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UNIQUENESS AND EXISTENCE OF SOLUTIONS FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH TWO FRACTIONAL ORDERS

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Abstract. In this work, we study the existence and uniqueness of solutions for integro-differential equations involving two fractional orders. By using the Banach's fixed point theorem, Leray-Schauder's nonlinear alternative and Leray-Schauder's degree theory, the existence and uniqueness of solutions are obtained. Some illustrative examples are also presented.

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Key words. Caputo derivative, Riemann-Liouville integral, existence, fixed point theorem, Banach's fixed point theorem.

1. INTRODUCTION AND PRELIMINARIES

The class of fractional differential equations arise in many scientific disciplines,such as physics, chemistry, control theory, signal processing and biophysics. For more details, we refer the reader to [1], [4]-[8], [17]. Recently, by applying different techniques of nonlinear analysis such as fixed-point theorems, Leray-Schauder theory, the upper and lower solution method etc., many authors have obtained results of the existence and multiplicity of solutions or positive solutions for various classes of fractional differential equations, for example, we refer the reader to [2, 3] [9]-[15], [18]-[21] and the references therein. In this work, we discuss the existence and uniqueness of solutions for the following integro-differential equations involving two fractional orders:

$$(1)\quad \begin{cases} \mathrm{D}^{\beta}\left(\mathrm{D}^{\alpha}+\lambda\right)x\left(t\right)=\theta f\left(t,x\left(t\right)\right)+AJ^{\delta}h\left(t,x\left(t\right)\right), & t\in\left[0,T\right], \\ J^{1-\alpha}\left(x\left(0\right)\right)=0,\ J^{2-\alpha-\beta}\left(x\left(T\right)\right)-BJ^{\alpha+\beta-1}\left(x\left(\eta\right)\right)=0, & 0<\eta<T, \end{cases}$$

where D^{q}, $q=\alpha,\beta$ denote the Riemann-Liouville fractional derivative, with $0<\alpha,\beta\leq 1$, $1<\alpha+\beta\leq 2$, $0<\delta\leq 1$, λ,θ,A,B are real constant and $B\neq\frac{\Gamma(2\alpha+2\beta-1)T}{\eta^{2\alpha+2\beta-2}}$, f, $h:[0,T]\times\mathbb{R}\to\mathbb{R}$, are continuous functions on $[0,T]$. The operator $J^{\vartheta},\vartheta\in\{\delta,1-\alpha,\alpha+\beta-1\}$ is The Riemann-Liouville fractional

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integral, defined by

$$J^{\vartheta} f(t) = \frac{1}{\Gamma(\vartheta)} \int_0^t (t-\tau)^{\vartheta-1} f(\tau)\, \mathrm{d}\tau, \vartheta > 0,$$

where $\Gamma(\vartheta) := \int_0^{\infty} e^{-u} u^{\vartheta-1} \mathrm{d}u$. The operator D^q is the fractional derivative in the sense of Riemann-Liouville, defined by

$$\mathrm{D}^q y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-\tau)^{n-q-1} y(\tau)\, \mathrm{d}\tau, n-1 < q < n.$$

LEMMA 1.1 ([16]). *For* $q > 0$, *the general solution of the fractional differential equation* $\mathrm{D}^q x(t) = 0$ *is given by*

$$x(t) = c_1 t^{q-1} + c_2 t^{q-2} + \ldots + c_n t^{q-n},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, $n-1 < \alpha \leq n$.

LEMMA 1.2 ([16]). *Let* $q > 0$. *Then for* $x \in L^1(0,T)$ *and* $\mathrm{D}^q x \in L^1(0,T)$,

$$J^q \mathrm{D}^q x(t) = x(t) + c_1 t^{q-1} + c_2 t^{q-2} + \ldots + c_n t^{q-n},$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, $n = [\alpha] + 1$.

In order to define the solution for the problem (1), we need the following lemma:

LEMMA 1.3. *For any* $g \in C[0,T] \cap L(0,T)$, *the unique solution of boundary value problem*

$$\begin{cases} \mathrm{D}^{\beta}(\mathrm{D}^{\alpha} + \lambda) x(t) = g(t), \quad t \in [0,T], 0 < \beta, \alpha < 1, \\ J^{1-\alpha}(x(0)) = 0, \ J^{2-\alpha-\beta}(x(T)) - BJ^{\alpha+\beta-1}(x(\eta)) = 0, 0 < \eta < T, \end{cases} \tag{2}$$

is given by:

$$\begin{aligned} x(t) = &\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} g(s)\, \mathrm{d}s - \lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)\, \mathrm{d}s \\ &- \frac{t^{\alpha+\beta-1}}{(T-\Lambda)\Gamma(\alpha+\beta)} \left[\int_0^T (T-s) g(s)\, \mathrm{d}s \right. \\ &\left. + \lambda \int_0^T \frac{(T-s)^{1-\beta}}{\Gamma(2-\beta)} x(s)\, \mathrm{d}s \right] + \frac{Bt^{\alpha+\beta-1}}{(T-\Lambda)\Gamma(\alpha+\beta)} \\ &\times \left[\int_0^{\eta} \frac{(\eta-s)^{2\alpha+2\beta-2}}{\Gamma(2\alpha+2\beta-1)} g(s)\, \mathrm{d}s - \lambda \int_0^{\eta} \frac{(\eta-s)^{2\alpha+\beta-2}}{\Gamma(2\alpha+\beta-1)} x(s)\, \mathrm{d}s \right]. \end{aligned} \tag{3}$$

where $\Lambda := \frac{B\eta^{2\alpha+2\beta-2}}{\Gamma(2\alpha+2\beta-1)}$ *and* $T \neq \Lambda$.

Proof. By applying Lemma 1.1 and Lemma 1.2, the solution of (2) is written as

$$x(t) = J^{\alpha+\beta} g(t) - \lambda J^{\alpha} x(t) - c_1 \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} t^{\alpha+\beta-1} - c_2 t^{\alpha-1}, \tag{4}$$

where c_0 and c_1 are arbitrary constants. By the boundary condition $J^{1-\alpha}x(0) = 0$, we conclude that $c_2 = 0$.

