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BOUNDARY VALUE PROBLEMS FOR HILFER FRACTIONAL DIFFERENTIAL EQUATIONS WITH KATUGAMPOLA FRACTIONAL INTEGRAL AND ANTI-PERIODIC CONDITIONS

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Abstract. The purpose of this paper is to investigate the existence and uniqueness of solutions for a new class of nonlinear fractional differential equations involving Hilfer fractional operator with fractional integral boundary conditions. Our analysis relies on classical fixed point theorems and the Boyd-Wong nonlinear contraction. At the end, an illustrative example is presented. The boundary conditions introduced in this work are of quite general nature and can be reduce to many special cases by fixing the parameters involved in the conditions.

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Key words. Fractional differential equation, Hilfer fractional derivative, Katugampola fractional integral, fixed point theorems.

1. INTRODUCTION

In the last decades, considerable interest in fractional differential equations has been stimulated due to their numerous applications in many fields of science and engineering. Important phenomena in finance, electromagnetics, acoustics, viscoelasticity, electrochemistry and material science are well described by differential equations of fractional order. For examples and recent development of the topic, see [2, 3, 4, 6, 9, 17, 18, 13, 20, 21, 22] and the references cited therein.

Boundary value problems of fractional differential equations and inclusions involve different kinds of boundary conditions such as nonlocal, integral, and multipoint boundary conditions. For fractional integral boundary conditions, see [4, 19], for nonlocal conditions one can consult [3, 4, 24], and anti-periodic conditions were presented in [12].

In 2008, Benchohra et al. [6] studied the existence and uniqueness of solutions of the following nonlinear fractional differential equation:

$$(1) \qquad \begin{cases} {}^{C}D^{\alpha}y(t) = f(t, y(t)), & t \in J := [0, T], \\ ay(0) + by(T) = c, \end{cases}$$

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where ${}^{C}D^{\alpha}$ is the Caputo fractional derivative of order α $(0 < \alpha < 1)$ $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and a, b, c are real constants with $a + b \neq 0$.

In 2017, Asghar Ahmadkhanlu. [5] studied the existence and uniqueness of solutions of the following boundary value problem:

$$(2) \qquad \begin{cases} {}^{C}D^{\alpha}y(t) = f(t, y(t)), & t \in J := [0,1], \\ y(0) = \eta I^{\beta} y(\tau), & 0 < \tau < 1. \end{cases}$$

where ${}^{C}D^{\alpha}$ is the Caputo fractional derivative of order α $(0 < \alpha < 1)$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $\eta \in \mathbb{R}$, I^{β},$0 < \beta < 1$, is the Riemman-Liouville fractional integral of order β.

In 2018, Benhamida et al. [7]. studied the existence of solutions to the boundary value problem for the following fractional-order differential equation

$$(3) \qquad \begin{cases} {}^{C}D^{\alpha}y(t) = f(t, y(t)), & t \in J := [0,T], \\ y(0) + y(T) = b\int_0^T y(s)\mathrm{d}s, & bT \neq 2, \end{cases}$$

where ${}^{C}D^{\alpha}$ is the Caputo fractional derivative of order α $(0 < \alpha < 1)$ and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and b are real constant.

In 2018, Abdo et al. [1]. discussed the existence and uniqueness of positive solutions of the following nonlinear fractional differential equation:

$$(4) \qquad \begin{cases} D^{\alpha}y(t) = f(t, y(t)), & t \in J := [0,1], \\ y(0) = b\int_0^1 y(s)\mathrm{d}s + d, \end{cases}$$

where $0 < \alpha \leq 1$, $\lambda \geq 0$, $d > 0$, D^{α} is the standard Caputo fractional operator and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a given continuous function.

Motivated by the studies above among others, we concentrate on the following boundary value problem of nonlinear Hilfer fractional differential equation

$$(5) \qquad D^{\alpha,\beta}x(t) = f(t, x(t)), \quad t \in J := [0,T],$$

supplemented with the boundary conditions of the form:

$$(6) \qquad aI^{1-\gamma}x(0) + bx(T) = \sum_{i=1}^{m} c_i \, {}^{\rho_i}I^{q_i}x(\eta_i) + d,$$

where $D^{\alpha,\beta}$ is the Hilfer fractional derivative $0 < \alpha < 1$,$0 \leq \beta \leq 1$, $\gamma = \alpha + \beta - \alpha\beta$, ${}^{\rho_i}I^{q_i}$ is the Katugampola integral of $q_i > 0$ and $I^{1-\gamma}$ is the Riemann-Liouville integral of order $1 - \gamma$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $a, b, d, c_i, i = 1, \ldots, m$ are real constants, and $0 < \eta_i < T, i = 1, \ldots, m$.

In the present paper we initiate the study of boundary value problems like (5-6), in which we combine Hilfer fractional differential equations subject to the Katugampola fractional integral boundary conditions.

The rest of the paper is organized into five sections. In Section 2, we recall some basic concepts of fractional calculus and introduce the integral operator associated with the given problem. In Section 3, the main existence and uniqueness results are obtained by using a variety of fixed point theorems, such as the Banach fixed point theorem, the Nonlinear Contractions Boyd and Wong, Schaefer's fixed point theorem, the Leray-Schauder Nonlinear Alternative. In Section 4, an example is provided, while the paper closes with some interesting observations.

