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AN EXTENSION SYSTEM OF SEQUENTIAL DIFFERENTIAL EQUATIONS OF ARBITRARY ORDER

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Abstract. We are concerned with an extension of a coupled sequential differential system of fractional type. Using the Banach contraction principle, we establish new results for the existence and uniqueness of solutions. Then, we prove another existence result via Schaefer's fixed point theorem. At the end, we illustrate one main result by an example.

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Key words. Caputo derivative, Hadamard derivative, fixed point, existence, sequential differential equation.

1. INTRODUCTION

The branch of differential equations is considered as an important branch in mathematics, especially differential equations of fractional order, after the spread of such equations in other scientific areas and it has many applications in physics, electrochemistry, biomathematics, aerodynamics, dynamics of earthquakes, viscoelasticity, electromagnetic, control theory of dynamical systems etc. For more details, we refer the reader to [3, 10, 11]. In particular, the existence and uniqueness problems of differential equations of fractional order have been investigated by many authors. For instance, we cite the papers [1, 2, 14, 15].

Recently, in [15], some existence and the uniqueness results have been given for the following system of sequential Caputo and Hadamard fractional differential equations

$$\begin{cases} c^C D^{\alpha H} D^{\beta} x\left(t\right) = f\left(t, x\left(t\right)\right), a \leq t \leq b, \\ \gamma_1 x\left(a\right) + \gamma_2^H D^{\beta} x\left(a\right) = 0, \quad \lambda_1 x\left(b\right) + \lambda_2^H D^{\beta} x\left(b\right) = 0, \end{cases}$$

where ${}^C D^{\alpha}, {}^H D^{\beta}$ denote the Caputo and Hadamard fractional derivatives of orders α and β, respectively with, $0 < \alpha, \beta \leq 1$ and $\gamma_i, \lambda_i \in \mathbb{R}\left(i = \overline{1, 2}\right), f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous function.

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Very recently, S. Asawasamrit et al. [1] studied the existence and uniqueness of solutions for the coupled system of nonlinear sequential Caputo and Hadamard fractional differential equations with coupled separated boundary conditions defined by:

$$(1)\qquad \begin{cases} {}^{C}D^{p_1\,H}D^{q_1}x(t) = f(t, x(t), y(t)), \ a \le t \le b, \\ {}^{H}D^{q_2\,C}D^{p_2}y(t) = g(t, x(t), y(t)), \ \ a \le t \le b, \\ \alpha_1 x(a) + \alpha_2^{C}D^{p_2}y(a) = 0, \ \beta_1 x(b) + \beta_2^{C}D^{p_2}y(b) = 0, \\ \alpha_3 y(a) + \alpha_4^{H}D^{q_1}x(a) = 0, \ \beta_3 y(b) + \beta_4^{H}D^{\beta_1}x(b) = 0, \end{cases}$$

where ${}^{C}D^{p_i}, {}^{H}D^{q_i}$ are the Caputo and Hadamard fractional derivatives of orders p_i and q_i, respectively with, $0 < \alpha_i, \beta_i \le 1$, $i = \overline{1,2}$ and $\alpha_i, \beta_i \left(i = \overline{1,4}\right)$ are real constants and $f, g : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$ are continuous functions.

Motivated by the above results, in this paper, we are concerned with extend the study of the work of S. Asawasamrit et al. [1], by considering the following sequential problem:

$$(2)\qquad \begin{cases} {}^{C}D^{\alpha_1\,H}D^{\beta_1}x(t) = f\left(t, x(t), y(t), {}^{H}D^{\alpha_2}y(t)\right), \ a \le t \le b, \\ {}^{H}D^{\beta_2\,C}D^{\alpha_2}y(t) = g\left(t, x(t), {}^{H}D^{\beta_1}x(t), y(t)\right), \ \ a \le t \le b, \\ \gamma_1 x(a) + \gamma_2^{C}D^{\alpha_2}y(a) = \theta_1, \ \lambda_1 x(b) + \lambda_2^{C}D^{\alpha_2}y(b) = \theta_2, \\ \gamma_3 y(a) + \gamma_4^{H}D^{\beta_1}x(a) = \theta_3, \ \lambda_3 y(b) + \lambda_4^{H}D^{\beta_1}x(b) = \theta_4, \end{cases}$$

where ${}^{C}D^{\alpha_i}, {}^{H}D^{\beta_i}$ denote the Caputo and Hadamard fractional derivatives of orders α_i and β_i, respectively with, $0 < \alpha_i, \beta_i \le 1$, $i = \overline{1,2}$ and γ_i, λ_i, θ_i, $\left(i = \overline{1,4}\right)$ are real numbers such that γ_i, λ_i are no zero numbers, $a, b \in \mathbb{R}$ with $a > 0$, and $f, g : [a;b] \times \mathbb{R}^3 \to \mathbb{R}$ are two given functions.

2. PRELIMINARY RESULTS

In this section, we present some definitions and lemmas for fractional derivatives which are used later, for more details, see [7, 8, 9, 11].

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha > 0$, for a continuous function f on $[a,b]$ is defined as:

$$I^{\alpha}f(t) = \tfrac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, \ \alpha > 0, \ a \le t \le b.$$

Definition 2.2. The fractional derivative of order $\alpha, n-1 < \alpha \le n$ for a continuous function $f : [a,b] \to \mathbb{R}$ is given by

$$D^{\alpha}f(t) = \tfrac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\,\mathrm{d}\tau, \quad a \le t \le b.$$

Lemma 2.3. *For $\alpha > 0$, the general solution of the fractional differential equation $D^{\alpha}x(t) = 0$ is given by*

$$x(t) = \sum_{i=0}^{n-1} c_i (t-a)^i,$$

where $c_i \in \mathbb{R}, i = 0,1,2,..,n-1,\ n = [\alpha]+1.$

LEMMA 2.4. *Let* $\alpha > 0$. *Then*

$$I^{\alpha} D^{\alpha} x(t) = x(t) + \sum_{i=0}^{n-1} c_i (t-a)^i, \quad n = [\alpha]+1.$$

DEFINITION 2.5. The Hadamard fractional integral of order $\alpha > 0$, for a continuous function f on $[a,b]$ is defined as:

$$^{H}I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{\log t}{\tau}\right)^{\alpha-1} f(\tau) \frac{\mathrm{d}\tau}{\tau},\ \alpha > 0,\ a \le t \le b.$$

DEFINITION 2.6. The Caputo-type Hadamard fractional derivative of order $\alpha, n = [\alpha]+1$ for an For an at least n-times differentiable function f is defined as:

$$^{H}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\frac{\log t}{\tau}\right)^{n-\alpha-1} \delta^n f(\tau) \frac{\mathrm{d}\tau}{\tau}, \quad a \le t \le b.$$

where $\delta = t\dfrac{\mathrm{d}}{\mathrm{d}t}$ and $\log(.) = \log_e(.)$.