Now, by taking the Riemann-Liouville fractional integral of order $2-\alpha-\beta$ and $\alpha+\beta-1$ for (4), we get

$$J^{2-\alpha-\beta}x(t) = J^2 g(t) - \lambda J^{2-\beta}x(t) - c_1\Gamma(\beta)\, t.$$

and

$$J^{\alpha+\beta-1}x(t) = J^{2\alpha+2\beta-1}g(t) - \lambda J^{2\alpha+\beta-1}x(t) - c_1\frac{\Gamma(\beta)}{\Gamma(2\alpha+2\beta-1)}t^{2\alpha+2\beta-2},$$

Using the boundary condition $J^{2-\alpha-\beta}x(T) - BJ^{\alpha+\beta-1}x(\eta) = 0$, we obtain that

$$\begin{aligned} c_1 &= \frac{1}{\Gamma(\beta)(T-\Lambda)}\Big[J^2 g(T) - \lambda J^{2-\beta}x(T) - BJ^{2\alpha+2\beta-1}g(\eta) \\ &\quad +\lambda BJ^{2\alpha+\beta-1}x(\eta)\Big], \end{aligned}$$

Substituting the value of c_0 and c_1 in (4), we obtain the solution (3). □

2. MAIN RESULT

We denote by $X = C([0,T],\mathbb{R})$ the Banach space of all continuous functions from $[0,T]$ to $\mathbb{R}$ endowed with a topology of uniform convergence with the norm defined by $\|x\| = \sup_{t\in[0,T]}|x(t)|$.

In view of Lemma 1.3, we can transform the problem (1) into an equivalent fixed point problem $\phi x = x$, where the operator $\phi : X \to X$ is defined by:

$$\begin{aligned} \phi x(t) &= \theta\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}f(s,x(s))\,\mathrm{d}s \\ &+ A\int_0^t \frac{(t-s)^{\alpha+\beta+\delta-1}}{\Gamma(\alpha+\beta+\delta)}h(s,x(s))\,\mathrm{d}s - \lambda\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)\,\mathrm{d}s \\ &- \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)(T-\Lambda)}\Big[\theta\int_0^T (T-s)f(s,x(s))\,\mathrm{d}s \\ &+ A\int_0^T \frac{(T-s)^{\delta+1}}{\Gamma(\delta+2)}h(s,x(s))\,ds + \lambda\int_0^T \frac{(T-s)^{1-\beta}}{\Gamma(2-\beta)}x(s)\,\mathrm{d}s\Big] \\ &+ \frac{Bt^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)(T-\Lambda)}\Big[\theta\int_0^\eta \frac{(\eta-s)^{2\alpha+2\beta-2}}{\Gamma(2\alpha+2\beta-1)}f(s,x(s))\,\mathrm{d}s \\ &+ A\int_0^\eta \frac{(\eta-s)^{2\alpha+2\beta+\delta-2}}{\Gamma(2\alpha+2\beta+\delta-1)}h(s,x(s))\,\mathrm{d}s \\ &+ \lambda\int_0^\eta \frac{(\eta-s)^{2\alpha+\beta-2}}{\Gamma(2\alpha+\beta-1)}x(s)\,\mathrm{d}s\Big]. \end{aligned} \tag{5}$$

Observe that the existence of a fixed point for the operator ϕ implies the existence of a solution for the problem (1).

For convenience we introduce the notations:

$$(6)\quad \nabla_1 \ : \ = \frac{|\theta|\,T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{|A|\,T^{\alpha+\beta+\delta}}{\Gamma(\alpha+\beta+\delta+1)} + \frac{T^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)\,|T-\Lambda|} \times \left[\frac{|\theta|\,T^2}{2} + \frac{|A|\,T^{\delta+2}}{\Gamma(\delta+3)} + \frac{|\theta B|\,\eta^{2\alpha+2\beta-1}}{\Gamma(2\alpha+2\beta)} + \frac{|AB|\,\eta^{2\alpha+2\beta+\delta-1}}{\Gamma(2\alpha+2\beta+\delta)}\right],$$

and

$$(7)\quad \nabla_2 := |\lambda| \left[\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{T^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)\,|T-\Lambda|}\left(\frac{T^{2-\beta}}{\Gamma(3-\beta)} + \frac{|B|\,\eta^{2\alpha+\beta-1}}{\Gamma(2\alpha+\beta)}\right)\right].$$

2.1. EXISTENCE AND UNIQUENESS SOLUTIONS VIA BANACH'S FIXED POINT THEOREM

The first results are based on Banach's fixed point theorem. We prove the following theorem:

THEOREM 2.1. *Let f, $h : [0,T] \times \mathbb{R} \to \mathbb{R}$ be continuous functions satisfying the hypothesis*

(H_1) *there exist nonnegative constants ω_i, $i = 1,2$, such that for all $t \in [0,T]$ and all x, $y \in \mathbb{R}$, we have*

$$|f(t,x) - f(t,y)| \le \omega_1 |x-y|, \quad |h(t,x) - h(t,y)| \le \omega_2 |x-y|.$$

Then the boundary value problem (1) has a unique solution provided by $\omega\nabla_1 < 1 - \nabla_2$, where $\omega = max\{\omega_i : i = 1,2\}$, ∇_1 and ∇_2 are defined by (6)*and* (7)*, respectively.*

Proof. Assume that $N = \max\{N_i : i = 1,2\}$, where N_i are finite numbers given by $N_1 = \sup_{t\in[0,T]} |f(t,0)|$, $N_2 = \sup_{t\in[0,T]} |h(t,0)|$. Setting $r \ge \frac{N\nabla_1}{1-(\nabla_1\omega+\nabla_2)}$, we show that $\phi B_r \subset B_r$, where $B_r = \{x \in X : \|x\| \le r\}$. For $x \in B_r$ we find the following estimates based on the hypothesis (H_1):

$$|f(s,x(s))| \le |f(s,x(s)) - f(s,0)| + |f(s,0)| \le \omega_1 r + N_1,$$

and

$$|h(s,x(s))| \le |h(s,x(s)) - h(s,0)| + |h(s,0)| \le \omega_2 r + N_2.$$

Using these estimates, we can write

$$\|\phi x\| \le \sup_{t\in[0,T]} \Bigg\{ |\theta| \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} |f(s,x(s))|\,\mathrm{d}s + |A| \int_0^t \frac{(t-s)^{\alpha+\beta+\delta-1}}{\Gamma(\alpha+\beta+\delta)} |h(s,x(s))|\,\mathrm{d}s + |\lambda| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x(s)|\,\mathrm{d}s + \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)\,|T-\Lambda|} \Bigg[|\theta| \int_0^T (T-s)\,|f(s,x(s))|\,\mathrm{d}s$$