2. PRELIMINARY LEMMAS

In what follows we introduce definitions, notations, and preliminary facts which will be used in the sequel. For more details, we refer to [2, 18, 13, 20, 22].

Definition 2.1. Let $J = [0, T]$ be a finite interval and $0 \leq \gamma < 1$. We introduce the weighted space $C_{1-\gamma}(J, E)$ of continuous functions f on $(0, T]$

$$C_{1-\gamma}(J, E) = \{f : (0, T] \to E : (t-a)^{1-\gamma} f(t) \in C(J, E)\}.$$

In the space $C_{1-\gamma}(J, E)$, we define the norm $\|f\|_{C_{1-\gamma}} = \|(t-a)^{1-\gamma} f(t)\|_C$.

Recall that $(C_{1-\gamma}(J, E), \|f\|_{C_{1-\gamma}})$ is a Banach space.

Now, we give some results and properties of fractional calculus.

Definition 2.2 ([18]). The Riemann-Liouville fractional integral of order $\alpha \in \mathbb{R}^+$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \mathrm{d}s, t > 0. \tag{7}$$

provided the right-hand side is point-wise defined on $(0, \infty)$, where $\Gamma(\alpha)$ is the Euler's Gamma function.

Definition 2.3 ([18]). The Riemann-Liouville fractional derivative of order $\alpha \in \mathbb{R}^+$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} f(s) \mathrm{d}s, n-1 < \alpha < n, \tag{8}$$

where $n = [\alpha] + 1$, and $[\alpha]$ means the integral part of α, provided the right hand side is point-wise defined on $(0, \infty)$.

Definition 2.4 ([18]). The Caputo derivative of order α for a function $f \in C^n[0, \infty)$, is given by

$$\begin{aligned} {}^C D^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) \mathrm{d}s, \\ &= I^{n-p} f^{(n)}(t),\ t > 0, , n-1 < \alpha < n. \end{aligned} \tag{9}$$

Definition 2.5 ([17]). Katugampola integral of order $q > 0$ and $\rho > 0$, of a function $f(t)$, for all $0 < t < \infty$, is defined as

$${}^{\rho} I^{q} f(t) = \frac{\rho^{1-q}}{\Gamma(q)} \int_0^t \frac{s^{\rho-1} f(s)}{(t^{\rho} - s^{\rho})^{1-q}} \mathrm{d}s, \tag{10}$$

provided the right-hand side is point-wise defined on $(0, \infty)$.

LEMMA 2.6 ([4]). *Let $\rho, q > 0$ and $p > 0$ be the given constants. Then the following formula holds:*

$$ {}^{\rho}I^{q}t^{p} = \frac{\Gamma(\frac{p+\rho}{\rho})}{\Gamma(\frac{p+\rho q+\rho}{\rho})}\frac{t^{p+\rho q}}{\rho^{q}}. \tag{11}$$

In [13], R. Hilfer studied applications of a generalized fractional operator having the Riemann-Liouville and Caputo derivatives as specific cases (see also [14, 15, 16]).

DEFINITION 2.7 ([13]). The Hilfer fractional derivative $D_{0^+}^{\alpha,\beta}$ of order α $(n-1<\alpha<n)$ and type β $(0 \le \beta \le 1)$ is defined by

$$ D_{0^+}^{\alpha,\beta} = I_{0^+}^{\beta(n-\alpha)}D^{n}I_{0^+}^{(1-\beta)(n-\alpha)}f(t) \tag{12}$$

where $I_{0^+}^{\alpha}$ and $D_{0^+}^{\alpha}$ are Riemann-Liouville fractional integral and derivative defined by (7) and (8), respectively.

REMARK 2.8 (See [13]). Hilfer fractional derivative interpolates between the Riemann-Liouville (8), if $\beta = 0$) and Caputo (9), if $\beta = 1$) fractional derivatives since $D_{0^+}^{\alpha,0} ={}^{R-L} D_{0^+}^{\alpha}$ and $D^{\alpha,1} ={}^{C} D_{0^+}^{\alpha}$.

LEMMA 2.9. *Let $0<\alpha<1, 0\le\beta\le1, \gamma=\alpha+\beta-\alpha\beta$, and $f \in L^{1}(J,E)$. The operator $D_{0^+}^{\alpha,\beta}$ can be written as*

$$D_{0^+}^{\alpha,\beta}f(t) = \left(I_{0^+}^{\beta(1-\alpha)}\tfrac{d}{dt}I_{0^+}^{(1-\gamma)}f\right)(t) = I_{0^+}^{\beta(1-\alpha)}D^{\gamma}f(t), \text{ for a.e. } t\in J.$$

Moreover, γ satisfies $0<\gamma\le1, \gamma\ge\alpha, \gamma>\beta, 1-\gamma<1-\beta(1-\alpha)$.