NOTATION 2.7. We denote by $AC_{\delta}^{n}[a,b] := \left\{g\colon [a,b] \to \mathbb{R}\colon \delta^{n-1} g \in AC[a,b]\right\}$ the space of functions g that have δ derivatives up to $(n-1)$ on $[a,b]$ and $\delta^{n-1} g \in AC[a,b]$, where, $AC[a,b]$ is the set of absolutely continuous functions on $[a,b]$ wich coincide with the space of primitives of Lebesgue measurable functions.

The Riemann-Liouville and Hadamard fractional integrals of a function with three variables are given by

$$^{H}I^{q}\left(^{RL}I^{p}(f_{x,y,z})\right)(\zeta) = \frac{1}{\Gamma(p)\Gamma(q)} \int_a^{\zeta} \int_a^{s} \left(\log\frac{\zeta}{s}\right)^{q-1} (s-r)^{p-1} f(r, x(r), y(r), z(r))\, \mathrm{d}r \frac{\mathrm{d}s}{s},$$

and

$$^{RL}I^{p}\left(^{H}I^{q}(f_{x,y,z})\right)(\zeta) = \frac{1}{\Gamma(p)\Gamma(q)} \int_a^{\zeta} \int_a^{s} (\zeta-r)^{p-1} \left(\log\frac{s}{r}\right)^{q-1} f(r, x(r), y(r), z(r)) \frac{\mathrm{d}r}{r} \mathrm{d}s,$$

where $0 < p, q \le 1$ and $\zeta \in \{t, b\}$.

As a special case that will be needed in this paper, we consider the following two quantities:

$$^{H}I^{q}\left(^{RL}I^{p}(1)\right)(\zeta) = \frac{1}{\Gamma(p)\Gamma(q)} \int_a^{\zeta} \int_a^{s} \left(\log\frac{\zeta}{s}\right)^{q-1} (s-r)^{p-1} \mathrm{d}r \frac{\mathrm{d}s}{s},$$

$$^{RL}I^{p}\left(^{H}I^{q}(1)\right)(\zeta) = \frac{1}{\Gamma(p)\Gamma(q)} \int_a^{\zeta} \int_a^{s} (\zeta-r)^{p-1} \left(\log\frac{s}{r}\right)^{q-1} \frac{\mathrm{d}r}{r} \mathrm{d}s.$$

We recall also the following lemma.

Lemma 2.8. *Let $\alpha > 0$ and $x \in AC_\delta^n[a,b]$. Then we have*

$$^{H}I^{\alpha}\left(^{H}D^{\alpha}\right)x(t) = x(t) - \sum_{i=0}^{n-1} c_i \left(\log\left(\frac{t}{a}\right)\right)^i,$$

where, $c_i \in \mathbb{R}, i = 1, 2, \dots, n-1, (n = [\alpha] + 1)$.

We introduce the following quantities

$$\Lambda_1 := \lambda_1 \frac{\left(\log\left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1 + 1)}, \quad \Lambda_2 := \lambda_1 - \lambda_2 \frac{\gamma_1}{\gamma_2},$$
$$\Lambda_3 := \lambda_4 - \lambda_3 \frac{\gamma_4}{\gamma_3}, \quad \Lambda_4 := \lambda_3 \frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2 + 1)}\left(\frac{\theta_1}{\gamma_2} - \frac{\gamma_1}{\gamma_2}\right),$$
$$\Sigma := \Lambda_4 \Lambda_1 - \Lambda_3 \Lambda_2.$$

In the following lemma, we prove a first auxiliary main result.

Lemma 2.9. *Let the functions $\varphi, \psi \in C([a,b], \mathbb{R})$. Then, the solution of the problem*

$$(3) \qquad \begin{cases} {}^{C}D^{\alpha_1 H}D^{\beta_1} x(t) = \varphi(t), \quad a \le t \le b, \\ {}^{H}D^{\beta_2 C}D^{\alpha_2} y(t) = \psi(t), \quad a \le t \le b, \\ \gamma_1 x(a) + \gamma_2^{C} D^{\alpha_2} y(a) = \theta_1, \ \lambda_1 x(b) + \lambda_2^{C} D^{\alpha_2} y(b) = \theta_2, \\ \gamma_3 y(a) + \gamma_4^{H} D^{\beta_1} x(a) = \theta_3, \ \lambda_3 y(b) + \lambda_4^{H} D^{\beta_1} x(b) = \theta_4 \end{cases}$$

is given by $(x(t), y(t))$, where

$$\begin{aligned} x(t) = \frac{1}{\Sigma}\Bigg[&- \Lambda_3\theta_2 + \left(\Lambda_3 - \Lambda_4 \frac{\left(\log\left(\frac{t}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}\right) \\ &\times \left(\lambda_1^{H} I^{\beta_1}\left(^{RL}I^{\alpha_1}\varphi\right)(b) + \lambda_2^{H} I^{\beta_2}\psi(b) + \lambda_2 \frac{\theta_1}{\gamma_2}\right)\Bigg] \\ &+ \frac{1}{\Sigma}\Bigg[(\Lambda_1 + \Lambda_4)\theta_4 - \left(\Lambda_1 + \Lambda_2 \frac{\left(\log\left(\frac{t}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}\right) \\ &\times \left(\lambda_3 I^{\alpha_2}\left(^{H}I^{\beta_2}\psi\right)(b) + \lambda_4^{RL} I^{\alpha_1}\varphi(b) + \lambda_3 \frac{\theta_3}{\gamma_3}\right)\Bigg] +^{H} I^{\beta_1}\left(^{RL}I^{\alpha_1}\varphi\right)(t) \end{aligned}$$

and

$$\begin{aligned} y(t) = &\frac{\theta_3}{\gamma_3} + \frac{\theta_1}{\gamma_2}\frac{(t-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} - \frac{1}{\Sigma}\left(\frac{\gamma_4}{\gamma_3}\Lambda_4 + \frac{\gamma_1}{\gamma_2}\Lambda_3 \frac{(t-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}\right) \\ &\times \left(\theta_2 - \left(\lambda_1^{H} I^{\beta_1}\left(^{RL}I^{\alpha_1}\varphi\right)(b) + \lambda_2^{H} I^{\beta_2}\psi(b)\right) - \lambda_2 \frac{\theta_1}{\gamma_2}\right) \\ &- \frac{1}{\Sigma}\Bigg[\frac{\gamma_4}{\gamma_3}\Lambda_4\theta_4 - \left(\frac{\gamma_4}{\gamma_3}\Lambda_4 + \frac{\gamma_1}{\gamma_2}\Lambda_3 \frac{(t-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}\right) \end{aligned}$$

$$\times \left(\lambda_3^{RL} I^{\alpha_2}\left({}^H I^{\beta_2}\psi\right)(b)+\lambda_4 I^{\alpha_1}\varphi(b)-\lambda_3\frac{\theta_3}{\gamma_3}\right)$$
$$\left.+\frac{\gamma_1}{\gamma_2}\theta_1\Lambda_1\frac{(t-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}\right]+^{RL} I^{\alpha_2}\left({}^H I^{\beta_2}\psi\right)(t).$$