$$+\left|A\right|\int_0^T\frac{(T-s)^{\delta+1}}{\Gamma(\delta+2)}\left|h(s,x(s))\right|\mathrm{d}s+\left|\lambda\right|\int_0^T\frac{(T-s)^{1-\beta}}{\Gamma(2-\beta)}\left|x(s)\right|\mathrm{d}s\Bigg]$$
$$+\frac{\left|B\right|t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)\left|T-\Lambda\right|}\Bigg[\left|\theta\right|\int_0^\eta\frac{(\eta-s)^{2\alpha+2\beta-2}}{\Gamma(2\alpha+2\beta-1)}\left|f(s,x(s))\right|\mathrm{d}s$$
$$+\left|A\right|\int_0^\eta\frac{(\eta-s)^{2\alpha+2\beta+\delta-2}}{\Gamma(2\alpha+2\beta+\delta-1)}\left|h(s,x(s))\right|\mathrm{d}s$$
$$+\left|\lambda\right|\int_0^\eta\frac{(\eta-s)^{2\alpha+\beta-2}}{\Gamma(2\alpha+\beta-1)}\left|x(s)\right|\mathrm{d}s\Bigg]\Bigg\}$$
$$\leq(\omega r+N)\left\{\frac{\left|\theta\right|T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{\left|A\right|T^{\alpha+\beta+\delta}}{\Gamma(\alpha+\beta+\delta+1)}+\frac{T^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)\left|T-\Lambda\right|}\right.$$
$$\left.\times\left[\frac{\left|\theta\right|T^2}{2}+\frac{\left|A\right|T^{\delta+2}}{\Gamma(\delta+3)}+\frac{\left|\theta B\right|\eta^{2\alpha+2\beta-1}}{\Gamma(2\alpha+2\beta)}+\frac{\left|AB\right|\eta^{2\alpha+2\beta+\delta-1}}{\Gamma(2\alpha+2\beta+\delta)}\right]\right\}$$
$$+\left|\lambda\right|\left[\frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{T^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)\left|T-\Lambda\right|}\left(\frac{T^{2-\beta}}{\Gamma(3-\beta)}+\frac{\left|B\right|\eta^{2\alpha+\beta-1}}{\Gamma(2\alpha+\beta)}\right)\right]r$$
$$=(\omega r+N)\nabla_1+\nabla_2 r\leq r,$$

which implies that $\phi B_r\subset B_r$. Now for $x,y\in B_r$ and for all $t\in[0,T]$, we obtain:

$$\left\|\phi x-\phi y\right\|\leq\sup_{t\in[0,T]}\Bigg\{\left|\theta\right|\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left|f(s,x(s))-f(s,y(s))\right|\mathrm{d}s$$
$$+\left|A\right|\int_0^t\frac{(t-s)^{\alpha+\beta+\delta-1}}{\Gamma(\alpha+\beta+\delta)}\left|h(s,x(x))-h(s,y(x))\right|\mathrm{d}s$$
$$+\left|\lambda\right|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|x(s)-y(s)\right|\mathrm{d}s+\frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)\left|T-\Lambda\right|}$$
$$\times\Bigg[\left|\theta\right|\int_0^T(T-s)\left|f(s,x(s))-f(s,y(s))\right|ds$$
$$+\left|A\right|\int_0^T\frac{(T-s)^{\delta+1}}{\Gamma(\delta+2)}\left|h(s,x(x))-h(s,y(x))\right|\mathrm{d}s$$
$$+\left|\lambda\right|\int_0^T\frac{(T-s)^{1-\beta}}{\Gamma(2-\beta)}\left|x(s)-y(s)\right|\mathrm{d}s\Bigg]$$
$$+\frac{\left|B\right|t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)\left|T-\Lambda\right|}\Bigg[\left|\theta\right|\int_0^\eta\frac{(\eta-s)^{2\alpha+2\beta-2}}{\Gamma(2\alpha+2\beta-1)}\left|f(s,x(s))-f(s,y(s))\right|\mathrm{d}s$$
$$+\left|A\right|\int_0^\eta\frac{(\eta-s)^{2\alpha+2\beta+\delta-2}}{\Gamma(2\alpha+2\beta+\delta-1)}\left|h(s,x(x))-h(s,y(x))\right|\mathrm{d}s$$

$$
\begin{aligned}
&+ |\lambda| \int_0^{\eta} \frac{(\eta - s)^{2\alpha+\beta-2}}{\Gamma(2\alpha+\beta-1)} |x(s) - y(s)| \,\mathrm{d}s \Bigg] \Bigg\} \\
&\leq \omega \Bigg\{ \frac{|\theta| T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{|A| T^{\alpha+\beta+\delta}}{\Gamma(\alpha+\beta+\delta+)} + \frac{T^{\alpha+\beta-1}}{|T-\Lambda| \Gamma(\alpha+\beta)} \\
&\times \left[\frac{|\theta| T^2}{2} + \frac{|A| T^{\delta+2}}{\Gamma(\delta+3)} + \frac{|\theta B| \eta^{2\alpha+2\beta-1}}{\Gamma(2\alpha+2\beta)} + \frac{|AB| \eta^{2\alpha+2\beta+\delta-1}}{\Gamma(2\alpha+2\beta+\delta)} \right] \Bigg\} \|x-y\| \\
&+ |\lambda| \left[\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{T^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)|T-\Lambda|} \left(\frac{T^{2-\beta}}{\Gamma(3-\beta)} + \frac{|B| \eta^{2\alpha+\beta-1}}{\Gamma(2\alpha+\beta)} \right) \right] \|x-y\| \\
&= (\omega \nabla_1 + \nabla_2) \|x-y\|,
\end{aligned}
$$

which leads to $\|\phi x - \phi y\| \leq (\omega \nabla_1 + \nabla_2) \|x-y\|$. Since $\omega \nabla_1 < 1 - \nabla_2$, ϕ is a contraction mapping. □

Now we give another existence and uniqueness result for problem (1) by using Banach's fixed point theorem and Hölder's inequality.