LEMMA 2.10. *Let $0<\alpha<1, 0\le\beta\le1, \gamma=\alpha+\beta-\alpha\beta$, If $D_{0^+}^{\beta(1-\alpha)}f$ exists and in $L^{1}(J,E)$, then $D_{0^+}^{\alpha,\beta}I_{0^+}^{\alpha}f(t) = I_{0^+}^{\beta(1-\alpha)}D_{0^+}^{\beta(1-\alpha)}f(t)$, for a.e. $t\in J$. Furthermore, if $f\in C_{1-\gamma}(J,E)$ and $I_{0^+}^{1-\beta(1-\alpha)}f \in C_{1-\gamma}^{1}(J,E)$, then $D_{0^+}^{\alpha,\beta}I_{0^+}^{\alpha}f(t) = f(t)$, for a.e. $t\in J$.*

LEMMA 2.11. *Let $0<\alpha<1, 0\le\beta\le1, \gamma=\alpha+\beta-\alpha\beta$, and $f \in L^{1}(J,E)$. If $D_{0^+}^{\gamma}f$ exists and in $L^{1}(J,E)$, then*

$$I_{0^+}^{\alpha}D_{0^+}^{\alpha,\beta}f(t) = I_{0^+}^{\gamma}D_{0^+}^{\gamma}f(t) = f(t) - \frac{I_{0^+}^{1-\gamma}f(0^+)}{\Gamma(\gamma)}(t-a)^{\gamma-1}, \text{ for a.e. } t\in J.$$

LEMMA 2.12 ([18]). *For $t>a$, we have*

$$\begin{aligned} I_{0^+}^{\alpha}(t-a)^{\beta-1}(t) &= \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(t-a)^{\beta+\alpha-1},\\ D_{0^+}^{\alpha}(t-a)^{\beta-1}(t) &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1}. \end{aligned} \tag{13}$$

LEMMA 2.13. *Let $\alpha>0, 0\le\beta\le1$, so the homogeneous differential equation with Hilfer fractional order*

$$D_{0^+}^{\alpha,\beta}h(t) = 0, \tag{14}$$

has a solution

$$h(t) = c_0 t^{\gamma-1} + c_1 t^{\gamma+2\beta-2} + c_2 t^{\gamma+2(2\beta)-3} + \cdots + c_n t^{\gamma+n(2\beta)-(n+1)}. \tag{15}$$

3. MAIN RESULTS

In this section we shall present and prove a preparatory lemma for a boundary value problem of linear fractional differential equations with Hilfer derivative.

DEFINITION 3.1. A function $x(t) \in C_{1-\gamma}(J,\mathbb{R})$ is said to be a solution of (5)-(6) if x satisfies the equation $D^{\alpha,\beta}x(t) = f(t, x(t))$ on J , and the conditions (6).

For the existence of solutions for the problem (5)-(6), we need the following auxiliary lemma.

LEMMA 3.2. *Let $h : J \times \mathbb{R} \to \mathbb{R}$ be a continuous function. A function x is a solution of the fractional integral equation*

$$x(t) = I^{\alpha}h(t) + \frac{t^{\gamma-1}}{\Lambda}\left\{\sum_{i=1}^{m} c_i\, {}^{\rho_i}I^{q_i} I^{\alpha}h(\eta_i) - bI^{\alpha}h(T) + d\right\}, \tag{16}$$

if and only if x is a solution of the fractional BVP

$$D^{\alpha,\beta}x(t) = h(t), t \in J, \tag{17}$$

$$aI^{1-\gamma}x(0) + bx(T) = \sum_{i=1}^{m} c_i^{\rho_i} I^{q_i} h(\eta_i) + d. \tag{18}$$

Proof. Assume x satisfies (17). Then Lemma 2.13 implies that

$$x(t) = I^{\alpha}h(t) + At^{\gamma-1}. \tag{19}$$

By applying the boundary conditions (18) in (19), we obtain

$$\begin{aligned} aA\Gamma(\gamma) + bI^{\alpha}h(T) + bAT^{\gamma-1} &= \sum_{i=1}^{m} c_i\, {}^{\rho_i}I^{q_i} I^{\alpha}h(\eta_i) \\ &\quad + \sum_{i=1}^{m} c_i A \frac{\Gamma(\frac{\gamma+\rho_i-1}{\rho_i})}{\Gamma(\frac{\gamma+\rho_i q_i+\rho_i-1}{\rho_i})} \frac{t^{\gamma+\rho_i q_i-1}}{\rho_i^{q_i}} + d. \end{aligned}$$

Thus,

$$\begin{aligned} A\left(a\Gamma(\gamma) + bT^{\gamma-1} - \sum_{i=1}^{m} c_i \frac{\Gamma(\frac{\gamma+\rho_i-1}{\rho_i})}{\Gamma(\frac{\gamma+\rho_i q_i+\rho_i-1}{\rho_i})} \frac{\eta_i^{\gamma+\rho_i q_i-1}}{\rho_i^{q_i}}\right) &= \sum_{i=1}^{m} c_i\, {}^{\rho_i}I^{q_i} I^{\alpha}h(\eta_i) \\ &\quad - bI^{\alpha}h(T) + d. \end{aligned}$$