Proof. We apply lemmas 4 to the first equation of (3), we can write

$$^H D^{\beta_1} x(t)=c_1+I^{\alpha_1}\varphi(t),\quad c_1\in\mathbb{R}. \tag{4}$$

We apply Lemma 8 to (4), we get

$$x(t)=c_2+c_1\frac{\left(\log\left(\frac{t}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}+^H I^{\beta_1}\left({}^{RL}I^{\alpha_1}\varphi\right)(t),\quad c_2\in\mathbb{R}. \tag{5}$$

By using the Hadamard fractional integral of order β_2 to the second equation of (3), it yields that

$$^C D^{\alpha_2} y(t)=c_3+^H I^{\beta_2}\psi(t),\quad c_3\in\mathbb{R}. \tag{6}$$

Thanks to Lemma 4 to (6), yields the following formula

$$y(t)=c_4+c_3\frac{(t-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}+^{RL} I^{\alpha_2}\left({}^H I^{\beta_2}\psi\right)(t),\quad c_4\in\mathbb{R}. \tag{7}$$

Thanks to the initial conditions of (3), we obtain

$$\begin{cases}\gamma_1 c_2+\gamma_2 c_3=\theta_1,\\ \lambda_1\left(c_2+c_1\frac{\left(\log\left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}+^H I^{\beta_1}\left({}^{RL}I^{\alpha_1}\varphi\right)(b)\right)\\ \qquad\qquad +\lambda_2\left(c_3+^H I^{\beta_2}\psi(b)\right)=\theta_2,\\ \gamma_3 c_4+\gamma_4 c_1=\theta_3,\\ \lambda_3\left(c_4+c_3\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}+^{RL} I^{\alpha_2}\left({}^H I^{\beta_2}\psi\right)(b)\right)+\lambda_4\left(c_1+I^{\alpha_1}\varphi(b)\right)=\theta_4\end{cases} \tag{8}$$

so, we have

$$\begin{cases}\lambda_1\frac{\left(\log\left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}c_1+\left(\lambda_1-\lambda_2\frac{\gamma_1}{\gamma_2}\right)c_2\\ \qquad =\theta_2-\lambda_1^H I^{\beta_1}\left({}^{RL}I^{\alpha_1}\varphi\right)(b)-\lambda_2^H I^{\beta_2}\psi(b)-\lambda_2\frac{\theta_1}{\gamma_2},\\ \left(\lambda_4-\lambda_3\frac{\gamma_4}{\gamma_3}\right)c_1+\lambda_3\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}\left(\frac{\theta_1}{\gamma_2}-\frac{\gamma_1}{\gamma_2}\right)c_2\\ \qquad =\theta_4-\lambda_3^{RL} I^{\alpha_2}\left({}^H I^{\beta_2}\psi\right)(b)-\lambda_4 I^{\alpha_1}\varphi(b)-\lambda_3\frac{\theta_3}{\gamma_3},\\ c_3=\frac{\theta_1}{\gamma_2}-\frac{\gamma_1}{\gamma_2}c_2,\\ c_4=\frac{\theta_3}{\gamma_3}-\frac{\gamma_4}{\gamma_3}c_1.\end{cases} \tag{9}$$

By solving $\begin{cases} \Lambda_1 c_1 + \Lambda_2 c_2 = \Delta_1 \\ \Lambda_3 c_1 + \Lambda_4 c_2 = \Delta_2, \end{cases}$ where

$$\Delta_1 := \theta_2 - \lambda_1^H I^{\beta_1}\left({}^{RL}I^{\alpha_1}\varphi\right)(b) - \lambda_2^H I^{\beta_2}\psi(b) - \lambda_2\frac{\theta_1}{\gamma_2},$$

$$\Delta_2 := \theta_4 - \lambda_3^{RL} I^{\alpha_2}\left({}^{H}I^{\beta_2}\psi\right)(b) - \lambda_4 I^{\alpha_1}\varphi(b) - \lambda_3\frac{\theta_3}{\gamma_3},$$

we obtain

$$c_1 = \frac{\Lambda_4}{\Sigma}\Delta_1 + \frac{\Lambda_2}{\Sigma}\Delta_2,$$
$$c_2 = \frac{\Lambda_1}{\Sigma}\Delta_2 - \frac{\Lambda_3}{\Sigma}\Delta_1.$$

Using (9), we get

$$c_3 = \frac{\theta_1}{\gamma_2} - \frac{\gamma_1\Lambda_1}{\gamma_2\Sigma}\Delta_2 + \frac{\gamma_1\Lambda_3}{\gamma_2\Sigma}\Delta_1,$$
$$c_4 = \frac{\theta_3}{\gamma_3} - \frac{\gamma_4\Lambda_2}{\gamma_3\Sigma}\Delta_2 - \frac{\gamma_4\Lambda_4}{\gamma_3\Sigma}\Delta_1.$$

Lemma 9 is thus proved. □

3. MAIN RESULT

We introduce the spaces

$$X := \left\{x \in C([a,b],\mathbb{R}), {}^{H}D^{\beta_1}x(t) \in C([a,b],\mathbb{R})\right\},$$
$$Y := \left\{y \in C([a,b],\mathbb{R}), {}^{H}D^{\alpha_2}y(t) \in C([a,b],\mathbb{R})\right\}.$$

We endowed the space X by the norm

$$\|u\|_X := \max\left(\|x\|, \left\|{}^{H}D^{\beta_1}x\right\|\right), \quad \|x\| = \sup_{a\le t\le b}|x(t)|,$$
$$\left\|{}^{H}D^{\beta_1}x\right\| = \sup_{a\le t\le b}\left|{}^{H}D^{\beta_1}x(t)\right|.$$

In the same manner with Y,

$$\|y\|_Y := \max\left(\|y\|, \left\|{}^{H}D^{\alpha_2}y\right\|\right), \quad \|y\| = \sup_{a\le t\le b}|y(t)|,$$
$$\left\|{}^{H}D^{\alpha_2}y\right\| = \sup_{a\le t\le b}\left|{}^{H}D^{\alpha_2}y(t)\right|.$$