Theorem 2.2. *Suppose that the continuous functions f, $h : [0,T] \times \mathbb{R} \to \mathbb{R}$. Assume that:*

(H$_2$) $|f(t,x) - f(t,y)| \leq a(t)|x-y|$, $|h(t,x) - h(t,y)| \leq b(t)|x-y|$, *for each $t \in [0,T]$, $x, y \in \mathbb{R}$, where $a, b \in L^{\frac{1}{\sigma}}([0,T], \mathbb{R}^+)$, and $\sigma \in (0,1)$.*

Denote $\|\vartheta\|_{\sigma} = \left(\int_0^T |\vartheta(s)|^{\frac{1}{\sigma}} \,\mathrm{d}s \right)^{\sigma}$.

If

$$
|\theta| \|a\|_{\sigma} \Lambda_1 + |A| \|b\|_{\sigma} \Lambda_2 < 1 - \nabla_2, \tag{8}
$$

where

$$
\begin{aligned}
\Lambda_1 :=& \frac{T^{\alpha+\beta-\sigma}}{\Gamma(\alpha+\beta)} \left(\frac{1-\sigma}{\alpha+\beta-\sigma} \right)^{1-\sigma} + \frac{T^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)|T-\Lambda|} \\
&\times \left(T^{2-\sigma} \left(\frac{1-\sigma}{2-\sigma} \right)^{1-\sigma} + \frac{|B| \eta^{2\alpha+2\beta-\sigma-1}}{\Gamma(2\alpha+2\beta-1)} \left(\frac{1-\sigma}{2\alpha+2\beta-\sigma-1} \right)^{1-\sigma} \right),
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_2 :=& \frac{T^{\alpha+\beta+\delta-\sigma}}{\Gamma(\alpha+\beta+\delta)} \left(\frac{1-\sigma}{\alpha+\beta+\delta-\sigma} \right)^{1-\sigma} \\
&+ \frac{T^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)|T-\Lambda|} \left(\frac{T^{\delta+2-\sigma}}{\Gamma(\delta+2)} \left(\frac{1-\sigma}{\delta+2-\sigma} \right)^{1-\sigma} \right. \\
&\left. + \frac{|B| \eta^{2\alpha+2\beta+\delta-\sigma-1}}{\Gamma(2\alpha+2\beta+\delta-1)} \left(\frac{1-\sigma}{2\alpha+2\beta+\delta-\sigma-1} \right)^{1-\sigma} \right).
\end{aligned} \tag{9}
$$

and ∇_2 is given by (7). *Then the boundary value problem* (1) *has a unique solution.*

Proof. For $x, y \in X$ and $t \in [0,T]$, by Hölder inequality and using (H_2), we have:

$$
\begin{aligned}
\|\phi x-\phi y\| \leq & \sup_{t\in[0,T]} \Bigg\{ \frac{|\theta|}{\Gamma(\alpha+\beta)} \int_0^t \left((t-s)^{\frac{\alpha+\beta-1}{1-\sigma}} \mathrm{d}s\right)^{1-\sigma} \left(\int_0^1 a(s)^{\frac{1}{\sigma}} \mathrm{d}s\right)^{\sigma} \\
& + \frac{|A|}{\Gamma(\alpha+\beta+\delta)} \int_0^t \left((t-s)^{\frac{\alpha+\beta+\delta-1}{1-\sigma}} \mathrm{d}s\right)^{1-\sigma} \left(\int_0^1 b(s)^{\frac{1}{\sigma}} \mathrm{d}s\right)^{\sigma} \\
& + \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)|T-\Lambda|} \Bigg[|\theta| \int_0^T \left((T-s)^{\frac{1}{1-\sigma}} \mathrm{d}s\right)^{1-\sigma} \left(\int_0^1 a(s)^{\frac{1}{\sigma}} \mathrm{d}s\right)^{\sigma} \\
& + \frac{|A|}{\Gamma(\delta+2)} \int_0^T \left((T-s)^{\frac{\delta+1}{1-\sigma}} \mathrm{d}s\right)^{1-\sigma} \left(\int_0^1 b(s)^{\frac{1}{\sigma}} \mathrm{d}s\right)^{\sigma} \Bigg] + \frac{|B|\, t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)|T-\Lambda|} \\
& \times \Bigg[\frac{|\theta|}{\Gamma(2\alpha+2\beta-1)} \int_0^{\eta} \left((\eta-s)^{\frac{2\alpha+2\beta-2}{1-\sigma}} \mathrm{d}s\right)^{1-\sigma} \times \left(\int_0^1 a(s)^{\frac{1}{\sigma}}\right)^{\sigma} \mathrm{d}s \\
& + \frac{|A|}{\Gamma(2\alpha+2\beta+\delta-1)} \int_0^{\eta} \left((\eta-s)^{\frac{2\alpha+2\beta+\delta-2}{1-\sigma}} \mathrm{d}s\right)^{1-\sigma} \left(\int_0^1 b(s)^{\frac{1}{\sigma}} \mathrm{d}s\right)^{\sigma} \Bigg] \\
& + |\lambda| \left[\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)|T-\Lambda|} \left(\frac{T^{2-\beta}}{\Gamma(3-\beta)} + \frac{|B|\,\eta^{2\alpha+\beta-1}}{\Gamma(2\alpha+\beta)} \right) \right] \Bigg\} \|x-y\| \\
\leq & |\theta| \|a\|_{\sigma} \Bigg\{ \Bigg[\frac{T^{\alpha+\beta-\sigma}}{\Gamma(\alpha+\beta)} \left(\frac{1-\sigma}{\alpha+\beta-\sigma}\right)^{1-\sigma} + \frac{T^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)|T-\Lambda|} \\
& \times \left(T^{2-\sigma} \left(\frac{1-\sigma}{2-\sigma}\right)^{1-\sigma} + \frac{|B|\,\eta^{2\alpha+2\beta-\sigma-1}}{\Gamma(2\alpha+2\beta-1)} \left(\frac{1-\sigma}{2\alpha+2\beta-\sigma-1}\right)^{1-\sigma} \right) \Bigg] \\
& + |A| \|b\|_{\sigma} \Bigg[\frac{T^{\alpha+\beta+\delta-\sigma}}{\Gamma(\alpha+\beta+\delta)} \left(\frac{1-\sigma}{\alpha+\beta+\delta-\sigma}\right)^{1-\sigma} + \frac{T^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)|T-\Lambda|} \\
& \times \Bigg(\frac{T^{\delta+2-\sigma}}{\Gamma(\delta+2)} \left(\frac{1-\sigma}{\delta+2-\sigma}\right)^{1-\sigma} \\
& + \frac{|B|\,\eta^{2\alpha+2\beta+\delta-\sigma-1}}{\Gamma(2\alpha+2\beta+\delta-1)} \left(\frac{1-\sigma}{2\alpha+2\beta+\delta-\sigma-1}\right)^{1-\sigma} \Bigg) \Bigg] \\
& + |\lambda| \left[\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{T^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)|T-\Lambda|} \left(\frac{T^{2-\beta}}{\Gamma(3-\beta)} + \frac{|B|\,\eta^{2\alpha+\beta-1}}{\Gamma(2\alpha+\beta)} \right) \right] \Bigg\} \|x-y\| \\
= & \left(|\theta| \|a\|_{\sigma} \Lambda_1 + |A| \|b\|_{\sigma} \Lambda_2 + \nabla_2\right) \|x-y\| .
\end{aligned}
$$