Consequently,

$$A = \frac{1}{\Lambda}\left\{\sum_{i=1}^{m} c_i\, {}^{\rho_i}I^{q_i} I^{\alpha} h(\eta_i) - bI^{\alpha}h(T) + d\right\},$$

where

$$\Lambda = \left(a\Gamma(\gamma) + bT^{\gamma-1} - \sum_{i=1}^{m} c_i \frac{\Gamma(\frac{\gamma+\rho_i-1}{\rho_i})}{\Gamma(\frac{\gamma+\rho_i q_i+\rho_i-1}{\rho_i})}\frac{\eta_i^{\gamma+\rho_i q_i-1}}{\rho_i^{q_i}}\right).$$

Finally, we obtain the desired equation (16). □

In the following subsections we prove existence, as well as existence and uniqueness results, for the boundary value problem (5), (6) by using a variety of fixed point theorems.

3.1. EXISTENCE AND UNIQUENESS RESULT VIA BANACH'S FIXED POINT THEOREM

THEOREM 3.3. *Assume the following hypothesis:*
(H1) *There exists a constant* $L > 0$ *such that* $|f(t,x) - f(t,y)| \le L|x-y|$.
If

$$L\Psi < 1, \tag{20}$$

where

$$\Psi := \left\{\frac{T^{\alpha-\gamma+1}}{\Gamma(\alpha+1)} + \frac{1}{|\Lambda|}\left\{\sum_{i=1}^{m} |c_i| \frac{\Gamma(\frac{\alpha+\rho_i}{\rho_i})}{\Gamma(\frac{\alpha+\rho_i q_i+\rho_i}{\rho_i})}\frac{\eta_i^{\alpha+\rho_i q_i}}{\rho_i^{q_i}} + |b|\frac{T^{\alpha}}{\Gamma(\alpha+1)}\right\}\right\},$$

then the problem (5) *has a unique solution on* J.

Proof. Transform the problem (5)-(6) into a fixed point problem for the operator Z defined by

$$\mathrm{Z}x(t) = I^{\alpha}h(t) + \frac{t^{\gamma-1}}{\Lambda}\left\{\sum_{i=1}^{m} c_i\, {}^{\rho_i}I^{q_i} I^{\alpha} h(\eta_i) - bI^{\alpha}h(T) + d\right\}. \tag{21}$$

Applying the Banach contraction mapping principle, we shall show that Z is a contraction.

We put $\sup_{t\in[0,T]}|f(t,0)| = M < \infty$ and choose $r \ge \frac{M\Psi}{1-L\Psi}$.

To show that $\mathrm{Z}B_r \subset B_r$, where $B_r = \{x \in C_{1-\gamma} : \|x\| \le r\}$, we have for any $x \in B_r$

$$\begin{aligned}
|((\mathrm{Z}x)(t))t^{1-\gamma}| \le \sup_{t\in[0,T]} &\Bigg\{t^{1-\gamma}I^{\alpha}|f(s,x(s))|(t) \\
&+ \frac{1}{|\Lambda|}\left\{\sum_{i=1}^{m} c_i\, {}^{\rho_i}I^{q_i} I^{\alpha}|f(s,x(s))|(\eta_i) + bI^{\alpha}|f(s,x(s))|(T) + d\right\}\Bigg\} \\
&\le T^{1-\gamma}I^{\alpha}(|f(s,x(s)) - f(t,0)| + |f(t,0)|)(T)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{|\Lambda|}\Bigg\{\sum_{i=1}^{m}|c_i|\ {}^{\rho_i}I^{q_i}I^{\alpha}(|f(s,x(s))-f(t,o)|+|f(t,0)|)(\eta_i)\\
&+|b|I^{\alpha}(|f(s,x(s))-f(t,0)|+|f(t,0)|)(T)\Bigg\}+\frac{|d|}{|\Lambda|}\\
&\leq (Lr+M)\Bigg\{T^{1-\gamma}I^{\alpha}(1)(T)\\
&+\frac{1}{|\Lambda|}\Bigg\{\sum_{i=1}^{m}|c_i|\ {}^{\rho_i}I^{q_i}(1)(\eta_i)+|b|I^{\alpha}(1)(T)\Bigg\}\Bigg\}+\frac{|d|}{|\Lambda|}\\
&:=(Lr+M)\Psi+\frac{|d|}{|\Lambda|}\leq r,
\end{aligned}
$$

which implies that $\mathrm{Z}B_r\subset B_r$.

