delta_m(\lambda t)}{a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)}\right|^p\mathrm{d}t\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1\left|f'\left(\lambda\left(\frac{ma_1+a_2}{2}\right)t+(1-t)a_2\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\\
&\le\left(\frac{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}{2\sqrt[q]{2}}\right)\sqrt[p]{B_{\Lambda_m}(\alpha,\lambda;p)}\\
&\times\left[\int_0^1\left(t\left|f'(ma_1)\right|^q+(1-t)\left|f'\left(\lambda\left(\frac{ma_1+a_2}{2}\right)\right)\right|^q\right)\mathrm{d}t\right]^{\frac{1}{q}}\\
&+\left(\frac{a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)}{2\sqrt[q]{2}}\right)\sqrt[p]{C_{\Delta_m}(\beta,\lambda;p)}\\
&\times\left[\int_0^1\left(t\left|f'\left(\lambda\left(\frac{ma_1+a_2}{2}\right)\right)\right|^q+(1-t)\left|f'(a_2)\right|^q\right)\mathrm{d}t\right]^{\frac{1}{q}}\\
&=\left(\frac{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}{2\sqrt[q]{2}}\right)\sqrt[p]{B_{\Lambda_m}(\alpha,\lambda;p)}\\
&\times\sqrt[q]{\left|f'(ma_1)\right|^q+\left|f'\left(\lambda\left(\frac{ma_1+a_2}{2}\right)\right)\right|^q}
\end{aligned}$$

$$+\left(\frac{a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)}{2\sqrt[q]{2}}\right)\sqrt[p]{C_{\Delta_m}(\beta,\lambda;p)}$$
$$\times\sqrt[q]{\left|f'\left(\lambda\left(\frac{ma_1+a_2}{2}\right)\right)\right|^q+|f'(a_2)|^q}.$$

The proof of Theorem 2.3 is completed. □

We point out some special cases of Theorem 2.3.

COROLLARY 2.4. *Taking $\lambda=m=1$ and $\phi(t)=t$ in* Theorem 2.3, *we get*

$$\left|\frac{\alpha f(a_1)+\beta f(a_2)}{2}+\frac{2-\alpha-\beta}{2}f\left(\frac{a_1+a_2}{2}\right)-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}f(t)\mathrm{d}t\right|$$
$$\le\left(\frac{a_2-a_1}{4\sqrt[q]{2}\sqrt[p]{p+1}}\right)$$
$$\times\left\{\sqrt[p]{\alpha^{p+1}+(1-\alpha)^{p+1}}\times\sqrt[q]{|f'(a_1)|^q+\left|f'\left(\frac{a_1+a_2}{2}\right)\right|^q}\right. \tag{20}$$
$$\left.+\sqrt[p]{\beta^{p+1}+(1-\beta)^{p+1}}\times\sqrt[q]{\left|f'\left(\frac{a_1+a_2}{2}\right)\right|^q+|f'(a_2)|^q}\right\}.$$

COROLLARY 2.5. *Taking $\alpha=\beta=1$ in* Corollary 2.4, *we have*

$$\left|\frac{f(a_1)+f(a_2)}{2}-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}f(t)\mathrm{d}t\right|\le\left(\frac{a_2-a_1}{4\sqrt[q]{2}\sqrt[p]{p+1}}\right)$$
$$\times\left\{\sqrt[q]{|f'(a_1)|^q+\left|f'\left(\frac{a_1+a_2}{2}\right)\right|^q}\right. \tag{21}$$
$$\left.+\sqrt[q]{\left|f'\left(\frac{a_1+a_2}{2}\right)\right|^q+|f'(a_2)|^q}\right\}.$$

COROLLARY 2.6. *Taking $\alpha=\beta=0$ in* Corollary 2.4, *we obtain*

$$\left|f\left(\frac{a_1+a_2}{2}\right)-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}f(t)\mathrm{d}t\right|\le\left(\frac{a_2-a_1}{4\sqrt[q]{2}\sqrt[p]{p+1}}\right)$$
$$\times\left\{\sqrt[q]{|f'(a_1)|^q+\left|f'\left(\frac{a_1+a_2}{2}\right)\right|^q}\right. \tag{22}$$
$$\left.+\sqrt[q]{\left|f'\left(\frac{a_1+a_2}{2}\right)\right|^q+|f'(a_2)|^q}\right\}.$$