Therefore,

$$\|\phi x-\phi y\| \leq \left(|\theta| \|a\|_{\sigma} \Lambda_1 + |A| \|b\|_{\sigma} \Lambda_2 + \nabla_2\right) \|x-y\| .$$

By the condition (8), it follows that ϕ is a contraction mapping. Hence, by the Banach's fixed point theorem ϕ has a unique fixed point which is the unique solution of the boundary value problem (1). Then, the proof is complete. $\square$

2.2. EXISTENCE SOLUTIONS VIA LERAY-SCHAUDER'S NONLINEAR ALTERNATIVE AND LERAY-SCHAUDER DEGREE

Now, we prove the existence of solutions of problem (1) by applying Leray-Schauder nonlinear alternative [21].

THEOREM 2.3 (Nonlinear alternative for single valued maps). *Let E be a Banach space, C a closed, convex subset of E, Ω an open subset of C and $0 \in \Omega$. Suppose that $\phi : \overline{\Omega} \to C$ is a continuous, compact (that is, $\phi(\overline{\Omega})$ is a relatively compact subset of C) map. Then either*

(i) *ϕ has a fixed point in $\overline{\Omega}$, or*

(ii) *there is a $x \in \partial\Omega$ (the boundary of Ω in C) and $\rho \in (0,1)$ with $x = \rho\phi x$.*

THEOREM 2.4. *Assume that f, $h : [0,T] \times \mathbb{R} \to \mathbb{R}$, are continuous functions. Suppose that:*

(H_3) *there exists nondecreasing functions $\psi_1, \psi_2 : \mathbb{R}^+ \to \mathbb{R}^+$, and functions $p, q \in C\left([0,T], \mathbb{R}^+\right)$, such that*

$$|f(t,x)| \leq p(t)\,\psi_1(\|x\|), \quad |h(t,x)| \leq q(t)\,\psi_2(\|x\|), \text{ for all } (t,x) \in [0,T] \times \mathbb{R}.$$

(H_4) *there exists a constant $L > 0$ such that*

$$\frac{L}{|\theta|\,\|p\|_{L^1}\,\psi_1(L)\,\Delta_1 + |A|\,\|q\|_{L^1}\,\psi_2(L)\,\Delta_2 + \nabla_2 L} > 1,$$

where

$$\begin{aligned}
\Delta_1 :=& \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \\
&+ \frac{T^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)\,|T-\Lambda|}\left(\frac{T^2}{2} + \frac{|B|\,\eta^{2\alpha+2\beta-1}}{\Gamma(2\alpha+2\beta)}\right), \\
\Delta_2 :=& \frac{T^{\alpha+\beta+\delta}}{\Gamma(\alpha+\beta+\delta+1)} \\
&+ \frac{T^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)\,|T-\Lambda|}\left(\frac{T^{\delta+2}}{\Gamma(\delta+3)} + \frac{|B|\,\eta^{2\alpha+2\beta+\delta-1}}{\Gamma(2\alpha+2\beta+\delta)}\right),
\end{aligned} \tag{10}$$

and ∇_2 is given by (7). Then the problem (1) has at least one solution on $[0,T]$.

Proof. Let the operator $\phi : X \to X$ be defined by (5). Firstly, we will show that ϕ maps bounded sets (balls) into bounded sets in X. For a number $r > 0$, let $B_r = \{x \in X : \|x\| \leq r\}$ be a bounded ball in X. Then, for $t \in [0,T]$, we have