Now let $x,y\in C_{1-\gamma}(J,\mathbb{R})$. Then, for $t\in J$, we have

$$
\begin{aligned}
|((\mathrm{Z}x)(t)-(\mathrm{Z}y)(t))t^{1-\gamma}|\leq \sup_{t\in[0,T]}&\Bigg\{t^{1-\gamma}I^{\alpha}|f(s,x(s))-f(s,y(s))|(t)\\
&+\frac{1}{|\Lambda|}\Bigg\{\sum_{i=1}^{m}c_i\ {}^{\rho_i}I^{q_i}I^{\alpha}|f(s,x(s))-f(s,y(s))|(\eta_i)\\
&+bI^{\alpha}|f(s,x(s))-f(s,y(s))|(T)\Bigg\}\Bigg\}\\
&\leq L\|x-y\|\Bigg\{T^{1-\gamma}I^{\alpha}(1)(T)\\
&+\frac{1}{|\Lambda|}\Bigg\{\sum_{i=1}^{m}|c_i|\ {}^{\rho_i}I^{q_i}I^{\alpha}(1)(\eta_i)+|b|I^{\alpha}(1)(T)\Bigg\}\Bigg\}\\
&\leq L\|x-y\|\Bigg\{\frac{T^{\alpha-\gamma+1}}{\Gamma(\alpha+1)}\\
&+\frac{1}{|\Lambda|}\Bigg\{\sum_{i=1}^{m}|c_i|\frac{\Gamma(\frac{\alpha+\rho_i}{\rho_i})}{\Gamma(\frac{\alpha+\rho_iq_i+\rho_i}{\rho_i})}\frac{\eta_i^{\alpha+\rho_iq_i}}{\rho_i^{q_i}}+|b|\frac{T^{\alpha}}{\Gamma(\alpha+1)}\Bigg\}\Bigg\}\\
&:=L\Psi\|x-y\|.
\end{aligned}
$$

Thus $\|((\mathrm{Z}x)(t)-(\mathrm{Z}y)(t))t^{1-\gamma}\|_{\infty}\leq L\Psi\|x-y\|_{\infty}$.

We deduce that Z is a contraction mapping. As a consequence of Banach contraction principle. the problem (5)-(6) has a unique solution on J. This completes the proof. □

3.2. EXISTENCE RESULT VIA SCHAEFER'S FIXED POINT THEOREM

LEMMA 3.4. *Let X be a Banach space. Assume that $T:X\to X$ is completely continuous operator and the set $\Omega=\{x\in X|x=\mu Tx, 0<\mu<1\}$ is bounded. Then T has a fixed point in X.*

THEOREM 3.5. *Assume the hypotheses:*
(H2) *The function* $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ *is continuous.*
(H3) *There exists a constant* $L_1 > 0$ *such that* $|f(t, x)| \leq L_1$, *for a.e.* $t \in J, x \in \mathbb{R}$.
Then the problem (5)-(6) *has at least one solution in* J.

Proof. We shall use Schaefer's fixed point theorem to prove that Z defined by (21) has a fixed point. The proof will be given in several steps.

Step 1. Z is continuous Let x_n be a sequence such that $x_n \to x$ in $C_{1-\gamma}(J, \mathbb{R})$. Then for each $t \in J$,

$$
\begin{aligned}
&|((\mathrm{Z}x_n)(t) - (\mathrm{Z}x)(t))t^{1-\gamma}| \leq t^{1-\gamma} I^{\alpha} \|f(s, x_n(s)) - f(s, x(s))\|(t) \\
&+ \frac{1}{|\Lambda|}\Bigg\{\sum_{i=1}^{m} c_i \,{}^{\rho_i}I^{q_i} I^{\alpha} \|f(s, x_n(s)) - f(s, x(s))\|(\eta_i) \\
&+ bI^{\alpha}\|f(s, x_n(s)) - f(s, x(s))\|(T)\Bigg\} \\
&\leq \Bigg\{T^{1-\gamma} I^{\alpha}(1)(T) + \frac{1}{|\Lambda|}\Bigg\{\sum_{i=1}^{m} c_i \,{}^{\rho_i}I^{q_i} I^{\alpha}(1)(\eta_i) + bI^{\alpha}(1)(T)\Bigg\}\Bigg\} \\
&\times \|f(s, x_n(s)) - f(s, x(s))\| := \Psi\|f(s, x_n(s)) - f(s, x(s))\|.
\end{aligned}
$$

Since f is continuous, $\|((\mathrm{Z}x_n)(t) - (\mathrm{Z}x)(t))t^{1-\gamma}\|_\infty \to 0$ as $n \to \infty$.

Step 2. Z maps bounded sets into bounded sets in $C_{1-\gamma}(J, \mathbb{R})$
Indeed, it is enough to show that for any $r > 0$, if we take $x \in B_r = \{x \in C(J, \mathbb{R}), \|x\|_\infty \leq r\}$, such that $\mathrm{Z}x(t)$ is bounded. Indeed, from (H3), Then for $x \in B_r$ and for each $t \in [0, T]$, we have

$$
\begin{aligned}
&|((\mathrm{Z}x)(t))t^{1-\gamma}| \leq t^{1-\gamma} I^{\alpha}|f(s, x(s))|(t) \\
&+ \frac{1}{|\Lambda|}\left\{\sum_{i=1}^{m} c_i \,{}^{\rho_i}I^{q_i} I^{\alpha}|f(s, x(s))|(\eta_i) + bI^{\alpha}|f(s, x(s))|(T) + d\right\} \\
&\leq L_1 T^{1-\gamma} I^{\alpha}(1)(T) + \frac{L_1}{|\Lambda|}\left\{\sum_{i=1}^{m} |c_i| \,{}^{\rho_i}I^{q_i} I^{\alpha}(1)(\eta_i) + |b| I^{\alpha}(1)(T)\right\} + \frac{|d|}{|\Lambda|} \\
&\leq L_1\Bigg\{T^{1-\gamma} I^{\alpha}(1)(T) + \frac{1}{|\Lambda|}\Bigg\{\sum_{i=1}^{m} |c_i| \,{}^{\rho_i}I^{q_i} I^{\alpha}(1)(\eta_i) + |b| I^{\alpha}(1)(T)\Bigg\}\Bigg\} + \frac{|d|}{|\Lambda|} \\
&:= L_1\Psi + \frac{|d|}{|\Lambda|}.
\end{aligned}
$$

Thus, $\|((\mathrm{Z}x)(t))T^{1-\gamma}\| \leq L_1\Psi + \frac{|d|}{|\Lambda|}$.

