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THE STRONG NORMAL SYSTEM OF SOME COMPACT RIGHT TOPOLOGICAL GROUPS

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Abstract. The aim of the present paper is to characterize the strong normal system of the Ellis groups of a well-known family of dynamical systems on the finite and infinite dimensional tori.
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1. INTRODUCTION

A *right topological group* is a group G endowed with a topology τ such that for each $t \in G$ the mapping $\rho_t : G \to G$ defined by $\rho_t(s) = st$ is continuous. On a right topological group G, the set of all continuous left translations $\lambda_s : G \to G$ (with $s \in G$), defined by $\lambda_s(t) = st$, is called the *topological center* of G and is denoted by $\Lambda(G)$. A right topological group (G, τ) is said to be (countably) *admissible* if there is a (countable) subset S of $\Lambda(G)$ such that S is dense in G. By a *CHART group* we mean a compact Hausdorff admissible right topological group. The *Furstenberg-Ellis-Namioka structure theorem* deals with the existence of a (transfinite) sequence of closed normal subgroups in a CHART group G characterizing the structure of G explicitly [2, 3, 8, 9]. The σ*-topology*, introduced by Namioka [9], on a right topological group (G, τ) is the quotient of the product topology $\tau \times \tau$ under the map $(G \times G, \tau \times \tau) \to G$ defined by $(x, y) \mapsto x^{-1}y$. For a σ*-closed* subgroup L of G let $N(L)$ denote the intersection of all σ-closed σ-neighborhoods of the identity element 1 in L, then $N(L)$ is a σ-closed normal subgroup of L. Furthermore, $N(G)$ is the smallest closed normal subgroup of G with the property that the quotient space $G/N(G)$ is a topological group. As a matter of fact, if we define $L_0 = G$, $L_1 = N(G)$, $L_2 = N(L_1), \dots,$ and $L_\xi = \cap_{\eta<\xi} L_\eta$ for any limit ordinal $\xi \leq \xi_0$ then the system of normal subgroups $\{L_\xi\}_\xi$ is exactly the strong normal system of G that is constructed in the Furstenberg-Ellis-Namioka structure theorem. Namioka [9] Showed that if a CHART group (G, τ) is countably admissible, and if $\mathcal{U}$ denotes the family of

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all τ-open neighborhoods of its identity 1, then $N(G) = \bigcap\{U^{-1}UU^{-1}U,\ U \in \mathcal{U}\}$. Later, Moors and Namioka [8] removed the countability condition and showed that the above result remains true for admissible compact Hausdorff right topological groups. More generally, it is a result of Milnes and Pym [7] that for a σ-closed normal subgroup L of a CHART group G, the family $\{U^{-1}U \cap L,\ U \in \mathcal{U}\}$ is a base of open neighborhoods of 1 in (L, σ) and $N(L)$ is a normal subgroup of G and that $N(L) = \bigcap\{(U^{-1}U \cap L)^{-1}(U^{-1}U \cap L),\ U \in \mathcal{U}\}$. They also proved the existence of a unique left invariant Haar measure on any CHART group, by using the strong normal system in the Furstenberg-Ellis-Namioka structure theorem.

For a dynamical system (X, T), the closure of the set $\{T^n : X \to X,\ n \in \mathbb{Z}\}$ in X^X with the product topology is a semigroup with composition as multiplication, is called the *enveloping semigroup* of the system and is denoted by $\Sigma(X, T)$. A dynamical system (X, T) is called *distal* if for any two points x, y in X and any net $\{n_\alpha\}_\alpha$ in $\mathbb{Z}$, the identity $\lim_\alpha T^{n_\alpha}x = \lim_\alpha T^{n_\alpha}y$ implies that $x = y$. Ellis [1] showed that a dynamical system is distal if and only if its enveloping semigroup is a group, called the *Ellis group* of the system.

