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FRACTIONAL ORDER DIFFERENTIAL INCLUSIONS ON AN UNBOUNDED DOMAIN WITH INFINITE DELAY

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Abstract. In this paper, we provide sufficient conditions for the existence of solutions to initial value problems, for partial hyperbolic differential inclusions of fractional order involving Caputo fractional derivative with infinite delay by applying the nonlinear alternative of Frigon type for multivalued admissible contraction in Fréchet spaces.

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1. INTRODUCTION

The idea of fractional calculus and fractional order differential equations and inclusions has been a subject of interest not only among mathematicians, but also among physicists and engineers. Indeed, we can find numerous applications in rheology, control, porous media, viscoelasticity, electrochemistry, electromagnetism, etc. [16, 21]. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Kilbas et al. [24], Miller and Ross [29], the papers of Agarwal [1, 2], Belarbi et al. [6], Benchohra et al. [7, 8, 9, 10], Kilbas and Marzan [25], Vityuk [31], Vityuk and Golushkov [32], and the references therein.

Differential inclusion is a generalization of the notion of an ordinary differential equation. Therefore all problems considered for differential equations, that is, existence of solutions, continuation of solutions, dependence on initial conditions and parameters, are present in the theory of differential inclusions. Since a differential inclusion usually has many solutions starting at a given point, new issues appear, such as investigation of topological properties of the set of solutions, selection of solutions with given properties, evaluation of the reachability sets, etc. To solve the above problems special mathematical techniques were developed. Differential inclusions have been the subject of an intensive study of many researchers in the recent decades, see [4, 5, 16, 23] and the references therein.

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Differential delay equations and inclusions, or functional differential equations and inclusions, have been used in modeling scientific phenomena for many years. Often, it has been assumed that the delay is either a fixed constant or is given as an integral in which case it is called a distributed delay; see for instance the books by Hale and Verduyn Lunel [18], Helal [19], Hino *et al.* [22], Kolmanovskii and Myshkis [27], Lakshmikantham et al. [28], Samko et al. [30] and the papers [17].

This paper initiates the existence of solutions for Darboux problem for fractional partial differential inclusions in Fréchet spaces with infinite delay, for the system

$$({}^cD_0^r u)(t,x) \in F(t,x,u_{(t,x)}), \text{ if } (t,x) \in J, \tag{1}$$

$$u(t,x) = \phi(t,x), \text{ if } (t,x) \in \tilde{J}, \tag{2}$$

$$\begin{cases} u(t,0) = \varphi(t), \\ u(0,x) = \psi(x), \end{cases} \quad (t,x) \in J, \tag{3}$$

where $\varphi(0) = \psi(0)$, $J := \mathbb{R}^2$, $\tilde{J} := (-\infty,+\infty) \times (-\infty,+\infty) \backslash [0,\infty) \times [0,\infty)$, ${}^cD_0^r$ is the standard Caputo's fractional derivative of order $r = (r_1, r_2) \in (0,1] \times (0,1]$, $F : J \times \mathcal{B} \to \mathcal{P}(\mathbb{R}^n)$ is a multivalued map with compact valued, $\mathcal{P}(\mathbb{R}^n)$ is the family of all subsets of $\mathbb{R}^n$, $\phi : \tilde{J} \to \mathbb{R}^n$ is a given continuous function with $\phi(t,0) = \varphi(t)$, $\phi(0,x) = \psi(x)$ for each $(t,x) \in J, \varphi : [0,\infty) \to \mathbb{R}^n, \psi : [0,\infty) \to \mathbb{R}^n$ are given absolutely continuous functions and $\mathcal{B}$ is called a phase space that will be specified in Section 3.

We denote by $u_{(t,x)}$ the element of $\mathcal{B}$ defined by

$$u_{(t,x)}(s,\tau) = u(t+s, x+\tau); \ (s,\tau) \in (-\infty,0] \times (-\infty,0],$$

here $u_{(t,x)}(\cdot,\cdot)$ represents the history of the state u.

In this paper, we present existence results for the problem (1)–(3). Our aim here is to give global existence results for the above problem. The fundamental tools applied here are essentialy multivalued version of nonlinear alternative of Frigon type [14].