Thus, $\left(X\times Y, \|.\|_{X\times Y}\right)$ is a Banach space with norm

$$\|(x,y)\|_{X\times Y} := \max\left(\|x\|_X, \|x\|_Y\right).$$

We consider the operator $\mathcal{T}$ defined as follows:

$$\begin{array}{rccc} \mathcal{T}: & X\times Y & \to & X\times Y \\ & (x,y)(t) & \longmapsto & \left(\mathcal{T}_1(x,y)(t), \mathcal{T}_2(x,y)(t)\right), \end{array}$$

where, $\forall t \in [a,b]$,

$$\begin{aligned}
\mathcal{T}_1(x,y)(t) &:= \frac{1}{\Sigma}\Bigg[-\Lambda_3\theta_2 + \left(\Lambda_3 - \Lambda_4\frac{\left(\log\left(\frac{t}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}\right) \\
&\times\Bigg(\lambda_1^H I^{\beta_1}\left({}^{RL}I^{\alpha_1} f\left(b, x(b), y(b), {}^H D^{\alpha_2} y(b)\right)\right) \\
&\times \lambda_2^H I^{\beta_2} g\left(b, x(b), y(b), {}^H D^{\beta_1} x(b)\right)\Bigg) + \lambda_2\frac{\theta_1}{\gamma_2}\Bigg] \\
&+ \frac{1}{\Sigma}\Bigg[(\Lambda_1+\Lambda_4)\theta_4 - \left(\Lambda_1 + \Lambda_2\frac{\left(\log\left(\frac{t}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}\right) \\
&\times\Bigg(\lambda_3^{RL} I^{\alpha_2}\left({}^H I^{\beta_2} g\left(b, x(b), y(b), {}^H D^{\beta_1} x(b)\right)\right) \\
&+ \lambda_4 I^{\alpha_1} f\left(b, x(b), y(b), {}^H D^{\alpha_2} y(b)\right)\Bigg) + \lambda_3\frac{\theta_3}{\gamma_3}\Bigg] \\
&+ {}^H I^{\beta_1}\left({}^{RL}I^{\alpha_1} f\left(t, x(t), y(t), {}^H D^{\alpha_2} y(t)\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{T}_2(x,y)(t) &:= \frac{\theta_3}{\gamma_3} + \frac{\theta_1}{\gamma_2}\frac{(t-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} - \frac{1}{\Sigma}\left(\frac{\gamma_4}{\gamma_3}\Lambda_4 + \frac{\gamma_1}{\gamma_2}\Lambda_3\frac{(t-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}\right) \\
&\times\Bigg(\theta_2 - \Big(\lambda_1^H I^{\beta_1}\left({}^{RL}I^{\alpha_1} f\left(b, x(b), y(b), {}^H D^{\alpha_2} y(b)\right)\right) \\
&+ \lambda_2^H I^{\beta_2} g\left(b, x(b), y(b), {}^H D^{\beta_1} x(b)\right)\Big) - \lambda_2\frac{\theta_1}{\gamma_2}\Bigg) \\
&- \frac{1}{\Sigma}\Bigg[\frac{\gamma_4}{\gamma_3}\Lambda_4\theta_4 - \left(\frac{\gamma_4}{\gamma_3}\Lambda_4 + \frac{\gamma_1}{\gamma_2}\Lambda_3\frac{(t-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}\right) \\
&\times\Big(\lambda_3^{RL} I^{\alpha_2}\left({}^H I^{\beta_2} g\left(b, x(b), y(b), {}^H D^{\beta_1} x(b)\right)\right) \\
&+ \lambda_4^{RL} I^{\alpha_1} f\left(b, x(b), y(b), {}^H D^{\alpha_2} y(b)\right) - \lambda_3\frac{\theta_3}{\gamma_3}\Big) \\
&+ \frac{\gamma_1}{\gamma_2}\theta_1\Lambda_1\frac{(t-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}\Bigg] + {}^{RL}I^{\alpha_2}\left({}^H I^{\beta_2} g\left(t, x(t), y(t), {}^H D^{\beta_1} x(t)\right)\right).
\end{aligned}$$

We need to consider the following hypothesis:

(H1): Suppose that there exist some constants $l_{ij} > 0, i = \overline{1,2}, j = \overline{1,3}$ such that

$$\begin{aligned}
|f(t,x_2,y_2,z_2) - f(t,x_1,y_1,z_1)| &\le l_{11}|x_2-x_1| + l_{12}|y_2-y_1| + l_{13}|z_2-z_1|, \\
|g(t,x_2,y_2,z_2) - g(t,x_1,y_1,z_1)| &\le l_{21}|x_2-x_1| + l_{22}|y_2-y_1| + l_{23}|z_2-z_1|,
\end{aligned}$$

for each $t \in [a,b]$ and all $x_i, y_i, z_i \in \mathbb{R}$.

Then, we introduce the quantities:

$$
\begin{aligned}
Q_1 &:= \frac{|\lambda_1|\, l_1}{|\Sigma|}\left(|\Lambda_3| + |\Lambda_4| \frac{\left(\log\left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}\right)^{H} I^{\beta_1}\left({}^{RL}I^{\alpha_1}(1)\right)(b) \\
&+ \frac{|\lambda_4|\, l_1}{|\Sigma|}\left(|\Lambda_1| + |\Lambda_2| \frac{\left(\log\left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}\right)^{RL} I^{\alpha_1}(1)(b) + l_1^{H} I^{\beta_1}\left({}^{RL}I^{\alpha_1}(1)\right)(b), \\
Q_2 &:= \frac{|\lambda_2|\, l_2}{|\Sigma|}\left(|\Lambda_3| + |\Lambda_4| \frac{\left(\log\left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}\right)^{H} I^{\beta_2}(1)(b) \\
&+ \frac{|\lambda_3|\, l_2}{|\Sigma|}\left(|\Lambda_1| + |\Lambda_2| \frac{\left(\log\left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}\right)^{RL} I^{\alpha_2}\left({}^{H}I^{\beta_2}(1)\right)(b), \\
Q_3 &:= \frac{|\lambda_1|\, l_1}{|\Sigma|}\left(\frac{|\gamma_4|}{|\gamma_3|}|\Lambda_4| + \frac{|\gamma_1|}{|\gamma_2|}|\Lambda_3| \frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}\right)^{H} I^{\beta_1}\left({}^{RL}I^{\alpha_1}(1)\right)(b) \\
&+ \frac{|\lambda_4|\, l_1}{|\Sigma|}\left(\frac{|\gamma_4|}{|\gamma_3|}|\Lambda_4| + \frac{|\gamma_1|}{|\gamma_2|}|\Lambda_3| \frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}\right)^{RL} I^{\alpha_1}(1)(b), \\
Q_4 &:= \frac{|\lambda_2|\, l_2}{|\Sigma|}\left(\frac{|\gamma_4|}{|\gamma_3|}|\Lambda_4| + \frac{|\gamma_1|}{|\gamma_2|}|\Lambda_3| \frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}\right)^{H} I^{\beta_2}(1)(b) \\
&+ \frac{|\lambda_3 l_2|}{|\Sigma|}\left(\frac{|\gamma_4|}{|\gamma_3|}|\Lambda_4| + \frac{|\gamma_1|}{|\gamma_2|}|\Lambda_3| \frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}\right)^{RL} \\
&\times I^{\alpha_2}\left({}^{H}I^{\beta_2}(1)\right)(b) + l_2^{RL} I^{\alpha_2}\left({}^{H}I^{\beta_2}(1)\right)(b), \\
M_1 &:= \frac{\left(\log \frac{b}{a}\right)^{\beta_1}}{\Gamma(1+\beta_1)}, M_2 := \frac{\left(\log \frac{b}{a}\right)^{\alpha_2}}{\Gamma(1+\alpha_2)},
\end{aligned}
$$

where $l_1 = \max(l_{11}, l_{12}, l_{13}), \quad l_2 = \max(l_{21}, l_{22}, l_{23})$.