Assume that $\mathbb{T}$ is the unit circle in the complex plane and let $E(\mathbb{T})$ denote the family of all endomorphisms of the group $\mathbb{T}$. Consider the dynamical systems $(\mathbb{T}^k, T_k)$ and $(\mathbb{T}^\infty, T_\infty)$ defined by

$$T_k(x_1, x_2, \ldots, x_k) = (\lambda x_1, x_1x_2, \ldots, x_{k-1}x_k), \text{ and}$$
$$T_\infty(x_1, x_2, x_3, \ldots) = (\lambda x_1, x_1x_2, x_2x_3, \ldots),$$

in which λ is an element of $\mathbb{T}$. Such systems are distal [4], hence their enveloping semigroups are actually groups. A characterization of the Ellis groups $\Sigma(\mathbb{T}^\infty, T_\infty)$, $\Sigma(\mathbb{T}^k, T_k)$ (for irrational λ), and $\Sigma(\mathbb{T}^k, T_k)$ (for rational λ) as closed subgroups of $E(\mathbb{T}^\infty)$, $E(\mathbb{T})^{k-1} \times \mathbb{T}$ and $E(\mathbb{T})^{k-1}$ is given in [6], [5] and [10], respectively, as follows: If $\sigma \in \Sigma(\mathbb{T}^\infty, T_\infty)$ and $\sigma = \lim_\alpha T^{n_\alpha}$, for some net $(n_\alpha)_\alpha$ in $\mathbb{Z}$, then for each $i = 1, 2, \ldots$ define $\theta_i \in E(\mathbb{T})$ by $\theta_i(x) = \lim_\alpha x^{P_i(n_\alpha)}$, for all $x \in \mathbb{T}$, in which for positive integer n, $P_i(n) = \binom{n}{i}$ and for negative integer n, $P_i(n) = (-1)^i\binom{i-n-1}{i}$, where $1 \le i \le |n|$. Then $\Theta_\infty : \Sigma(\mathbb{T}^\infty, T_\infty) \to E(\mathbb{T})^\infty$ defined by $\Theta_\infty(\Sigma) = (\theta_1, \theta_2, \theta_3, \ldots)$ is an embedding isomorphism onto its range. If $\lambda \in \mathbb{T}$ is irrational and $\sigma \in \Sigma(\mathbb{T}^k, T_k)$, and $\sigma = \lim_\alpha T^{n_\alpha}$, for some net $(n_\alpha)_\alpha$ in $\mathbb{Z}$, then define $\Theta_k : \Sigma(\mathbb{T}^k, T_k) \to E(\mathbb{T})^{k-1} \times \mathbb{T}$ by $\Theta_k(\sigma) = (\theta_1, \ldots, \theta_{k-1}, u)$, where $u = \lim_\alpha \lambda^{P_k(n_\alpha)}$ and $\theta_1, \ldots, \theta_{k-1}$ are defined as above and the mapping Θ_k is an embedding isomorphism onto its range. Finally, if $\lambda \in \mathbb{T}$ is rational and $\sigma \in \Sigma(\mathbb{T}^k, T_k)$, and $\sigma = \lim_\alpha T^{n_\alpha}$, for some net $(n_\alpha)_\alpha$ in $\mathbb{Z}$, then define $\Theta_k : \Sigma(\mathbb{T}^k, T_k) \to E(\mathbb{T})^{k-1}$ by $\Theta_k(\sigma) = (\theta_1, \ldots, \theta_{k-1})$, where $\theta_1, \ldots, \theta_{k-1}$ are defined as above and the mapping Θ_k is an embedding isomorphism onto its range. Notice that the products in the groups $E(\mathbb{T})^\infty$, $E(\mathbb{T})^{k-1}$ and $E(\mathbb{T})^{k-1} \times \mathbb{T}$ are given by

$$(\theta_1, \theta_2, \ldots)(\theta_1', \theta_2', \ldots) = (\varphi_1, \varphi_2, \ldots),$$

$$(\theta_1, \ldots, \theta_{k-1})(\theta_1^{'}, \ldots, \theta_{k-1}^{'}) = (\varphi_1, \ldots, \varphi_{k-1}) \text{ and}$$
$$(\theta_1, \ldots, \theta_{k-1}, u)(\theta_1^{'}, \ldots, \theta_{k-1}^{'}, u^{'}) = (\varphi_1, \ldots, \varphi_{k-1}, z),$$

where for $i = 1, 2, \ldots$, one has $\varphi_i = \prod_{j=0}^{i} \theta_{i-j} \circ \theta_j^{'}$ with $\theta_i, \theta_i^{'} \in E(\mathbb{T})$, and also $z = u^{'} \prod_{j=1}^{k-1} \theta_{k-j} \circ \theta_j^{'}(\lambda)u$, for $\theta_i, \theta_i^{'} \in E(\mathbb{T})$ and $u, u^{'} \in \mathbb{T}$.

In this paper, we characterize the Furstenberg-Ellis-Namioka structure of the groups $\Sigma(\mathbb{T}^k, T_k)$ and $\Sigma(\mathbb{T}^\infty, T_\infty)$ (Theorem 2.2) explicitly. In [11] the structure of the group $\Sigma(\mathbb{T}^3, T_3)$ is discussed in detail.