Corollary 2.7. *Taking* $p = q = 2$ *in* Theorem 2.3*, we get*

$$\begin{aligned}
\left|T_{f,\Lambda_m,\Delta_m}(\lambda,\alpha,\beta,a_1,a_2)\right| &\leq \left(\frac{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}{2\sqrt{2}}\right)\sqrt{B_{\Lambda_m}(\alpha,\lambda;2)}\\
&\times\sqrt{\left|f'(ma_1)\right|^2+\left|f'\left(\lambda\left(\frac{ma_1+a_2}{2}\right)\right)\right|^2}\\
&+\left(\frac{a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)}{2\sqrt{2}}\right)\sqrt{C_{\Delta_m}(\beta,\lambda;2)}\\
&\times\sqrt{\left|f'\left(\lambda\left(\frac{ma_1+a_2}{2}\right)\right)\right|^2+\left|f'(a_2)\right|^2}.
\end{aligned}\tag{23}$$

Corollary 2.8. *Taking* $|f'| \leq K$ *in* Theorem 2.3*, we get*

$$\begin{aligned}
&\left|T_{f,\Lambda_m,\Delta_m}(\lambda,\alpha,\beta,a_1,a_2)\right|\\
&\leq \frac{K}{2}\left\{\left(\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1\right)\sqrt[p]{B_{\Lambda_m}(\alpha,\lambda;p)}\right.\\
&\left.+\left(a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)\right)\sqrt[p]{C_{\Delta_m}(\beta,\lambda;p)}\right\}.
\end{aligned}\tag{24}$$

Theorem 2.9. *Let* $f : P \to \Re$ *be a differentiable mapping on* P° *and* $\alpha,\beta \in [0,1]$*. If* $|f'|^q$ *is convex on* P *for* $q \geq 1$*, then the following inequality for generalized fractional integrals holds:*

$$\begin{aligned}
\left|T_{f,\Lambda_m,\Delta_m}(\lambda,\alpha,\beta,a_1,a_2)\right| &\leq \left(\frac{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}{2}\right)\left[B_{\Lambda_m}(\alpha,\lambda;1)\right]^{1-\frac{1}{q}}\\
&\times\sqrt[q]{D_{\Lambda_m}(\alpha,\lambda)\left|f'(ma_1)\right|^q+E_{\Lambda_m}(\alpha,\lambda)\left|f'\left(\lambda\left(\frac{ma_1+a_2}{2}\right)\right)\right|^q}\\
&+\left(\frac{a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)}{2}\right)\left[C_{\Delta_m}(\beta,\lambda;1)\right]^{1-\frac{1}{q}}\\
&\times\sqrt[q]{F_{\Delta_m}(\beta,\lambda)\left|f'\left(\lambda\left(\frac{ma_1+a_2}{2}\right)\right)\right|^q+G_{\Delta_m}(\beta,\lambda)\left|f'(a_2)\right|^q},
\end{aligned}\tag{25}$$

where

$$D_{\Lambda_m}(\alpha,\lambda)=\int_0^1 t\left|\frac{\lambda\Lambda_m((1-t)\lambda)}{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}-\alpha\right|\mathrm{d}t,\tag{26}$$

$$E_{\Lambda_m}(\alpha,\lambda)=\int_0^1 t\left|\frac{\lambda\Lambda_m(\lambda t)}{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}-\alpha\right|\mathrm{d}t, \tag{27}$$

$$F_{\Delta_m}(\beta,\lambda)=\int_0^1 t\left|\beta-\frac{\lambda\Delta_m(\lambda t)}{a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)}\right|\mathrm{d}t, \tag{28}$$

$$G_{\Delta_m}(\beta,\lambda)=\int_0^1 (1-t)\left|\beta-\frac{\lambda\Delta_m(\lambda t)}{a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)}\right|\mathrm{d}t \tag{29}$$

and $B_{\Lambda_m}(\alpha,\lambda;1)$, $C_{\Delta_m}(\beta,\lambda;1)$ *are defined as in Theorem* 2.3.