Theorem 3.1. *Assume that* (H1) *is satisfied. Then, the problem* (2) *has a unique solution on* $[a,b]$, *provided that* $Q < 1$, *where*

$$Q := \max\{\max((Q_1+Q_2), M_1(Q_1+Q_2)), \max((Q_3+Q_4), M_2(Q_3+Q_4))\}.$$

Proof. We show that the operator $\mathcal{T}$ is contractive. Let $(x_1,y_1), (x_2,y_2) \in X \times Y$. Then, for each $t \in [a,b]$, we have

$$
\begin{aligned}
&|\mathcal{T}_1(x_2,y_2)(t) - \mathcal{T}_1(x_1,y_1)(t)| \\
&\le \frac{1}{|\Sigma|}\left[\left(|\Lambda_3| + |\Lambda_4| \frac{\left(\log\left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}\right)(|\lambda_1|\, l_1 (\|x_2 - x_1\|\right. \\
&+ \|y_2 - y_1\| + \left\|{}^{H}D^{\alpha_2}(y_2 - y_1)\right\|)^{H} I^{\beta_1}\left({}^{RL}I^{\alpha_1}(1)\right)(b)
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\lambda_2\right| l_2\left(\left\|x_2-x_1\right\|+\left\|y_2-y_1\right\|+\left\|{}^H D^{\beta_1}\left(x_2-x_1\right)\right\|\right) \\
&\times{}^H I^{\beta_2}(1)(b)\Big)\Big]+\frac{1}{|\Sigma|}\left[\left(\left|\Lambda_1\right|+\left|\Lambda_2\right| \frac{\left(\log \left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma\left(\beta_1+1\right)}\right)\right. \\
&\times\left(\left|\lambda_3\right| l_2\left\|x_2-x_1\right\|+\left\|y_2-y_1\right\|+\left\|{}^H D^{\beta_1}\left(x_2-x_1\right)\right\|\right) \\
&\times{}^{RL} I^{\alpha_2}\left({}^H I^{\beta_2}(1)\right)(b)+\left|\lambda_4\right| l_1\left(\left\|x_2-x_1\right\|+\left\|y_2-y_1\right\|\right. \\
&\left.\left.+{}^H D^{\alpha_2}\left\|y_2-y_1\right\|\right){}^{RL} I^{\alpha_1}(1)(b)\right]+l_1\left(\left\|x_2-x_1\right\|+\left\|y_2-y_1\right\|\right. \\
&\left.+{}^H D^{\alpha_2}\left\|y_2-y_1\right\|\right){}^H I^{\beta_1}\left({}^{RL} I^{\alpha_1}(1)\right)(b) .
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\left|\mathcal{T}_1\left(x_2, y_2\right)(t)-\mathcal{T}_1\left(x_1, y_1\right)(t)\right| \\
&\leq \frac{\left|\lambda_1\right| l_1}{|\Sigma|}\left(\left|\Lambda_3\right|+\left|\Lambda_4\right| \frac{\left(\log \left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma\left(\beta_1+1\right)}\right) \\
&\times\left(\left\|x_2-x_1\right\|_X+\left\|y_2-y_1\right\|_Y\right){}^H I^{\beta_1} I^{\alpha_1}(1)(b)+\frac{\left|\lambda_2\right| l_2}{|\Sigma|} \\
&\times\left(\left|\Lambda_3\right|+\left|\Lambda_4\right| \frac{\left(\log \left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma\left(\beta_1+1\right)}\right)\left(\left\|x_2-x_1\right\|_X+\left\|y_2-y_1\right\|_Y\right){}^H I^{\beta_2}(1)(b) \\
&+\frac{\left|\lambda_3\right| l_2}{|\Sigma|}\left(\left|\Lambda_1\right|+\left|\Lambda_2\right| \frac{\left(\log \left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma\left(\beta_1+1\right)}\right)\left(\left\|x_2-x_1\right\|_X+\left\|y_2-y_1\right\|_Y\right) \\
&\times I^{\alpha_2 H} I^{\beta_2}(1)(b)+\frac{\left|\lambda_4\right| l_1}{|\Sigma|}\left(\left|\Lambda_1\right|+\left|\Lambda_2\right| \frac{\left(\log \left(\frac{b}{a}\right)\right)^{\beta_1}}{\Gamma\left(\beta_1+1\right)}\right) \\
&+l_1\left(\left\|x_2-x_1\right\|_X+\left\|y_2-y_1\right\|_Y\right){}^H I^{\beta_1} I^{\alpha_1}(1)(b) \\
&\leq\left(Q_1+Q_2\right) \max \left(\left\|x_2-x_1\right\|_{\infty},\left\|y_2-y_1\right\|_{\infty}\right) .
\end{aligned}
\tag{10}
$$

On other hand, we have

$$
\begin{aligned}
&\left|{}^H D^{\beta_1} \mathcal{T}_1\left(x_2, y_2\right)(t)-{}^H D^{\beta_1} \mathcal{T}_1\left(x_1, y_1\right)(t)\right| \\
&\leq \frac{1}{\Gamma\left(1-\beta_1\right)}\left(t \frac{\mathrm{d}}{\mathrm{d} t}\right) \int_a^t\left(\log \frac{t}{s}\right)^{-\beta_1}\left|\mathcal{T}_1\left(x_2, y_2\right)(t)-\mathcal{T}_1\left(x_1, y_1\right)(s)\right| \frac{\mathrm{d} s}{s} \\
&\leq M_1\left(Q_1+Q_2\right) \max \left(\left\|x_2-x_1\right\|_{\infty},\left\|y_2-y_1\right\|_{\infty}\right) .
\end{aligned}
\tag{11}
$$

Similarly, we can write

$$
\left|\mathcal{T}_2\left(x_2, y_2\right)(t)-\mathcal{T}_2\left(x_1, y_1\right)(t)\right| \leq\left(Q_3+Q_4\right) \max \left(\left\|x_2-x_1\right\|_{\infty},\left\|y_2-y_1\right\|_{\infty}\right) .
$$