2. THE MAIN RESULT

Assume that $\mathbb{Q}$ is the set of all rational numbers. Let $\mathbb{T}_\mathbb{Q}$ denote the torsion subgroup of $\mathbb{T}$, that is $\mathbb{T}_\mathbb{Q} = \{x \in \mathbb{T};\ x^n = 1, \text{ for some } n \in \mathbb{Z}\} = \{e^{2\pi i q};\ q \in \mathbb{Q}\}$. For given $t \in \mathbb{T}$ and $U \subseteq \mathbb{T}$, put $B(t, U) = \{\varphi \in E(\mathbb{T});\ \varphi(t) \in U\}$. Then the family $\{B(t, U), t \in \mathbb{T},\ U \subseteq \mathbb{T}$ an open set containing $1\}$ forms a sub-base of $E(\mathbb{T})$ around the element $1_\mathbb{T}$, where $1_\mathbb{T}(t) = 1$, for all $t \in \mathbb{T}$. In fact, the family $\{B(t, U);\ t \in \mathbb{T}_\mathbb{Q},\ U \subseteq \mathbb{T}$ an open set containing $1\}$ forms a sub-base of $E(\mathbb{T})$ around $1_\mathbb{T}$ [11]. We need the next lemma in the sequel.

LEMMA 2.1. *Let $W = B(t_1, U_1) \times \cdots \times B(t_{k-1}, U_{k-1}) \times U$, where for $j = 1, \ldots, k-1$, $t_j \in \mathbb{T}_\mathbb{Q}$ and U and U_j are open sets in $\mathbb{T}$ containing 1. Let $(1_\mathbb{T}, \psi_2, \ldots, \psi_{k-1}, v) \in E(\mathbb{T})^{k-1} \times \mathbb{T}$ with $\psi_j(t_j) = 1$, for all $j = 2, 3, \ldots, k-1$, then $(1_\mathbb{T}, \psi_2, \ldots, \psi_{k-1}, v) \in W^{-1}WW^{-1}W$, where the product in $E(\mathbb{T})^{k-1} \times \mathbb{T}$ is given above with $\lambda \in \mathbb{T} - \mathbb{T}_\mathbb{Q}$.*

Proof. Let $W \subset E(\mathbb{T})^{k-1} \times \mathbb{T}$ and $(1_\mathbb{T}, \psi_2, \ldots, \psi_{k-1}, v) \in E(\mathbb{T})^{k-1} \times \mathbb{T}$ be as stated in the lemma. Let $y = (\theta_1, \ldots, \theta_{k-1}, u) \in W^{-1}W$ be arbitrary. Let $x = (\theta_1^{'}, \ldots, \theta_{k-1}^{'}, u^{'}) \in W$ with $u^{'}v \in U$ and

$$xy = (\theta_1^{'}, \ldots, \theta_{k-1}^{'}, u^{'})(\theta_1, \ldots, \theta_{k-1}, u) \in W.$$

Then (by taking $\theta_0^{'} = id_\mathbb{T}$) one has

$$z = \left(\frac{1}{\theta_1}, \frac{\psi_2}{(\theta_1^{'} \circ \theta_1)\theta_2}, \ldots, \frac{\psi_{k-1}}{\prod_{r=1}^{k-1} \theta_{(k-1)-r}^{'} \circ \theta_r}, \frac{v}{u \prod_{r=1}^{k-1} \theta_{k-r}^{'} \circ \theta_r(\lambda)}\right) \in W^{-1}W.$$

In fact, it is enough to show that $(xy)z \in W$. A straightforward computation shows that

$$(xy)z = (\theta_1^{'}, \theta_2^{'}\psi_2, \theta_3^{'}\psi_3, \ldots, \theta_{k-1}^{'}\psi_{k-1}, u^{'}v) \in W,$$

because $\psi_j(t_j) = 1$, for $j = 2, 3, \ldots, k-1$. Thus we derive that $z \in W^{-1}W$ and hence $(1_\mathbb{T}, \psi_2, \ldots, \psi_{k-1}, v) = yz \in (W^{-1}W)(W^{-1}W)$. □

We are ready to prove our main result. In what follows $(\mathbb{T}^\infty, T_\infty)$ and $(\mathbb{T}^k, T_k)$ are the dynamical systems stated in the introduction.