Proof. From Lemma 2.1, convexity of $|f'|^q$, the well–known power mean inequality and properties of the modulus, we have

$$\begin{aligned}
&\left|T_{f,\Lambda_m,\Delta_m}(\lambda,\alpha,\beta,a_1,a_2)\right|\le\left(\frac{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}{2}\right)\\
&\times\int_0^1\left|\frac{\lambda\Lambda_m((1-t)\lambda)}{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}-\alpha\right|\left|f'\left(ma_1t+\lambda(1-t)\left(\frac{ma_1+a_2}{2}\right)\right)\right|\mathrm{d}t\\
&+\left(\frac{a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)}{2}\right)\int_0^1\left|\beta-\frac{\lambda\Delta_m(\lambda t)}{a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)}\right|\\
&\times\left|f'\left(\lambda\left(\frac{ma_1+a_2}{2}\right)t+(1-t)a_2\right)\right|\mathrm{d}t\\
&\le\left(\frac{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}{2}\right)\left(\int_0^1\left|\frac{\lambda\Lambda_m((1-t)\lambda)}{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}-\alpha\right|\mathrm{d}t\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\left|\frac{\lambda\Lambda_m((1-t)\lambda)}{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}-\alpha\right|\left|f'\left(ma_1t+\lambda(1-t)\left(\frac{ma_1+a_2}{2}\right)\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\\
&+\left(\frac{a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)}{2}\right)\left(\int_0^1\left|\beta-\frac{\lambda\Delta_m(\lambda t)}{a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)}\right|\mathrm{d}t\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\left|\beta-\frac{\lambda\Delta_m(\lambda t)}{a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)}\right|\left|f'\left(\lambda\left(\frac{ma_1+a_2}{2}\right)t+(1-t)a_2\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\\
&\le\left(\frac{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}{2}\right)\left[B_{\Lambda_m}(\alpha,\lambda;1)\right]^{1-\frac{1}{q}}\\
&\times\Bigg[\int_0^1\left|\frac{\lambda\Lambda_m((1-t)\lambda)}{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}-\alpha\right|\Bigg(t\left|f'(ma_1)\right|^q+(1-t)\Bigg|
\end{aligned}$$

$$f'\left(\lambda\left(\frac{ma_1+a_2}{2}\right)\right)\Big|^q\Bigg)\mathrm{d}t\Bigg]^{\frac{1}{q}} + \left(\frac{a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)}{2\sqrt[q]{2}}\right)\Big[C_{\Delta_m}(\beta,\lambda;1)\Big]^{1-\frac{1}{q}}$$

$$\times\left[\int_0^1\left|\beta-\frac{\lambda\Delta_m(\lambda t)}{a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)}\right|\left(t\left|f'\left(\lambda\left(\frac{ma_1+a_2}{2}\right)\right)\right|^q\right.\right.$$

$$\left.\left.+(1-t)\big|f'(a_2)\big|^q\right)\mathrm{d}t\right]^{\frac{1}{q}} = \left(\frac{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}{2}\right)\Big[B_{\Lambda_m}(\alpha,\lambda;1)\Big]^{1-\frac{1}{q}}$$

$$\times\sqrt[q]{D_{\Lambda_m}(\alpha,\lambda)\big|f'(ma_1)\big|^q+E_{\Lambda_m}(\alpha,\lambda)\left|f'\left(\lambda\left(\frac{ma_1+a_2}{2}\right)\right)\right|^q}$$

$$+\left(\frac{a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)}{2}\right)\Big[C_{\Delta_m}(\beta,\lambda;1)\Big]^{1-\frac{1}{q}}$$

$$\times\sqrt[q]{F_{\Delta_m}(\beta,\lambda)\left|f'\left(\lambda\left(\frac{ma_1+a_2}{2}\right)\right)\right|^q+G_{\Delta_m}(\beta,\lambda)\big|f'(a_2)\big|^q}.$$

The proof of Theorem 2.9 is completed. □

We point out some special cases of Theorem 2.9.

Corollary 2.10. *Taking* $\lambda = m = 1$ *and* $\phi(t) = t$ *in* Theorem 2.9, *we get* [16, Theorem 3.1].

Corollary 2.11. *Taking* $q = 1$ *in* Theorem 2.9, *we get*

$$\begin{aligned}
&\big|T_{f,\Lambda_m,\Delta_m}(\lambda,\alpha,\beta,a_1,a_2)\big| \le \left(\frac{\lambda\left(\frac{ma_1+a_2}{2}\right)-ma_1}{2}\right)\\
&\times\left[D_{\Lambda_m}(\alpha,\lambda)\big|f'(ma_1)\big|+E_{\Lambda_m}(\alpha,\lambda)\left|f'\left(\lambda\left(\frac{ma_1+a_2}{2}\right)\right)\right|\right]\\
&+\left(\frac{a_2-\lambda\left(\frac{ma_1+a_2}{2}\right)}{2}\right)\left[F_{\Delta_m}(\beta,\lambda)\left|f'\left(\lambda\left(\frac{ma_1+a_2}{2}\right)\right)\right|\right.\\
&\left.+G_{\Delta_m}(\beta,\lambda)\big|f'(a_2)\big|\right].
\end{aligned} \tag{30}$$