Also, we have

$$
\left|{}^H D^{\alpha_2} \mathcal{T}_2\left(x_2, y_2\right)(t)-{}^H D^{\alpha_2} \mathcal{T}_2\left(x_1, y_1\right)(t)\right|
$$

$$\leq \frac{1}{\Gamma(1-\alpha_2)} \left(t\frac{\mathrm{d}}{\mathrm{d}t} \right) \int_a^t \left(\log \frac{t}{s} \right)^{-\alpha_2}$$
$$\times \left| \mathcal{T}_2(x_2,y_2)(t) - \mathcal{T}_2(x_1,y_1)(s) \right| \frac{\mathrm{d}s}{s}$$
$$\leq M_2(Q_3+Q_4) \max\left(\|x_2-x_1\|_\infty, \|y_2-y_1\|_\infty \right).$$

Thanks to $(10),(11)$, we obtain

$$\|\mathcal{T}_1(x_2,y_2) - \mathcal{T}_1(x_1,y_1)\|_X \leq \max\left((Q_1+Q_2), M_1(Q_1+Q_2)\right) \times \max\left(\|x_2-x_1\|_X, \|y_2-y_1\|_Y\right).$$

With the same arguments as before, we have

$$\|\mathcal{T}_2(x_2,y_2) - \mathcal{T}_2(x_1,y_1)\|_Y \leq \max\left((Q_3+Q_4), M_2(Q_3+Q_4)\right) \times \max\left(\|x_2-x_1\|_X, \|y_2-y_1\|_Y\right),$$

consequently, we obtain

$$\|\mathcal{T}(x_2,y_2) - \mathcal{T}(x_1,y_1)\|_{X\times Y} \leq Q \max\left(\|x_2-x_1\|_X, \|y_2-y_1\|_Y\right).$$

Using the fact that $Q<1$, we conclude that $\mathcal{T}$ is a contraction mapping. As consequence of Banach's fixed point theorem, the problem (2) admits a unique solution over $[a,b]$. □

The second main result is based on Schaefer's fixed point theorem. We prove the following existence result.

Theorem 3.2. *Assume that the following two hypotheses are valid:*
(H_2) : *The functions* $f,g:[a,b]\times\mathbb{R}^3\to\mathbb{R}$ *are continuous*
(H_3) : *There exist two constants* $K_1, K_2, >0$ *such that,* $\forall t\in[a,b], x,y,z\in\mathbb{R}$, $|f(t,x,y,z)|\leq K_1$, $|g(t,x,y,z)|\leq K_2$. *Then, the problem* (2) *has at least one solution on* $[a,b]$.

Proof. First of all, it is to note that the operator is continuous since the given functions of our problem are also continuous. Then, the following steps are needed to achieve the proof of this results.

Step 1. We show that the operator $\mathcal{T}$ maps bounded sets into bounded sets in $X\times Y$. Let Ω bounded in $X\times Y$. For each $t\in[a,b]$ and $(x,y)\in\Omega$, we have

$$\begin{aligned}\left|\mathcal{T}_1(x,y)(t)\right| &\leq \frac{1}{|\Sigma|}\Bigg[\left|\Lambda_3\theta_2\right| + \left(\left|\Lambda_3\right| + \left|\Lambda_4\right|\frac{\left(\log\left(\frac{t}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}\right) \\ &\times \left(K_1\left|\lambda_1\right| I^{\beta_1} I^{\alpha_1}(b) + K_2\left|\lambda_2\right|^{H} I^{\beta_2}(b)\right) + \left|\lambda_2\right|\left|\frac{\theta_1}{\gamma_2}\right|\Bigg] \\ &+ \frac{1}{|\Sigma|}\Bigg[\left(\left|\Lambda_1\right| + \left|\Lambda_4\right|\right)\left|\theta_4\right| + \left(\left|\Lambda_1\right| + \left|\Lambda_2\right|\frac{\left(\log\left(\frac{t}{a}\right)\right)^{\beta_1}}{\Gamma(\beta_1+1)}\right) \\ &\times \left(K_2\left|\lambda_3\right| I^{\alpha_2 H} I^{\beta_2}(b) + \left|\lambda_4\right| K_1 I^{\alpha_1}(b)\right) + \left|\lambda_3\right|\left|\frac{\theta_3}{\gamma_3}\right|\Bigg] + K_1^{H} I^{\beta_1} I^{\alpha_1}(b) \\ &\leq K_1 Q_1 + K_2 Q_2 + \frac{1}{|\Sigma|}\left(\left|\Lambda_3\theta_2\right| + \left(\left|\Lambda_1\right| + \left|\Lambda_4\right|\right)\left|\theta_4\right| + \left|\lambda_3\right|\left|\frac{\theta_3}{\gamma_3}\right|\right).\end{aligned} \tag{12}$$

On the other hand, we have

$$\begin{aligned}\left|{}^{H}D^{\beta_1}\mathcal{T}_1(x,y)(t)\right| &\leq M_1\Bigg(K_1 Q_1 + K_2 Q_2 \\ &+ \frac{1}{|\Sigma|}\left(\left|\Lambda_3\theta_2\right| + \left(\left|\Lambda_1\right| + \left|\Lambda_4\right|\right)\left|\theta_4\right| + \left|\lambda_3\right|\left|\frac{\theta_3}{\gamma_3}\right|\right)\Bigg).\end{aligned} \tag{13}$$

So, (12) and (13) give us

$$\left\|\mathcal{T}_1(x,y)(t)\right\|_X \leq \max\left(\begin{array}{c} K_1 Q_1 + K_2 Q_2 + \frac{1}{|\Sigma|}\Big(\left|\Lambda_3\theta_2\right| \\ + \left(\left|\Lambda_1\right| + \left|\Lambda_4\right|\right)\left|\theta_4\right| + \left|\lambda_3\right|\left|\frac{\theta_3}{\gamma_3}\right|\Big), \\ M_1\Bigg(K_1 Q_1 + K_2 Q_2 + \frac{1}{|\Sigma|}\Big(\left|\Lambda_3\theta_2\right| \\ + \left(\left|\Lambda_1\right| + \left|\Lambda_4\right|\right)\left|\theta_4\right| + \left|\lambda_3\right|\left|\frac{\theta_3}{\gamma_3}\right|\Big)\Bigg) \end{array}\right). \tag{14}$$

Similarly, we get

$$\begin{aligned}\left|\mathcal{T}_2(x,y)(t)\right| \leq &\left|\frac{\theta_3}{\gamma_3}\right| + \left|\frac{\theta_1}{\gamma_2}\right|\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} \\ &+ \frac{1}{|\Sigma|}\Bigg(\left|\theta_2\right| + \left|\lambda_2\right|\left|\frac{\theta_1}{\gamma_2}\right| + \left|\frac{\gamma_4}{\gamma_3}\right|\left|\Lambda_4\right|\left|\theta_4\right| \\ &+ \left|\frac{\gamma_1}{\gamma_2}\right|\left|\theta_1\right|\left|\Lambda_1\right|\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \left|\lambda_3\right|\left|\frac{\theta_3}{\gamma_3}\right|\Bigg) \\ &+ K_1 Q_3 + K_2 Q_4.\end{aligned}$$