THEOREM 2.2. (i) *The strong normal system* $\{N_i\}_{i\in\mathbb{N}}$ *of* $\Sigma(\mathbb{T}^\infty, T_\infty)$ *is characterized by:* $(\theta_1, \theta_2, \theta_3, \ldots) \in N_i(\Sigma(\mathbb{T}^\infty, T_\infty))$ *if and only if* $\theta_1 = \theta_2 = \ldots = \theta_i = 1_{\mathbb{T}}$.

(ii) *The strong normal system* $\{N_i, 1 \le i \le k\}$ *of* $\Sigma(\mathbb{T}^k, T_k)$ *with* $\lambda \in \mathbb{T} - \mathbb{T}_{\mathbb{Q}}$ *is characterized by:* $(\theta_1, \ldots, \theta_{k-1}, u) \in N_i(\Sigma(\mathbb{T}^k, T_k))$ *if and only if* $u \in \mathbb{T}$ *is arbitrary and* $\theta_1 = \theta_2 = \ldots = \theta_i = 1_{\mathbb{T}}$ *(for* $1 \le i \le k-1$*) and* $N_k = \{(1_{\mathbb{T}}, \ldots, 1_{\mathbb{T}}, 1)\}$.

(iii) *The strong normal system* $\{N_i, 1 \le i \le k-1\}$ *of* $\Sigma(\mathbb{T}^k, T_k)$ *with* $\lambda \in \mathbb{T}_{\mathbb{Q}}$ *is characterized by:* $(\theta_1, \ldots, \theta_{k-1}) \in N_i(\Sigma(\mathbb{T}^k, T_k))$ *if and only if* $\theta_1 = \theta_2 = \ldots = \theta_i = 1_{\mathbb{T}}$ *(for* $1 \le i \le k-1$*).*

Proof. We prove part (ii) by induction. The other two cases are proved similarly. Fix $k \ge 2$ in $\mathbb{N}$. Let $\lambda \in \mathbb{T}$ be irrational, i.e. $\lambda \in \mathbb{T} - \mathbb{T}_{\mathbb{Q}}$. Define $\Theta_k : \Sigma(\mathbb{T}^k, T_k) \to E(\mathbb{T})^{k-1} \times \mathbb{T}$ by $\Theta_k(\sigma) = (\theta_1, \ldots, \theta_{k-1}, u)$, where $u = \lim_\alpha \lambda^{P_k(n_\alpha)}$ and $\theta_1, \ldots, \theta_{k-1}$ are defined as in the introduction. The mapping Θ_k is an embedding isomorphism onto its range. Clearly $\Sigma = \Sigma(\mathbb{T}^k, T_k)$ is a CHART group. Thus

$$N(\Sigma) = \bigcap (W\cap\Sigma)^{-1}(W\cap\Sigma)(W\cap\Sigma)^{-1}(W\cap\Sigma) = \bigcap (W^{-1}W\cap\Sigma)(W^{-1}W\cap\Sigma)$$

where W runs over the local sub-basis $\mathcal{W}$ of all open sets of the form

$$W = B(t_1, U_1) \times \cdots \times B(t_{k-1}, U_{k-1} \times U),$$

in which for $j = 1, \ldots, k-1$, $t_j \in \mathbb{T}_{\mathbb{Q}}$ and U and U_j are open sets in $\mathbb{T}$ containing 1. (Actually, it is easily verified that $(W\cap\Sigma)^{-1}(W\cap\Sigma) = W^{-1}W\cap\Sigma$.) Hence,

$$N(\Sigma) = \bigcap_{W\in\mathcal{W}} (W^{-1}W \cap \Sigma)(W^{-1}W \cap \Sigma).$$

Let $(\psi_1, \ldots, \psi_{k-1}, v) \in N_1 = N(\Sigma)$. Thus for $W \in \mathcal{W}$, $(\psi_1, \ldots, \psi_{k-1}, v) \in (W^{-1}W \cap \Sigma)(W^{-1}W \cap \Sigma)$. It follows that $\psi_1(t_1) \in U_1^{-1}U_1U_1^{-1}U_1$, for each $t_1 \in \mathbb{T}_{\mathbb{Q}}$, and for any open set U_1 in $\mathbb{T}$ containing 1. Hence $\psi_1 = 1_{\mathbb{T}}$. Therefore,

$$N(\Sigma) = N_1 \subseteq \{(1_{\mathbb{T}}, \theta_2, \ldots, \theta_{k-1}, u) \in \Sigma;\ u \in \mathbb{T},\ \theta_j \in E(\mathbb{T}),\ j = 2, \ldots, k-1\}.$$