2. PRELIMINARY LEMMAS

In this section we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $J_0 = [0,n] \times [0,n]$; $n > 0$. By $C(J_0,\mathbb{R})$ we denote the Banach space of all continuous functions from J_0 into $\mathbb{R}^n$ with the norm

$$\|u\|_\infty = \sup_{(t,x)\in J_0} \|u(t,x)\|,$$

where $\|\cdot\|$ denotes a suitable complete norm on $\mathbb{R}^n$. As usual, by $AC(J_0,\mathbb{R})$ we denote the space of absolutely continuous functions from J_0 into $\mathbb{R}^n$ and

$L^1(J_0, \mathbb{R})$ is the space of Lebesgue-integrable functions $u : J_0 \to \mathbb{R}^n$ with the norm

$$\|u\|_{L^1} = \int_0^n \int_0^n \|u(t,x)\| \mathrm{d}t\mathrm{d}x.$$

Now we give some definitions and properties of fractional calculus.

Definition 2.1 ([32]). Let $r = (r_1, r_2) \in (0,\infty) \times (0,\infty), \theta = (0,0)$ and $u \in L^1(J, \mathbb{R}^n)$. The left-sided mixed Riemann-Liouville integral of order r of u is defined by

$$(I_\theta^r u)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} u(s,\tau)\mathrm{d}\tau\mathrm{d}s.$$

In particular,

$$(I_\theta^\theta u)(t,x) = u(t,x),\ (I_\theta^\sigma u)(t,x) = \int_0^t \int_0^x u(s,\tau)\mathrm{d}\tau\mathrm{d}s;$$

for almost all $(t,x) \in J$, where $\sigma = (1,1)$.

For instance, $I_\theta^r u$ exists for all $r_1, r_2 \in (0,\infty) \times (0,\infty)$, when $u \in L^1(J, \mathbb{R}^n)$. Note also that when $u \in C(J, \mathbb{R}^n)$, then $(I_\theta^r u) \in C(J, \mathbb{R}^n)$, moreover

$$(I_\theta^r u)(t,0) = (I_\theta^r u)(0,x) = 0;\ (t,x) \in J.$$

Example 2.2. Let $\lambda, \omega \in (-1,\infty)$ and $r = (r_1, r_2) \in (0,\infty) \times (0,\infty)$, then

$$I_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} t^{\lambda+r_1} x^{\omega+r_2},\ \text{for almost all } (t,x) \in J.$$

By $1-r$ we mean $(1-r_1, 1-r_2) \in [0,1) \times [0,1)$. Denote by $D_{tx}^2 := \frac{\partial^2}{\partial t \partial x}$, the mixed second order partial derivative.

Definition 2.3 ([32]). Let $r \in (0,1] \times (0,1]$ and $u \in L^1(J, \mathbb{R}^n)$. The mixed fractional Riemann-Liouville derivative of order r of u is defined by the expression

$$D_\theta^r u(t,x) = (D_{tx}^2 I_\theta^{1-r} u)(t,x)$$

and the Caputo fractional-order derivative of order r of u is defined by the expression

$$({}^c D_0^r u)(t,x) = \left(I_\theta^{1-r} \frac{\partial^2}{\partial t \partial x} u \right)(t,x).$$

The case $\sigma = (1,1)$ is included and we have

$$(D_\theta^\sigma u)(t,x) = ({}^c D_\theta^\sigma u)(t,x) = (D_{tx}^2 u)(t,x),\ \text{for almost all } (t,x) \in J.$$

Example 2.4. Let $\lambda, \omega \in (-1,\infty)$ and $r = (r_1, r_2) \in (0,1] \times (0,1]$, then

$$D_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} t^{\lambda-r_1} x^{\omega-r_2},\ \text{for almost all } (t,x) \in J.$$

In the sequel we will make use of the following generalization of Gronwall's lemma for two independent variables and singular kernel.

Lemma 2.5 ([20]). *Let $\upsilon : J \to [0,\infty)$ be a real function and $\omega(\cdot,\cdot)$ be a nonnegative, locally integrable function on J. If there are constants $c > 0$ and $0 < r_1, r_2 < 1$, such that*

$$\upsilon(t,x) \le \omega(t,x) + c\int_0^t\int_0^x \frac{\upsilon(s,\tau)}{(t-s)^{r_1}(x-\tau)^{r_2}}\mathrm{d}\tau\mathrm{d}s,$$

then there exists a constant $\delta = \delta(r_1, r_2)$, such that

$$\upsilon(t,x) \le \omega(t,x) + \delta c\int_0^t\int_0^x \frac{\omega(s,\tau)}{(t-s)^{r_1}(x-\tau)^{r_2}}\mathrm{d}\tau\mathrm{d}s,$$

for every $(t,x) \in J$.

3. THE PHASE SPACE $\mathcal{B}$

The notation of the phase space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theory for functional differential equations. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato (see [17]). For further applications see for instance the books [18, 22, 28] and their references.