Corollary 2.12. *Taking* $\alpha = \beta$ *in* Theorem 2.9, *we have*

$$
\begin{aligned}
&\left|T_{f,\Lambda_m,\Delta_m}(\lambda,\alpha,\alpha,a_1,a_2)\right| \leq \left(\frac{\lambda\left(\frac{ma_1+a_2}{2}\right) - ma_1}{2}\right)\left[B_{\Lambda_m}(\alpha,\lambda;1)\right]^{1-\frac{1}{q}} \\
&\times \sqrt[q]{D_{\Lambda_m}(\alpha,\lambda)\left|f'(ma_1)\right|^q + E_{\Lambda_m}(\alpha,\lambda)\left|f'\left(\lambda\left(\frac{ma_1+a_2}{2}\right)\right)\right|^q} \\
&+ \left(\frac{a_2 - \lambda\left(\frac{ma_1+a_2}{2}\right)}{2}\right)\left[C_{\Delta_m}(\alpha,\lambda;1)\right]^{1-\frac{1}{q}} \\
&\times \sqrt[q]{F_{\Delta_m}(\alpha,\lambda)\left|f'\left(\lambda\left(\frac{ma_1+a_2}{2}\right)\right)\right|^q + G_{\Delta_m}(\alpha,\lambda)\left|f'(a_2)\right|^q}.
\end{aligned}
\tag{31}
$$

Corollary 2.13. *Taking* $|f'| \leq K$ *in* Theorem 2.9, *we obtain*

$$
\begin{aligned}
&\left|T_{f,\Lambda_m,\Delta_m}(\lambda,\alpha,\beta,a_1,a_2)\right| \leq \frac{K}{2} \\
&\times \left\{\left(\lambda\left(\frac{ma_1+a_2}{2}\right) - ma_1\right)\left[B_{\Lambda_m}(\alpha,\lambda;1)\right]^{1-\frac{1}{q}}\right. \\
&\times \sqrt[q]{D_{\Lambda_m}(\alpha,\lambda) + E_{\Lambda_m}(\alpha,\lambda)} \\
&+ \left(a_2 - \lambda\left(\frac{ma_1+a_2}{2}\right)\right)\left[C_{\Delta_m}(\beta,\lambda;1)\right]^{1-\frac{1}{q}} \\
&\left.\times \sqrt[q]{F_{\Delta_m}(\beta,\lambda) + G_{\Delta_m}(\beta,\lambda)}\right\}.
\end{aligned}
\tag{32}
$$

Remark 2.14. Applying our Theorems 2.3 and 2.9 for special parameter values $\lambda = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1;\ \alpha, \beta = 0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1$ and various suitable choices of function $\phi(t) = t,\ \frac{t^\alpha}{\Gamma(\alpha)}, \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, where $\alpha, k > 0$, $\phi(t) = t(a_2 - t)^{\alpha-1}$ and $\phi(t) = \frac{t}{\alpha}\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)t\right]$ for $\alpha \in (0,1)$, such that $|f'|^q$ to be convex, we can deduce some new fascinating general fractional integral inequalities. We omit their proofs and the details are left to the interested reader.

3. APPLICATIONS

Consider the following special means for different real numbers $a_1 < a_2$:

(1) The arithmetic mean:

$$A(a_1,a_2) = \frac{a_1+a_2}{2}.$$

(2) The harmonic mean:

$$H(a_1,a_2) = \frac{2}{\frac{1}{a_1} + \frac{1}{a_2}}.$$

(3) The logarithmic mean:

$$\mathcal{L}(a_1, a_2) = \frac{a_2 - a_1}{\ln|a_2| - \ln|a_1|}.$$

(4) The r–generalized log–mean:

$$\mathcal{L}_r(a_1, a_2) = \left[\frac{a_2^{r+1} - a_1^{r+1}}{(r+1)(a_2 - a_1)}\right]^{\frac{1}{r}}; \quad r \in \mathbb{Z} \setminus \{-1, 0\}.$$

Now, using the theory results in Section 2, we give some applications to special means for different real numbers.