To prove the converse inclusion, let $(\psi_1, \psi_2, \ldots, \psi_{k-1}, v) \in \Sigma$ with $\psi_1 = 1_{\mathbb{T}}$. It follows from [10, Theorem 3.9 (iii)] that for each $j = 1, 2, \ldots, k-1$,

$$\psi_j(t^{j!}) = \prod_{l=1}^{j} \psi_1^{(l)}(t^{s(j,l)}) = 1, \text{ for all } t \in \mathbb{T}_{\mathbb{Q}},$$

in which $\psi_1^{(l)}$ denotes the composition of l instances of ψ_1, and $s(j,l)$ is a Stirling number of the first kind. Furthermore, for $1 \le j \le k-1$, we know that every element s of $\mathbb{T}_{\mathbb{Q}}$ can be written in the form $s = t^{j!}$, for some $t \in \mathbb{T}_{\mathbb{Q}}$. Hence $\psi_2(t) = \ldots = \psi_{k-1}(t) = 1$, for all $t \in \mathbb{T}_{\mathbb{Q}}$. Let $W = B(t_1, U_1) \times \cdots \times B(t_{k-1}, U_{k-1}) \times U$, where for $j = 1, \ldots, k-1$ $t_j \in \mathbb{T}_{\mathbb{Q}}$ and U and U_j are open sets in $\mathbb{T}$ containing 1. By Lemma 2.1, $(1_{\mathbb{T}}, \psi_2, \ldots, \psi_{k-1}, v) \in (W^{-1}W \cap$

$\Sigma)(W^{-1}W \cap \Sigma)$. In fact, it is easily verified that if $y = (\theta_1, \ldots, \theta_{k-1}, u) \in W^{-1}W \cap \Sigma$, then

$$z = \left(\frac{1}{\theta_1}, \frac{\psi_2}{(\theta_1^{'} \circ \theta_1)\theta_2}, \ldots, \frac{\psi_{k-1}}{\prod_{r=1}^{k-1} \theta^{'}_{(k-1)-r} \circ \theta_r}, \frac{v}{u\prod_{r=1}^{k-1} \theta^{'}_{k-r} \circ \theta_r(\lambda)}\right)$$

is in $W^{-1}W \cap \Sigma$. Hence $(1_{\mathbb{T}}, \psi_2, \ldots, \psi_{k-1}, v) = yz \in (W^{-1}W \cap \Sigma)(W^{-1}W \cap \Sigma)$. It follows that $(1_{\mathbb{T}}, \psi_2, \ldots, \psi_{k-1}, v) \in N(\Sigma)$. Therefore

$$N_1 = N(\Sigma) = \{(1_{\mathbb{T}}, \theta_2, \ldots, \theta_{k-1}, u) \in \Sigma;\ u \in \mathbb{T},\ \theta_j \in E(\mathbb{T}),\ j = 2, \ldots, k-1\}.$$

To continue the proof by induction, fix $1 \leq i < k-1$, and assume that

$$N_i = \{(1_{\mathbb{T}}, \ldots, 1_{\mathbb{T}}, \theta_{i+1}, \ldots, u) \in \Sigma;\ u \in \mathbb{T},\ \theta_j \in E(\mathbb{T}),\ j = i+1, \ldots, k-1\}.$$

We have to show that $N_{i+1}(\Sigma)$ coincides with the set

$$\{(1_{\mathbb{T}}, \ldots, 1_{\mathbb{T}}, \theta_{i+2}, \ldots, \theta_{k-1}, u) \in \Sigma;\ u \in \mathbb{T},\ \theta_j \in E(\mathbb{T}),\ j = i+2, \ldots, k-1\}.$$

To this end, with relativization of the σ-topology of Σ to $N_i(\Sigma)$, recall that

$$\begin{aligned} N_{i+1}(\Sigma) = N(N_i(\Sigma)) &= \bigcap_{W \in \mathcal{W}} (W^{-1}W \cap N_i(\Sigma))^{-1}(W^{-1}W \cap N_i(\Sigma)) \\ &= \bigcap_{W \in \mathcal{W}} (W^{-1}W \cap N_i(\Sigma))(W^{-1}W \cap N_i(\Sigma)). \end{aligned}$$