Step 3. G maps bounded sets into equicontinuous sets of $C_{1-\gamma}(J, \mathbb{R})$.
Let $t_1, t_2 \in J, t_1 < t_2$, B_r be a bounded set of $C_{1-\gamma}(J, \mathbb{R})$ as in Step 2, and let

$x \in B_r$. Then

$$\begin{aligned}\|(\mathrm{Z}x(t_2)-\mathrm{Z}x(t_1))t^{1-\gamma}\| &\leq I^{\alpha}|t_2^{1-\gamma}f(s,x(s))(t_2)-t_1^{1-\gamma}f(s,x(s))(t_1)|\\ &\leq \frac{L_1}{\Gamma(\alpha)}\left|t_2^{1-\gamma}\int_1^{t_1}(t_2-s)^{\alpha-1}(1)\mathrm{d}s\ -t_1^{1-\gamma}\int_1^{t_1}(t_1-s)^{\alpha-1}(1)\mathrm{d}s\right|\\ &+\frac{L_1}{\Gamma(\alpha)}\left|t_2^{1-\gamma}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}(1)\mathrm{d}s\right| \leq \frac{L_1}{\Gamma(\alpha+1)}(t_2^{\alpha-\gamma+1}-t_1^{\alpha-\gamma+1}).\end{aligned}$$

which implies $\|\mathrm{Z}(t_2)-\mathrm{Z}x(t_1)\|_\infty \to 0$ as $t_1 \to t_2$. As a consequence of Step1 to Step 3, together with the Arzela-Ascoli theorem, we can conclude that Z is continuous and completely continuous.

Step 4: A priori bounds.

Now it remains to show that the set $\Omega = \{x \in C(J,\mathbb{R}) : x = \mu\mathrm{Z}(x)$ for some $0<\mu<1\}$ is bounded.

Let $x \in \Omega$. Then, for each $t \in J$, we have

$$x(t) \leq \mu\left\{I^{\alpha}h(t)+\frac{t^{\gamma-1}}{\Lambda}\left\{\sum_{i=1}^{m}c_i\ {}^{\rho_i}I^{q_i}I^{\alpha}h(\eta_i)-bI^{\alpha}h(T)+d\right\}\right\}.$$

For $\mu \in [0,1]$, let x be such that for each $t \in J$

$$\begin{aligned}\|(\mathrm{Z}x(t))t^{1-\gamma}\| &\leq L_1\Bigg\{T^{1-\gamma}I^{\alpha}(1)(T)\\ &+\frac{1}{|\Lambda|}\left\{\sum_{i=1}^{m}|c_i|\ {}^{\rho_i}I^{q_i}I^{\alpha}(1)(\eta_i)+|b|I^{\alpha}(1)(T)\right\}\Bigg\}+\frac{|d|}{|\Lambda|} := L_1\Psi+\frac{|d|}{|\Lambda|}.\end{aligned}$$

Thus

$$\|(\mathrm{Z}x(t))t^{1-\gamma}\| \leq \infty.$$

This implies that the set Ω is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that Z has a fixed point, which is a solution on J of the problem (5)-(6). □

3.3. EXISTENCE RESULT VIA THE LERAY-SCHAUDER NONLINEAR ALTERNATIVE

THEOREM 3.6. *Assume the following hypotheses:*

(H4) *There exist* $\omega \in L^1(J,\mathbb{R}^+)$ *and* $\Phi : [0,\infty) \to (0,\infty)$ *continuous and nondecreasing such that*

$$|f(t,x)| \leq \omega(t)\Phi(\|x\|),\ \textit{for a.e. } t \in J \textit{ and each } x \in \mathbb{R}.$$

(H5) *There exists a constant* $\epsilon > 0$ *such that*

$$\frac{\epsilon-\frac{|d|}{|\Lambda|}}{\|\omega\|\Phi(\epsilon)\Psi} > 1.$$

Then the boundary value problem (5)-(6) *has at least one solution on* J.