PROPOSITION 3.1. *Let $a_1, a_2 \in \Re \setminus \{0\}$, where $a_1 < a_2$ and $\alpha, \beta \in [0,1]$. Then for $r \geq 2$ and $r \in \mathbb{N}$, where $q > 1$ and $p^{-1} + q^{-1} = 1$, the following inequality holds:*

$$\left|A(\alpha a_1^r, \beta a_2^r) + \frac{2-\alpha-\beta}{2} A^r(a_1, a_2) - \mathcal{L}_r^r(a_1, a_2)\right| \leq \frac{r(a_2 - a_1)}{4\sqrt[p]{p+1}}$$
$$\times \left\{ \sqrt[p]{\alpha^{p+1} + (1-\alpha)^{p+1}} \times \sqrt[q]{A\left(|a_1|^{q(r-1)}, \left|\frac{a_1+a_2}{2}\right|^{q(r-1)}\right)} \right. \tag{33}$$
$$\left. + \sqrt[p]{\beta^{p+1} + (1-\beta)^{p+1}} \times \sqrt[q]{A\left(\left|\frac{a_1+a_2}{2}\right|^{q(r-1)}, |a_2|^{q(r-1)}\right)} \right\}.$$

Proof. Taking $f(t) = t^r$ in Corollary 2.4, one can obtain the result immediately. □

PROPOSITION 3.2. *Let $a_1, a_2 \in \Re \setminus \{0\}$, where $a_1 < a_2$ and $\alpha, \beta \in [0,1]$. Then for $q > 1$ and $p^{-1} + q^{-1} = 1$, the following inequality holds:*

$$\left|\frac{1}{H(\beta a_1, \alpha a_2)} + \frac{2-\alpha-\beta}{2A(a_1, a_2)} - \frac{1}{\mathcal{L}(a_1, a_2)}\right| \leq \frac{(a_2 - a_1)}{4\sqrt[p]{p+1}}$$
$$\times \left\{ \frac{\sqrt[p]{\alpha^{p+1} + (1-\alpha)^{p+1}}}{\sqrt[q]{H\left(|a_1|^{2q}, \left|\frac{a_1+a_2}{2}\right|^{2q}\right)}} + \frac{\sqrt[p]{\beta^{p+1} + (1-\beta)^{p+1}}}{\sqrt[q]{H\left(\left|\frac{a_1+a_2}{2}\right|^{2q}, |a_2|^{2q}\right)}} \right\}. \tag{34}$$

Proof. Taking $f(t) = \frac{1}{t}$ in Corollary 2.4, one can obtain the result immediately. □

Next, we provide some new error estimates for the mixed trapezium and midpoint formula. Let $\mathcal{P}$ be the partition of the points $a_1 = x_0 < x_1 < \ldots < x_k = a_2$ of the interval $[a_1, a_2]$. Let consider the following quadrature formula:

$$\int_{a_1}^{a_2} f(x)\mathrm{d}x = TM(f, \mathcal{P}; \alpha, \beta) + E(f, \mathcal{P}; \alpha, \beta),$$

where

$$
\begin{aligned}
&TM(f,\mathcal{P};\alpha,\beta)\\
&\quad=\sum_{i=0}^{k-1}\left[\frac{\alpha f(x_i)+\beta f(x_{i+1})}{2}+\frac{2-\alpha-\beta}{2}f\left(\frac{x_i+x_{i+1}}{2}\right)\right](x_{i+1}-x_i)
\end{aligned}
$$

is the mixed trapezium and midpoint version and $E(f,\mathcal{P};\alpha,\beta)$ is denote their associated approximation error.

Proposition 3.3. *Let $f:[a_1,a_2]\to\Re$ be a differentiable function on (a_1,a_2), where $a_1<a_2$ and α, $\beta\in[0,1]$. If $|f'|^q$ is convex on $[a_1,a_2]$ for $q>1$ and $p^{-1}+q^{-1}=1$, then the following inequality holds:*

$$
\begin{aligned}
&\big|E(f,\mathcal{P};\alpha,\beta)\big|\le\frac{1}{4\sqrt[q]{2}\sqrt[p]{p+1}}\times\sum_{i=0}^{k-1}(x_{i+1}-x_i)^2\\
&\times\left\{\sqrt[p]{\alpha^{p+1}+(1-\alpha)^{p+1}}\times\sqrt[q]{|f'(x_i)|^q+\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q}\right.\\
&\left.+\sqrt[p]{\beta^{p+1}+(1-\beta)^{p+1}}\times\sqrt[q]{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q+|f'(x_{i+1})|^q}\right\}.
\end{aligned}\tag{35}
$$