Similar to the proof given above, for the fact that

$$N_1(\Sigma) = \{(1_{\mathbb{T}}, \theta_2, \ldots, \theta_{k-1}, u) \in \Sigma;\ u \in \mathbb{T},\ \theta_j \in E(\mathbb{T}),\ j = 2, \ldots, k-1\},$$

by looking at the $i+1$-th component of the product of two elements in $N_i(\Sigma)$, it is straightforward to show that $N_{i+1}(\Sigma)$ is contained in the set

$$\{(1_{\mathbb{T}}, \ldots, 1_{\mathbb{T}}, \theta_{i+2}, \ldots, \theta_{k-1}, u) \in \Sigma;\ u \in \mathbb{T},\ \theta_j \in E(\mathbb{T}),\ j = i+2, \ldots, k-1\}.$$

For the converse inclusion, let $(\psi_1, \psi_2, \ldots, \psi_{k-1}, v) \in \Sigma$ with $\psi_1 = \psi_2 = \ldots = \psi_{i+1} = 1_{\mathbb{T}}$. We have to show that $(1_{\mathbb{T}}, \ldots, 1_{\mathbb{T}}, \psi_{i+2}, \ldots, \psi_{k-1}, v) \in N_{i+1}(\Sigma)$. Recall that for each $j = 1, 2, \ldots, k-1$,

$$\psi_j(t^{j!}) = \prod_{l=1}^{j} \psi_1^{(l)}(t^{s(j,l)}) = 1, \text{ for all } t \in \mathbb{T}_{\mathbb{Q}},$$

Hence

$$\psi_{i+2}(t) = \ldots = \psi_{k-1}(t) = 1, \text{ for all } t \in \mathbb{T}_{\mathbb{Q}}.$$

Let $W = B(t_1, U_1) \times \cdots \times B(t_{k-1}, U_{k-1}) \times U$, where for $j = 1, \ldots, k-1$, $t_j \in \mathbb{T}_{\mathbb{Q}}$ and U and U_j are open sets in $\mathbb{T}$ containing 1. To prove the result it is enough to show that

$$(1_{\mathbb{T}}, \ldots, 1_{\mathbb{T}}, \psi_{i+2}, \ldots, \psi_{k-1}, v) \in (W^{-1}W \cap N_i(\Sigma))(W^{-1}W \cap N_i(\Sigma)).$$

If $y = (1_{\mathbb{T}}, \ldots, 1_{\mathbb{T}}, \theta_{i+1}, \ldots, \theta_{k-1}, u) \in W^{-1}W \cap N_i(\Sigma)$, then

$$xy = (\theta_1^{'}, \ldots, \theta_{k-1}^{'}, u^{'})(1_{\mathbb{T}}, \ldots, 1_{\mathbb{T}}, \theta_{i+1}, \ldots, \theta_{k-1}, u) \in W,$$

for some $x=(\theta_1^{'},\dots,\theta_{k-1}^{'},u^{'})\in W$. Let

$$z=\left(1_{\mathbb{T}},\dots,1_{\mathbb{T}},\frac{\psi_{i+1}}{\prod_{r=1}^{i+1}\theta^{'}_{(i+1)-r}\circ\theta_r},\dots,\right.$$
$$\left.\frac{\psi_{k-1}}{\prod_{r=1}^{k-1}\theta^{'}_{(k-1)-r}\circ\theta_r},\frac{v}{u\prod_{r=1}^{k-1}\theta^{'}_{k-r}\circ\theta_r(\lambda)}\right).$$

To show that $z\in W^{-1}W$, it is enough to show that $(xy)z\in W$. A straightforward computation shows that

$$(xy)z=(\theta_1^{'},\theta_2^{'},\dots,\theta_i^{'},\theta_{i+1}^{'}\psi_{i+1},\dots,\theta_3^{'}\psi_3,\dots,\theta_{k-1}^{'}\psi_{k-1},u^{'}v).$$

But $(\theta_1^{'},\theta_2^{'},\dots,\theta_i^{'},\theta_{i+1}^{'}\psi_{i+1},\dots,\theta_3^{'}\psi_3,\dots,\theta_{k-1}^{'}\psi_{k-1},u^{'}v)\in W$, since $\psi_j(t_j)=1$, for $j=i+1,\dots,k-1$. Hence $z\in W^{-1}W\cap N_i(\Sigma)$. Furthermore,

$$(1_{\mathbb{T}},\dots,1_{\mathbb{T}},\psi_{i+2},\dots,\psi_{k-1},v)=yz\in(W^{-1}W\cap N_i(\Sigma))(W^{-1}W\cap N_i(\Sigma)).$$

Hence part (ii) follows by induction. □

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