Also, we have

$$\left|{}^{H}D^{\alpha_2}\mathcal{T}_2(x,y)(t)\right| \leq M_2\Bigg(K_1Q_3+K_2Q_4+\left|\frac{\theta_3}{\gamma_3}\right|+\left|\frac{\theta_1}{\gamma_2}\right|\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}+\frac{1}{|\Sigma|}$$

$$\times\left(|\theta_2|+|\lambda_2|\left|\frac{\theta_1}{\gamma_2}\right|+\left|\frac{\gamma_4}{\gamma_3}\right||\Lambda_4||\theta_4|+\left|\frac{\gamma_1}{\gamma_2}\right||\theta_1||\Lambda_1|\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}+|\lambda_3|\left|\frac{\theta_3}{\gamma_3}\right|\right)\Bigg).$$

Then,

$$\|\mathcal{T}_2(x,y)(t)\|_Y \leq \max\left(\begin{array}{c} \left|\frac{\theta_3}{\gamma_3}\right|+\left|\frac{\theta_1}{\gamma_2}\right|\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}+\frac{1}{|\Sigma|}\Big(|\theta_2| \\ +|\lambda_2|\left|\frac{\theta_1}{\gamma_2}\right|+\left|\frac{\gamma_4}{\gamma_3}\right||\Lambda_4||\theta_4| \\ +\left|\frac{\gamma_1}{\gamma_2}\right||\theta_1||\Lambda_1|\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} \\ +|\lambda_3|\left|\frac{\theta_3}{\gamma_3}\right|\Big)+K_1Q_3+K_2Q_4, \\ M_2\Big(K_1Q_3+K_2Q_4+\left|\frac{\theta_3}{\gamma_3}\right|+\left|\frac{\theta_1}{\gamma_2}\right|\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} \\ +\frac{1}{|\Sigma|}\Big(|\theta_2|+|\lambda_2|\left|\frac{\theta_1}{\gamma_2}\right|+\left|\frac{\gamma_4}{\gamma_3}\right||\Lambda_4||\theta_4| \\ +\left|\frac{\gamma_1}{\gamma_2}\right||\theta_1||\Lambda_1|\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}+|\lambda_3|\left|\frac{\theta_3}{\gamma_3}\right|\Big)\Big) \end{array}\right). \tag{15}$$

Hence, from (14) and (15), we deduce that $\mathcal{T}\Omega$ is a uniformly bounded set.

Step 2. We prove that $\mathcal{T}$ maps bounded sets into equicontinuous sets. Let $t_1, t_2 \in [a,b]$ such that $t_1 < t_2$, and let $(x,y) \in \Omega$, then

$$|\mathcal{T}_1(x,y)(t_2)-\mathcal{T}_1(x,y)(t_1)| \leq \frac{|\Lambda_4|}{|\Sigma|\Gamma(\beta_1+1)}\left|\left(\log\frac{t_2}{a}\right)^{\beta_1}-\left(\log\frac{t_1}{a}\right)^{\beta_1}\right|$$

$$\times\Big(|\lambda_1|^{H}I^{\beta_1}\left({}^{RL}I^{\alpha_1}\left|f\left(b,x(b),y(b),{}^{H}D^{\alpha_2}y(b)\right)\right|\right)$$

$$+|\lambda_2|^{H}I^{\beta_2}\left|g\left(b,x(b),y(b),{}^{H}D^{\beta_1}x(b)\right)\right|\Big)$$

$$+\frac{|\Lambda_2|}{|\Sigma|\Gamma(\beta_1+1)}\left|\left(\log\frac{t_2}{a}\right)^{\beta_1}-\left(\log\frac{t_1}{a}\right)^{\beta_1}\right|$$

$$\times\Big(|\lambda_3|^{RL}I^{\alpha_2}\left({}^{H}I^{\beta_2}\left|g\left(b,x(b),y(b),{}^{H}D^{\beta_1}x(b)\right)\right|\right)$$

$$+|\lambda_4|^{RL}I^{\alpha_1}\left|f\left(b,x(b),y(b),{}^{H}D^{\alpha_2}y(b)\right)\right|\Big)$$

$$+{}^{H}I^{\beta_1 RL}I^{\alpha_1}\left(\left|f\left(t_2,x(t_2),y(t_2),{}^{H}D^{\alpha_2}y(t_2)\right)\right.\right.$$

$$\left.\left.-f\left(t_1,x(t_1),y(t_1),{}^{H}D^{\alpha_2}y(t_1)\right)\right|\right)$$

$$\leq \frac{|\Lambda_4|}{|\Sigma|\,\Gamma(\beta_1+1)}\left|\left(\log\frac{t_2}{a}\right)^{\beta_1}-\left(\log\frac{t_1}{a}\right)^{\beta_1}\right|$$
$$\times\left(K_1|\lambda_1|^H I^{\beta_1}\left({}^{RL}I^{\alpha_1}(1)\right)(b)+K_2|\lambda_2|^H I^{\beta_2}(1)(b)\right)$$
$$+\frac{|\Lambda_2|}{|\Sigma|\,\Gamma(\beta_1+1)}\left|\left(\log\frac{t_2}{a}\right)^{\beta_1}-\left(\log\frac{t_1}{a}\right)^{\beta_1}\right|$$
$$\times\left(K_2|\lambda_3|^{RL} I^{\alpha_2}\left({}^H I^{\beta_2}(1)\right)(b)+|\lambda_4|\,K_1^{RL}I^{\alpha_1}(1)(b)\right)$$
$$+\frac{K_1(b-a)^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\beta_1+1)}\left[2\left(\log\frac{t_2}{t_1}\right)^{\beta_1}+\left|\left(\log\frac{t_2}{a}\right)^{\beta_1}-\left(\log\frac{t_1}{a}\right)^{\beta_1}\right|\right].$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero.

On the other hand, we obtain

$$\left|{}^H D^{\beta_1}\mathcal{T}_1(x,y)(t_2)-{}^H D^{\beta_1}\mathcal{T}_1(x,y)(t_1)\right|\leq M_1\left|\mathcal{T}_1(x,y)(t_2)-\mathcal{T}_1(x,y)(t_1)\right|.$$

Therefore, we obtain $|\mathcal{T}_1(x,y)(t_2)-\mathcal{T}_1(x,y)(t_1)|\to 0,$ as $t_1\to t_2$. With the same manner, we can show that $|\mathcal{T}_2(x,y)(t_2)-\mathcal{T}_2(x,y)(t_1)|\to 0,$ as $t_1\to t_2$. Thanks to Steps 1 and 2 and using Arzela-Ascoli theorem, we conclude that the operator $\mathcal{T}$ is completely continuous.