$$\left|\phi x\left(t\right)\right| \leq \left|\theta\right| \int_0^t \frac{\left(t-s\right)^{\alpha+\beta-1}}{\Gamma\left(\alpha+\beta\right)} p\left(s\right) \psi_1\left(\left\|x\right\|\right) \mathrm{d}s$$
$$+\left|A\right| \int_0^t \frac{\left(t-s\right)^{\alpha+\beta+\delta-1}}{\Gamma\left(\alpha+\beta+\delta\right)} q\left(s\right) \psi_2\left(\left\|x\right\|\right) \mathrm{d}s + \left|\lambda\right| \int_0^t \frac{\left(t-s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)} \left|x\left(s\right)\right| \mathrm{d}s$$
$$+\frac{t^{\alpha+\beta-1}}{\Gamma\left(\alpha+\beta\right)\left|T-\Lambda\right|}\left[\left|\theta\right| \int_0^T \left(T-s\right) p\left(s\right) \psi_1\left(\left\|x\right\|\right) \mathrm{d}s\right.$$
$$\left.+\left|A\right| \int_0^T \frac{\left(T-s\right)^{\delta+1}}{\Gamma\left(\delta+2\right)} q\left(s\right) \psi_2\left(\left\|x\right\|\right) \mathrm{d}s + \left|\lambda\right| \int_0^T \frac{\left(T-s\right)^{1-\beta}}{\Gamma\left(2-\beta\right)} \left|x\left(s\right)\right| \mathrm{d}s\right]$$
$$+\frac{\left|B\right| t^{\alpha+\beta-1}}{\Gamma\left(\alpha+\beta\right)\left|T-\Lambda\right|}\left[\left|\theta\right| \int_0^\eta \frac{\left(\eta-s\right)^{2\alpha+2\beta-2}}{\Gamma\left(2\alpha+2\beta-1\right)} p\left(s\right) \psi_1\left(\left\|x\right\|\right) \mathrm{d}s\right.$$
$$\left.+\left|A\right| \int_0^\eta \frac{\left(\eta-s\right)^{2\alpha+2\beta+\delta-2}}{\Gamma\left(2\alpha+2\beta+\delta-1\right)} q\left(s\right) \psi_2\left(\left\|x\right\|\right) \mathrm{d}s + \left|\lambda\right| \int_0^\eta \frac{\left(\eta-s\right)^{2\alpha+\beta-2}}{\Gamma\left(2\alpha+\beta-1\right)} \left|x\left(s\right)\right| \mathrm{d}s\right]$$
$$\leq \left|\theta\right| \left\|p\right\|_{L^1} \psi_1\left(r\right)\left[\frac{T^{\alpha+\beta}}{\Gamma\left(\alpha+\beta+1\right)} + \frac{T^{\alpha+\beta-1}}{\Gamma\left(\alpha+\beta\right)\left|T-\Lambda\right|}\left(\frac{T^2}{2} + \frac{\left|B\right|\eta^{2\alpha+2\beta-1}}{\Gamma\left(2\alpha+2\beta\right)}\right)\right]$$
$$+\left|A\right| \left\|q\right\|_{L^1} \psi_2\left(r\right)\left[\frac{T^{\alpha+\beta+\delta}}{\Gamma\left(\alpha+\beta+\delta+1\right)} + \frac{T^{\alpha+\beta-1}}{\Gamma\left(\alpha+\beta\right)\left|T-\Lambda\right|}\right.$$
$$\left.\times\left(\frac{T^{\delta+2}}{\Gamma\left(\delta+3\right)} + \frac{\left|B\right|\eta^{2\alpha+2\beta+\delta-1}}{\Gamma\left(2\alpha+2\beta+\delta\right)}\right)\right]$$
$$+\left|\lambda\right|\left[\frac{T^{\alpha}}{\Gamma\left(\alpha+1\right)} + \frac{T^{\alpha+\beta-1}}{\Gamma\left(\alpha+\beta\right)\left|T-\Lambda\right|}\left(\frac{T^{2-\beta}}{\Gamma\left(3-\beta\right)} + \frac{\left|B\right|\eta^{2\alpha+\beta-1}}{\Gamma\left(2\alpha+\beta\right)}\right)\right] r.$$

Consequently,

$$\left\|\phi x\right\| \leq \left|\theta\right| \left\|p\right\|_{L^1} \psi_1\left(r\right)\Delta_1 + \left|A\right| \left\|q\right\|_{L^1} \psi_2\left(r\right)\Delta_2 + \nabla_2 r.$$

Next, we show that ϕ maps bounded sets into equicontinuous sets of X. Let $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ and $x \in B_r$. Then, we have

$$\left|\phi x\left(t_2\right) - \phi x\left(t_1\right)\right| \leq \frac{\left|\theta\right| \left\|p\right\|_{L^1} \psi_1\left(r\right)}{\Gamma\left(\alpha+\beta+1\right)}\left(t_2^{\alpha+\beta} - t_1^{\alpha+\beta}\right)$$
$$+\frac{\left|A\right| \left\|q\right\|_{L^1} \psi_2\left(r\right)}{\Gamma\left(\alpha+\beta+\delta+1\right)}\left(t_2^{\alpha+\beta+\delta} - t_1^{\alpha+\beta+\delta}\right) + \frac{\left|\lambda\right|}{\Gamma\left(\alpha+1\right)}\left(t_2^{\alpha} - t_1^{\alpha}\right)$$
$$+\left[\left|\theta\right| \left\|p\right\|_{L^1} \psi_1\left(r\right)\left(\frac{T^2}{2} + \frac{\left|B\right|\eta^{2\alpha+2\beta-1}}{\Gamma\left(2\alpha+2\beta\right)}\right) +\right.$$
$$+\left|A\right| \left\|q\right\|_{L^1} \psi_2\left(r\right)\left(\frac{T^{\delta+1}}{\Gamma\left(\delta+3\right)} + \frac{\left|B\right|\eta^{2\alpha+2\beta+\delta-1}}{\Gamma\left(2\alpha+2\beta+\delta\right)}\right)$$
$$\left.+\left|\lambda\right|\left(\frac{T^{2-\beta}}{\Gamma\left(3-\beta\right)} + \frac{\left|B\right|\eta^{2\alpha+\beta-1}}{\Gamma\left(2\alpha+\beta\right)}\right) r\right]\left(t_2^{\alpha+\beta-1} - t_1^{\alpha+\beta-1}\right).$$

Obviously, the right-hand side of the above inequality tends to zero independently of $x \in B_r$ as $t_2 - t_1 \to 0$. Therefore, $\phi : X \to X$ is completely continuous by application of the Arzela-Ascoli theorem.

Now, we can conclude the result by using the Leray-Schauder's nonlinear alternative theorem. Consider the equation $x = \rho\phi x$ for $0 < \rho < 1$ and assume that x be a solution. Then, using the computations in proving that ϕ is bounded, we have

$$\begin{aligned}
\|x\| &= \|\rho\phi x\| \leq |\theta| \|p\|_{L^1} \psi_1 (\|x\|) \\
&\times \left[\frac{T^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \frac{T^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)|T-\Lambda|} \left(T + \frac{|B| \eta^{2\alpha+2\beta-2}}{\Gamma(2\alpha+2\beta-1)} \right) \right] \\
&+ |A| \|q\|_{L^1} \psi_2 (\|x\|) \left[\frac{T^{\alpha+\beta+\delta-1}}{\Gamma(\alpha+\beta+\delta)} + \frac{T^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)|T-\Lambda|} \right. \\
&\times \left. \left(\frac{T^{\delta+2}}{\Gamma(\delta+3)} + \frac{|B| \eta^{2\alpha+2\beta+\delta-2}}{\Gamma(2\alpha+2\beta+\delta-1)} \right) \right] \\
&+ |\lambda| \left[\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{T^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)|T-\Lambda|} \left(\frac{T^{2-\beta}}{\Gamma(3-\beta)} + \frac{|B| \eta^{2\alpha+\beta-1}}{\Gamma(2\alpha+\beta)} \right) \right] \|x\| \\
&= |\theta| \|p\|_{L^1} \psi_1 (\|x\|) \Delta_1 + |A| \|q\|_{L^1} \psi_2 (\|x\|) \Delta_2 + \nabla_2 \|x\| .
\end{aligned}$$