Proof. We shall use the Leray-Schauder theorem to prove that Z defined by (21) has a fixed point. As shown in Theorem 3.6, we see that the operator Z is continuous, uniformly bounded, and maps bounded sets into equicontinuous sets. So by the Arzela-Ascoli theorem Z is completely continuous.
Let x be such that for each $t \in J$, we take the equation $x = \rho Z x$ for $\rho \in (0,1)$ and let x be a solution. After that, the following is obtained:

$$
\begin{aligned}
&|(x)(t))t^{1-\gamma}| \leq t^{1-\gamma} I^{\alpha}|f(s,x(s))|(t)\\
&+\frac{1}{|\Lambda|}\left\{\sum_{i=1}^{m} c_i\, {}^{\rho_i}I^{q_i} I^{\alpha}|f(s,x(s))|(\eta_i) + bI^{\alpha}|f(s,x(s))|(T) + d\right\}\\
&\leq \Phi(\|x\|)T^{\gamma-1} I^{\alpha}\omega(s)(T)\\
&+\frac{\Phi(\|x\|)}{|\Lambda|}\left\{\sum_{i=1}^{m} |c_i|\, {}^{\rho_i}I^{q_i} I^{\alpha}\omega(s)(\eta_i) + |b|I^{\alpha}\omega(s)(T)\right\} + \frac{|d|}{|\Lambda|}\\
&\leq \Phi(\|x\|)\|\omega\|\left\{I^{\alpha}(1)(T) + \frac{T^{\gamma-1}}{|\Lambda|}\left\{\sum_{i=1}^{m} |c_i|\, {}^{\rho_i}I^{q_i} I^{\alpha}(1)(\eta_i) + |b|I^{\alpha}(1)(T)\right\}\right\} + \frac{|d|}{|\Lambda|}\\
&:= \Phi(\|x\|)\|\omega\|\Psi + \frac{|d|}{|\Lambda|},
\end{aligned}
$$

which leads to $\frac{\|x\| - \frac{|d|}{|\Lambda|}}{\|\omega\|\Phi(\|x\|)\Psi} \leq 1$. In view of (H5), there exists ϵ such that $\|x\| \neq \epsilon$. Let us set $U = \{x \in C_{1-\gamma}(J,\mathbb{R}) : \|x\| < \epsilon\}$.

Obviously, the operator $Z : \overline{U} \to C_{1-\gamma}(J,\mathbb{R})$ is completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \lambda Z(x)$ for some $\lambda \in (0,1)$. As a result, by the Leray-Schauder's nonlinear alternative theorem, Z has a fixed point $x \in U$ which is a solution of the (5)-(6). The proof is completed. □

3.4. EXISTENCE AND UNIQUENESS RESULT VIA BOYD-WONG NONLINEAR CONTRACTION

DEFINITION 3.7. Assume that E is a Banach space and $T : E \to E$ is a mapping. If there exists a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\psi(0) = 0$ and $\psi(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$ with the property:

$$\|Tx - Ty\| \leq \psi(\|x-y\|), \forall x, y \in E,$$

then we say that T is a nonlinear contraction.

THEOREM 3.8 (Boyd-Wong Nonlinear Contraction). *Suppose that E is a Banach space and $T : E \to E$ is a nonlinear contraction. Then T has a unique fixed point in E.*

THEOREM 3.9. *Assume that $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous function and suppose that there exists $H > 0$ such that*

$$|f(t,x) - f(t,y)| \leq z(t)\frac{|x-y|}{H + |x-y|}, \text{ for } t \in J, x, y \in \mathbb{R}, \tag{22}$$

where $z : [0,T] \to \mathbb{R}^+$ *is continuous and* H *the constant defined by*

$$H = I^{\alpha} z(T) + \frac{T^{\gamma-1}}{\Lambda}\left\{\sum_{i=1}^{m} |c_i|\ {}^{\rho_i}I^{q_i} I^{\alpha} z(\eta_i) + |b| I^{\alpha} z(T)\right\}.$$

Then the fractional BVP (5)-(6) *has a unique solution on* J *.*

Proof. The operator Z is as defined in (21) and consider a continuous non-decreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\psi(\varepsilon) = \frac{H\varepsilon}{H+\varepsilon}, \forall \varepsilon > 0$. Notice that the function ψ satisfies $\psi(0) = 0$ and $\psi(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$. For any $x, y \in \tau$, and for each $t \in J$, we obtain

$$\begin{aligned}
&|((\mathrm{Z}x)(t) - (\mathrm{Z}y)(t))t^{1-\gamma}| \le \sup_{t\in[0,T]}\Big\{t^{1-\gamma} I^{\alpha} |f(s,x(s)) - f(s,y(s))|(t) \\
&+ \frac{1}{|\Lambda|}\Big\{\sum_{i=1}^{m} c_i\ {}^{\rho_i}I^{q_i} I^{\alpha} |f(s,x(s)) - f(s,y(s))|(\eta_i) \\
&+ bI^{\alpha}|f(s,x(s)) - f(s,y(s))|(T)\Big\}\Big\} \le T^{1-\gamma} I^{\alpha}(z(t)\frac{|x-y|}{H+|x-y|})(T) \\
&+ \frac{1}{\Lambda}\left\{\sum_{i=1}^{m} |c_i|\ {}^{\rho_i}I^{q_i} I^{\alpha}(z(t)\frac{|x-y|}{H+|x-y|})(\eta_i)\ + |b| I^{\alpha}(z(t)\frac{|x-y|}{H+|x-y|})(T)\right\} \\
&\le \frac{\psi(\|x-y\|)}{H}\Big\{T^{1-\gamma} I^{\alpha} z(T) + \frac{1}{\Lambda}\Big\{\sum_{i=1}^{m} |c_i|\ {}^{\rho_i}I^{q_i} I^{\alpha} z(\eta_i) + |b| I^{\alpha} z(T)\Big\}\Big\} \\
&:= \psi(\|x-y\|).
\end{aligned}$$