Step 3. Now, we show that the set

$$\mathcal{E}=\{(x,y)\in X\times Y:(x,y)=\lambda\mathcal{T}(x,y),\ 0<\lambda<1\}$$

is bounded.

If $(x,y)\in\mathcal{E}$, this yields that $\begin{cases} x(t)=\lambda\mathcal{T}_1(x,y)(t)\\ y(t)=\lambda\mathcal{T}_2(x,y)(t)\end{cases}$, $\forall t\in[a,b]$ Hence, we have $|x(t)|\leq\lambda\|\mathcal{T}_1(x,y)(t)\|\leq\|\mathcal{T}_1(x,y)\|$ and $|y(t)|\leq\lambda\|\mathcal{T}_2(x,y)(t)\|\leq\|\mathcal{T}_2(x,y)\|$. Thus, we get

$$\|(x,y)\|_{X\times Y}=\max(|x(t)|,|y(t)|)\leq\max(\|\mathcal{T}_1(x,y)\|,\|\mathcal{T}_2(x,y)\|).$$

Using the condition $(H2)$ of Theorem (3.2), we deduce that $\mathcal{E}$ is bounded.

At the end, in view of Schaefer's fixed point theorem, we conclude that $\mathcal{T}$ has a fixed point which is a solution of the problem (2). □

4. AN ILLUSTRATIVE EXAMPLE

Let us consider the example:

$$(16)\qquad\begin{cases} {}^C D^{\frac{2}{3}}\,{}^H D^{\frac{1}{4}}x(t)=f\left(t,x(t),y(t),{}^H D^{\frac{6}{7}}y(t)\right),\ 1\leq t\leq 3,\\ {}^H D^{\frac{4}{5}}\,{}^C D^{\frac{6}{7}}y(t)=g\left(t,x(t),{}^H D^{\frac{1}{4}}x(t),y(t)\right),\ \ 1\leq t\leq 3,\\ 0.2x(1)+1.2\,{}^C D^{\frac{6}{7}}y(1)=1.3,\quad 0.6x(3)+2.6\,{}^C D^{\frac{6}{7}}y(3)=0.9,\\ 2.15y(1)+1.6\,{}^H D^{\frac{1}{4}}x(1)=1.7,\quad 0.3y(1)+1.6\,{}^H D^{\frac{1}{4}}x(1)=3.2.\end{cases}$$

Here, $\alpha_1 = \frac{2}{3}$; $\alpha_2 = \frac{6}{7}$; $\beta_1 = \frac{1}{4}$; $\beta_2 = \frac{4}{5}$; $\gamma_1 = 0.2$; $\gamma_2 = 1.2$; $\gamma_3 = 2.15$; $\gamma_4 = 1.6$; $\theta_1 = 1.3$; $\theta_2 = 0.9$; $\theta_3 = 1.7$; $\theta_4 = 3.2$; $\lambda_1 = 0.6$; $\lambda_2 = 2.6$; $\lambda_3 = 0.3$; $\lambda_4 = 1.6$, and the functions $f, g : [1; 3] \times \mathbb{R}^3 \to \mathbb{R}$ are given by

$$\begin{cases} f\left(t, x(t), y(t), {}^H D^{\frac{6}{7}} y(t)\right) = \frac{1}{2}t + \frac{t^2 x(t)}{54(1 + x(t))} + \frac{1}{6}\cos y(t) \\ \qquad + \frac{t}{18}\tan^{-1}\left({}^H D^{\frac{6}{7}} y(t)\right) \\ g\left(t, x(t), {}^H D^{\frac{1}{4}} x(t), y(t)\right) = \frac{1}{7}\tan^{-1}(x(t)) + \frac{t y(t)}{21(1 + y(t))} \\ \qquad + \frac{t}{29}\frac{{}^H D^{\frac{1}{4}} x(t)}{\left(1 + {}^H D^{\frac{1}{4}} x(t)\right)}. \end{cases}$$

It is clear that f, g are continuous functions and we have: $|f(t, x, y, z)| \leq \dfrac{13}{21} = K_1$ and $|g(t, x, y, z)| \leq \frac{79}{203} = K_2$. Thanks to Theorem 3.2, the system (2) has at least one solution $(x(t), y(t)), t \in [1, 3]$.

REFERENCES

[1] S. Asawasamrit, S. Ntouyas, J. Tariboon and W. Nithiarayaphaks, *Coupled systems of sequential caputo and hadamard fractional differential equations with coupled separated boundary conditions*, Symmetry, **10** (2018), 1–17.

[2] B. Ahmad, A. Alsaedi, S.K. Ntouyas and J. Tariboon, *Hadamard-type fractional differential equations, inclusions and inequalities*, Springer, Cham, 2017.

[3] R.L. Bagley and P.J. Torvik, *A theoritical basis for the application of fractional calculus in viscoelasticity*, Journal of Rheology, **27** (1983), 201–210.

[4] M. Benchohra and J.E. Lazreg, *Existence and Ulam stability for non-linear implicit fractional differential equations with Hadamard derivative*, Stud. Univ. Babeş-Bolyai Math., **62** (2017), 27–38.

[5] K. Diethelm and N.J. Ford, *Analysis of fractional differential equations*, J. Math. Anal. Appl., **265** (2002), 229–248.

[6] A. Granas and J. Dugundji, *Fixed point theory*, Springer-Verlag, New York, 2005.

[7] A.A. Kilbas, *Hadamard-type fractional calculus*, J. Korean Math. Soc., **38** (2001), 1191–1204.

[8] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Vol. 204, Elsevier, Amsterdam, 2006.

[9] K.S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, Wiley and Sons, New York, 1993.

[10] H. Rudolf, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[11] B. Ross, *Fractional Calculus and its Applications*, Springer-Verlag, Berlin, 1975.

[12] J. Sabatier, O.P. Agrawal and J.A. Machado, *Advances in fractional calculus: theoretical developments and applications in physics and engineering*, Springer, Dordrecht, 2007.

[13] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional integrals and derivatives*, Gordon and Breach Science Publishers, Yverdon, 1993.

[14] J. Tariboon, A. Cuntavepanit, S.K. Ntouyas and W. Nithiarayaphaks, *Separated boundary value problems of sequential Caputo and Hadamard fractional differential equations*, J. Funct. Spaces, **2018**, Article 6974046, 1–8.

[15] P. Thiramanus, S.K. Ntouyas and J. Tariboon, *Positive solutions for Hadamard fractional differential equations on infinite domain*, Adv. Difference Equ., **2016**, Article 83, 1–18.

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