Therefore,

$$\frac{\|x\|}{|\theta| \|p\|_{L^1} \psi_1 (\|x\|) \Delta_1 + |A| \|q\|_{L^1} \psi_2 (\|x\|) \Delta_2 + \nabla_2 \|x\|} \leq 1.$$

By (H_4) there exists L such that $L \neq \|x\|$. Let us set

$$\Omega := \{x \in X : \|x\| < L\}.$$

We see that the operator $\phi : \overline{\Omega} \to X$ is continuous and completely continuous. From the choice of $\overline{\Omega}$, there is no $x \in \partial\Omega$ such that $x = \rho\phi x$ for some $0 < \rho < 1$. Consequently, by the nonlinear alternative of Leray-Schauder's type, we deduce that ϕ has a fixed point $x \in \overline{\Omega}$ which is a solution of the problem (1). This completes the proof. □

THEOREM 2.5. *Let f, $h : [0,T] \times \mathbb{R} \to \mathbb{R}$ be continuous functions. Suppose that*

(H_5) *there exist constants $0 \leq m < \frac{1-\nabla_2}{|\theta|\Delta_1 + |A|\Delta_2}$ and $M_i > 0$, $i = 1, 2$ such that*

$$|f(t,x)| \leq m_1(|x|) + M_1, \quad |h(t,x)| \leq m_2(|x|) + M_2, \quad (t,x) \in [0,T] \times \mathbb{R},$$

where $m = \max\{m_i : i = 1,2\}$, $M = \max\{M_i : i = 1,2\}$. Then the problem (1) has at least one solution on $[0,T]$.

Proof. We define an operator $\phi : X \to X$ as in (5) and consider the fixed point equation $x = \phi x$. We shall prove that there exists a fixed point $x \in X$ satisfying (1). It is sufficient to show that $\phi : \overline{B}_r \to X$ satisfies

$$x \neq \mu\phi x, \ \forall (x,\mu) \in \partial B_r \times [0,1], \tag{11}$$

where

$$B_r := \left\{ x \in X : \max_{t\in[0,T]} |x(t)| < r,\ r > 0 \right\}.$$

We define $W(\mu, x) = \mu\phi x, (x, \mu) \in X \times [0,1]$. As shown in Theorem 2.4, the operator ϕ is continuous, uniformly bounded, and equicontinuous. Then, by the Arzela- Ascoli theorem, a continuous map w_μ defined by $w_\mu = x - W(\mu, x) = x - \mu\phi x$ is completely continuous. If (11) holds, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

$$\begin{aligned} \deg(w_\mu, B_r, 0) &= \deg(I - \mu\phi, B_r, 0) = \deg(w_1, B_r, 0) = \deg(w_0, B_r, 0) \\ &= \deg(I, B_r, 0) = 1 \neq 0,\ 0 \in B_r, \end{aligned}$$

where I denotes the identity operator. By the nonzero property of Leray-Schauder's degree, $w_1(x) = x - \phi x = 0$ for at least one $x \in B_r$. In order to prove (11), we assume that $x = \mu\phi x$ for some $\mu \in [0,1]$ and for all $t \in [0,T]$. Then

$$\begin{aligned} \phi x(t) &= |\mu\phi x(t)| \leq (m|x(t)| + M)|\theta| \\ &\times \left[\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{T^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)|T-\Lambda|} \left(\frac{T^2}{2} + \frac{|B|\eta^{2\alpha+2\beta-1}}{\Gamma(2\alpha+2\beta)} \right) \right] \\ &+ (m|x(t)| + M)|A| \left[\frac{T^{\alpha+\beta+\delta}}{\Gamma(\alpha+\beta+\delta+1)} + \frac{T^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)|T-\Lambda|} \right. \\ &\times \left. \left(\frac{T^{\delta+2}}{\Gamma(\delta+3)} + \frac{|B|\eta^{2\alpha+2\beta+\delta-1}}{\Gamma(2\alpha+2\beta+\delta)} \right) \right] \\ &+ |\lambda| \left[\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{T^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)|T-\Lambda|} \left(\frac{T^{2-\beta}}{\Gamma(3-\beta)} + \frac{|B|\eta^{2\alpha+\beta-1}}{\Gamma(2\alpha+\beta)} \right) \right] |x(t)| \\ &= (m|x(t)| + M)\left[|\theta|\Delta_1 + |A|\Delta_2\right] + \nabla_2 |x(t)|. \end{aligned}$$

Taking norm $\sup_{t\in[0,T]} |x(t)| = \|x\|$, we get

$$\|x\| \leq (m\|x\| + M)\left[|\theta|\Delta_1 + |A|\Delta_2\right] + \|x\|\nabla_2,$$

which implies that

$$\|x\| \leq \frac{L\left[|\theta|\Delta_1 + |A|\Delta_2\right]}{1 - m\left[|\theta|\Delta_1 + |A|\Delta_2\right] - \nabla_2}.$$

If $r = \frac{L[|\theta|\Delta_1 + |A|\Delta_2]}{1 - m[|\theta|\Delta_1 + |A|\Delta_2] - \nabla_2} + 1$, then inequality (11) holds. This completes the proof. $\square$

Remark 2.6. If $p(t) = q(t) = 1$ and $\psi_1(x) = m_1(|x|) + M_1, \psi_2(x) = m_2(|x|) + M_2$, then the theorem 2.4 can be reduced to theorem 2.5.

3. EXAMPLES

To illustrate our main results, we treat the following examples.