Proof. Applying Theorem 2.3 for $\lambda=m=1$ and $\phi(t)=t$, on the subintervals $[x_i,x_{i+1}]$ $(i=0,\dots,k-1)$ of the partition $\mathcal{P}$, we have

$$
\begin{aligned}
&\left|\frac{\alpha f(x_i)+\beta f(x_{i+1})}{2}+\frac{2-\alpha-\beta}{2}f\left(\frac{x_i+x_{i+1}}{2}\right)\right.\\
&\left.-\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}f(x)\mathrm{d}x\right|\le\frac{(x_{i+1}-x_i)}{4\sqrt[q]{2}\sqrt[p]{p+1}}\\
&\times\left\{\sqrt[p]{\alpha^{p+1}+(1-\alpha)^{p+1}}\times\sqrt[q]{|f'(x_i)|^q+\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q}\right.\\
&\left.+\sqrt[p]{\beta^{p+1}+(1-\beta)^{p+1}}\times\sqrt[q]{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q+|f'(x_{i+1})|^q}\right\}.
\end{aligned}\tag{36}
$$

Hence from (36), we get

$$
\begin{aligned}
\big|E(f,\mathcal{P};\alpha,\beta)\big|&=\left|\int_{a_1}^{a_2}f(x)\mathrm{d}x-TM(f,\mathcal{P};\alpha,\beta)\right|\\
&\le\left|\sum_{i=0}^{k-1}\left\{\int_{x_i}^{x_{i+1}}f(x)\mathrm{d}x-\left[\frac{\alpha f(x_i)+\beta f(x_{i+1})}{2}\right.\right.\right.
\end{aligned}
$$

$$+\frac{2-\alpha-\beta}{2}f\left(\frac{x_i+x_{i+1}}{2}\right)\Bigg](x_{i+1}-x_i)\Bigg\}\Bigg|$$

$$\leq \sum_{i=0}^{k-1}\Bigg|\Bigg\{\int_{x_i}^{x_{i+1}}f(x)\mathrm{d}x-\Bigg[\frac{\alpha f(x_i)+\beta f(x_{i+1})}{2}$$

$$+\frac{2-\alpha-\beta}{2}f\left(\frac{x_i+x_{i+1}}{2}\right)\Bigg](x_{i+1}-x_i)\Bigg\}\Bigg|$$

$$\leq \frac{1}{4\sqrt[q]{2}\sqrt[p]{p+1}}\times\sum_{i=0}^{k-1}(x_{i+1}-x_i)^2$$

$$\times\Bigg\{\sqrt[p]{\alpha^{p+1}+(1-\alpha)^{p+1}}\times\sqrt[q]{|f'(x_i)|^q+\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q}$$

$$+\sqrt[p]{\beta^{p+1}+(1-\beta)^{p+1}}\times\sqrt[q]{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q+|f'(x_{i+1})|^q}\Bigg\}.$$

The proof of Proposition 3.3 is completed. □

REMARK 3.4. Applying our Theorems 2.3 and 2.9 for $m=1$, for special parameter values $\lambda=\frac{1}{3},\frac{1}{2},\frac{2}{3},1$; $\alpha,\beta=0,\frac{1}{3},\frac{1}{2},\frac{2}{3},1$, and various suitable choices of function $\phi(t)=t$, $\frac{t^\alpha}{\Gamma(\alpha)}$, $\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, where α, $k>0$; $\phi(t)=t(a_2-t)^{\alpha-1}$ and $\phi(t)=\frac{t}{\alpha}\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)t\right]$ for $\alpha\in(0,1)$, such that $|f'|^q$ to be convex, we can deduce some new inequalities for special means and new bounds for the mixed trapezium and midpoint formula using above ideas and techniques. We omit their proofs and the details are left to the interested reader.

4. CONCLUSION

Since convexity has large applications in many mathematical areas, then it can be applied to obtain several results in convex analysis, special functions, related optimization theory, mathematical inequalities and may stimulate further research in different areas of pure and applied sciences. Also, from our results using above ideas and techniques we can deduce some new fascinating inequalities using special means and we can find out new refinement bounds for the above mixed trapezium and midpoint formula for different choices of parameters.

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