Then, we get $\|\mathrm{Z}x - \mathrm{Z}y\| \le \psi(\|x-y\|)$. Hence, Z is a nonlinear contraction. Thus, by Boyd-Wong nonlinear contraction theorem, the operator Z has a unique fixed point which is the unique solution of the fractional BVP (5)-(6). The proof is completed. □

4. EXAMPLE

EXAMPLE 4.1. We consider the problem for Hilfer fractional differential equations of the form

$$\left\{\begin{array}{l} D^{\frac{2}{3}} x(t) = f(t,x(t)), (t,x) \in ([0,\pi], \mathbb{R}), \\ I^{\frac{2}{6}} x(0) + x(\pi) = \left(\frac{1}{6} I^{\frac{1}{4}} x(1)\right). \end{array}\right. \tag{23}$$

Here

$$\begin{array}{llll}
a = 1, & b = 1, & c = 1, & d = 0, \\
\alpha = \dfrac{2}{3}, & \beta = \dfrac{1}{2}, & \gamma = \dfrac{4}{6}, & q = \dfrac{1}{4}, \\
\rho = \dfrac{1}{6}, & \eta = 1, & T = \pi, & m = 1.
\end{array}$$

With

$$f(t,x) = \left(\frac{\sin^2(\pi t)}{(\mathrm{e}^t+10)}\right)\left(\frac{|x|}{|x|+1}+1\right)+\left(\frac{\sqrt{3}}{4}\right), \qquad t \in [0,\pi],$$

clearly, the function f is continuous.

For each $x \in \mathbb{R}^+$ and $t \in [0,\pi]$, we have

$$|f(t,x(t)) - f(t,y(t))| \leq \frac{1}{10}|x-y|.$$

Hence, the hypothesis (H1) is satisfied with $L = \frac{1}{10}$. Further,

$$\Psi := \left\{\frac{T^{\alpha-\gamma+1}}{\Gamma(\alpha+1)} + \frac{1}{|\Lambda|}\left\{\sum_{i=1}^{m}|c_i|\frac{\Gamma(\frac{\alpha+\rho_i}{\rho_i})}{\Gamma(\frac{\alpha+\rho_i q_i+\rho_i}{\rho_i})}\frac{\eta_i^{\alpha+\rho_i q_i}}{\rho_i^{q_i}} + |b|\frac{T^\alpha}{\Gamma(\alpha+1)}\right\}\right\} \simeq 5.003$$

and $L\Psi \simeq 0.5003 < 1$. Therefore, by the conclusion of Theorem 3.3, it follows that the problem (23) has a unique solution defined on $[0,\pi]$.

5. CONCLUSION

In this paper, we have obtained some existence results for nonlinear Hilfer fractional differential equations with Katugampola integral boundary conditions by means of some standard fixed point theorems and nonlinear alternative of Leray-Schauder type. Though the technique applied to establish the existence results for the problem at hand is a standard one, yet its exposition in the present framework is new. Our results are new and generalize some available results on the topic.

In all these cases we choose $m = \rho_i = 1$:

✓We remark that when $a = 1$, $b = c_1 = d = 0$, problem (5)-(6) reduces to the case initial value problem considered in [25].

✓We remark that when $\beta = 1$, $c_1 = 0$, problem (5)-(6) reduces to the case initial value problem considered in [6].

✓If we take $a = b = \beta = q = 1$, $d = 0$, in (5)-(6), then our results correspond to the case integral boundary conditions considered in [7].

✓If we take $a = \beta = q = 1$, $b = 0$, in (5)-(6), then our results correspond to the case integral boundary conditions considered in [1].

✓If we take $\alpha = 1$, $\beta = \delta = 0$, in (6), then our results correspond to the case fractional integral boundary conditions considered in [5].

✓By fixing ($a = 0$, $b = 1$) or ($a = 1$, $b = 0$) and $c_1 = 0$, $\beta = 1$ in (6), our results correspond to the ones for initial value problem take the form: $x(T) = d$ or $x(0) = d$.

✓By fixing $a = 1$, $b = c_1 = 0$, in (6), our results correspond to the ones for initial value problem take the form: $I^{1-\gamma}x(0) = d$ considered in [8].

✓In case we choose $a = b = \beta = 1$, $d = c_1 = 0$, in (6), our results correspond to anti-periodic type boundary conditions take the form: $x(0) = -x(T)$.

✓When, $a = b = \beta = 1$, $d = 0$, in (6), our results correspond to fractional integral and anti-periodic type boundary conditions.

On the other hand, if $m \geq 1$ and $\rho = 1$, we have the case:

✓When, $a = d = 0$, in (6), our results correspond to a initial value problem with m-point fractional integral conditions.

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