EXAMPLE 3.1. Let us consider the following fractional difference nonlocal boundary value problem:

$$(12)\qquad \begin{cases} \mathrm{D}^{\frac{1}{2}}\left(\mathrm{D}^{\frac{2}{3}}+\frac{1}{13}\right)x\left(t\right)=\theta f\left(t,x\left(t\right)\right)+AJ^{\frac{1}{3}}h\left(t,x\left(t\right)\right),\ t\in\left[0,1\right], \\ J^{1-\alpha}\left(x\left(0\right)\right)=0,\ J^{2-\alpha-\beta}\left(x\left(1\right)\right)-2J^{\alpha+\beta-1}\left(x\left(\frac{3}{5}\right)\right)=0. \end{cases}$$

For this example, we have $\theta=A=1$ and $f\left(t,x\right)=\frac{\mathrm{e}^{-\pi t^{2}}x(t)}{\left(32\sqrt{\pi}+e^{-\pi t}\right)\left(1+x(t)\right)}$, $h\left(t,x\right)=\frac{\sin(2\pi x(t))}{16\pi(t+2)^{2}}$. Also for $x,y\in\mathbb{R}$ and $t\in\left[0,1\right]$, we have

$$\begin{aligned} \left|f\left(t,x\right)-f\left(t,y\right)\right| &\leq \frac{1}{32\sqrt{\pi}+1}\left|x-y\right|, \\ \left|h\left(t,x\right)-h\left(t,y\right)\right| &\leq \frac{1}{32\pi}\left|x-y\right|. \end{aligned}$$

Hence,

$$\omega_{1}=\frac{1}{32\sqrt{\pi}+1},\omega_{2}=\frac{1}{32\pi},\ \nabla_{1}=2.687\,2,\nabla_{2}=0.162\,89,$$

and

$$\omega=\max\left\{\omega_{i},i=1,2\right\}=\frac{1}{32\sqrt{\pi}+1}.$$

Therefore, we have$\omega\nabla_{1}=4.655\,7\times10^{-2}<1-\nabla_{2}=1-0.162\,89$.Hence, all the hypotheses of Theorem 2.1 are satisfied. Thus, by the conclusion of Theorem 2.1, problem (12) has a unique solution.

EXAMPLE 3.2. As a second illustrative example, let us take

$$(13)\qquad \begin{cases} \mathrm{D}^{\frac{2}{5}}\left(\mathrm{D}^{\frac{1}{4}}+\frac{1}{10}\right)x\left(t\right)=\frac{1}{3}f\left(t,x\left(t\right)\right)+\frac{1}{7}J^{\frac{1}{5}}h\left(t,x\left(t\right)\right),\ t\in\left[0,1\right], \\ J^{1-\alpha}\left(x\left(0\right)\right)=0,\ J^{2-\alpha-\beta}\left(x\left(1\right)\right)-\sqrt{3}J^{\alpha+\beta-1}\left(x\left(\frac{2}{7}\right)\right)=0. \end{cases}$$

Here,

$$\begin{aligned} f\left(t,x\right) &= \frac{\sin x\left(t\right)}{16\pi+\cos^{2}x\left(t\right)}+\frac{3+\sinh\left(\mathrm{e}^{t^{2}}\right)}{2\pi}, \\ h\left(t,x\right) &= \frac{2\sin\left(\frac{x(t)}{2}\right)}{20\sqrt{\pi}+\cos^{2}x\left(t\right)}+\frac{2+\cosh\left(\pi t+1\right)}{\sqrt{\pi}+3}. \end{aligned}$$

Then we can find that$\Delta_{1}=2.058\,1,\ \Delta_{2}=1.813\,7,\ \nabla_{2}=0.139\,6$. Clearly,

$$\left|f\left(t,x\right)\right|=\left|\frac{\sin x\left(t\right)}{16\pi+\cos^{2}x\left(t\right)}+\frac{3+\sinh\left(\mathrm{e}^{t^{2}}\right)}{2\pi}\right|\leq\left(\frac{3+\sinh\left(\mathrm{e}^{t^{2}}\right)}{16\pi}\right)\left(\left|x\right|+8\right),$$

$$|h(t,x)| = \left| \frac{2\sin\left(\frac{x(t)}{2}\right)}{15\sqrt{\pi}+\cos^2 x(t)} + \frac{2+\cosh\left(\sqrt{\pi}t+1\right)}{3\sqrt{\pi}+3} \right|$$
$$\leq \left(\frac{2+\cosh\left(\sqrt{\pi}t+1\right)}{15\sqrt{\pi}}\right)(|x|+5).$$

Choosing $p(t) = \frac{3+\sinh\left(e^{t^2}\right)}{16\pi}$, $q(t) = \frac{2+\cosh(\pi t+1)}{15\sqrt{\pi}}$ and $\psi_1(|x|) = \frac{|x|+2}{16\pi}, \psi_2(|x|)$ $= |x|+5$, we can show that $\frac{L}{|\theta|\|p\|_{L^1}(L+5)\Delta_1+|A|\|q\|_{L^1}(L+5)\Delta_2+\nabla_2 L} > 1$, which implies $L > 12,061$. Hence, by Theorem 2.4, the problem (13) has at least one solution on $[0,1]$.

Example 3.3. Our third example is the following:

$$\text{(14)} \qquad \begin{cases} \mathrm{D}^{\frac{3}{4}}\left(\mathrm{D}^{\frac{4}{5}}+\frac{1}{17}\right)x(t) = \frac{2}{3}f(t,x(t)) + \frac{1}{4}J^{\frac{5}{6}}h(t,x(t)),\ t\in[0,1], \\ J^{1-\alpha}(x(0)) = 0,\ J^{2-\alpha-\beta}(x(1)) - \frac{5}{3}J^{\alpha+\beta-1}\left(x\left(\frac{1}{3}\right)\right) = 0, \end{cases}$$

where, $f(t,x) = \frac{3}{8}\sin\left(\frac{|x|}{3}\right) + \frac{2|x|}{1+|x|}$, $h(t,x) = \frac{3}{16\pi}\sin\left(\frac{2\pi}{3}|x|\right) + \frac{|x|}{1+2|x|} + \frac{1}{2}$,

$$\begin{aligned} f(t,x) &= \left|\frac{3}{8}\sin\left(\frac{|x|}{3}\right) + \frac{2|x|}{1+|x|}\right| \leq \frac{1}{8}|x|+2, \\ h(t,x) &= \left|\frac{3}{16\pi^2}\sin\left(\frac{2\pi}{3}|x|\right) + \frac{|x|}{1+2|x|} + \frac{1}{2}\right| \leq \frac{1}{8\pi}|x|+1, \end{aligned}$$

$$\Delta_1 = 1.960\,6,\ \Delta_2 = 0.806\,34,\ \nabla_2 = 0.214\,22.$$

Clearly $L = \max\{L_i, i=1,2\} = 2$ and $m = \max\{m_i, i=1,2\} = \frac{1}{8} < \frac{1-\nabla_2}{|\theta|\Delta_1+|A|\Delta_2} = 0.520\,85$. Thus, all the conditions of Theorem 2.5 are satisfied and consequently the problem (14) has at least one solution.

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