For any $(t,x) \in J$ denote $E_{(t,x)} := [0,t] \times \{0\} \cup \{0\} \times [0,x]$, furthermore in case $t = a,\ x = b$ we simply write E. Consider the space $(\mathcal{B}, \|(\cdot,\cdot)\|_{\mathcal{B}})$ is a seminormed linear space of functions mapping $(-\infty,0] \times (-\infty,0]$ into $\mathbb{R}^n$, and satisfying the following fundamental axioms which were adapted from those introduced by Hale and Kato for ordinary differential functional equations:

(A_1) If $y : (-\infty,a] \times (-\infty,b] \to \mathbb{R}^n$ continuous on J and $y_{(t,x)} \in \mathcal{B}$, for all $(t,x) \in E$, then there are constants $H, K, M > 0$ such that for any $(t,x) \in J$ the following conditions hold:

(i) $y_{(t,x)}$ is in $\mathcal{B}$;

(ii) $\|y(t,x)\| \le H\|y_{(t,x)}\|_{\mathcal{B}}$,

(iii) $\|y_{(t,x)}\|_{\mathcal{B}} \le K\sup_{(s,\tau)\in[0,t]\times[0,x]}\|y(s,\tau)\| + M\sup_{(s,\tau)\in E_{(t,x)}}\|y_{(s,\tau)}\|_{\mathcal{B}}$,

(A_2) For the function $y(\cdot,\cdot)$ in (A_1), $y_{(t,x)}$ is a $\mathcal{B}$-valued continuous function on J.

(A_3) The space $\mathcal{B}$ is complete.

Now, we present some examples of phase spaces [12, 11].

Example 3.1. Let $\mathcal{B}$ be the set of all functions $\phi : (-\infty,0] \times (-\infty,0] \to \mathbb{R}^n$ which are continuous on $[-\alpha,0] \times [-\beta,0],\ \alpha,\beta \ge 0$, with the seminorm

$$\|\phi\|_{\mathcal{B}} = \sup_{(s,\tau)\in[-\alpha,0]\times[-\beta,0]}\|\phi(s,\tau)\|.$$

Then we have $H = K = M = 1$. The quotient space $\widehat{\mathcal{B}} = \mathcal{B}/\|.\|_{\mathcal{B}}$ is isometric to the space $C([-\alpha,0] \times [-\beta,0], \mathbb{R}^n)$ of all continuous functions from $[-\alpha,0] \times [-\beta,0]$ into $\mathbb{R}^n$ with the supremum norm, this means that partial differential functional equations with finite delay are included in our axiomatic model.

EXAMPLE 3.2. Let $\gamma \in \mathbb{R}$ and let C_γ be the set of all continuous functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}^n$ for which a limit $\lim_{\|(s,\tau)\| \to \infty} e^{\gamma(s+\tau)}\phi(s,\tau)$ exists, with the norm

$$\|\phi\|_{C_\gamma} = \sup_{(s,\tau)\in(-\infty,0]\times(-\infty,0]} e^{\gamma(s+\tau)}\|\phi(s,\tau)\|.$$

Then we have $H = 1$ and $K = M = \max\{e^{-\gamma(a+b)}, 1\}$.

EXAMPLE 3.3. Let $\alpha, \beta, \gamma \geq 0$ and let

$$\|\phi\|_{CL_\gamma} = \sup_{(s,\tau)\in[-\alpha,0]\times[-\beta,0]} \|\phi(s,\tau)\| + \int_{-\infty}^{0}\int_{-\infty}^{0} e^{\gamma(s+\tau)}\|\phi(s,\tau)\|\mathrm{d}\tau\mathrm{d}s.$$

be the seminorm for the space CL_γ of all functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}^n$ which are continuous on $[-\alpha, 0] \times [-\beta, 0]$ measurable on $(-\infty, -\alpha] \times (-\infty, 0] \cup (-\infty, 0] \times (-\infty, -\beta]$, and such that $\|\phi\|_{CL_\gamma} < \infty$. Then

$$H = 1,\ K = \int_{-\alpha}^{0}\int_{-\beta}^{0} e^{\gamma(s+\tau)}\mathrm{d}\tau\mathrm{d}s,\ M = 2.$$

4. SOME PROPERTIES OF SET-VALUED MAPS

Let $(X, \|\cdot\|)$ be a Banach space. Denote

- $\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$,
- $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$,
- $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$,
- $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \ \text{ compact}\}$,
- $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact and convex}\}$.

For each $u \in C(J, \mathbb{R}^n)$, define the set of selections of F by

$$S_{F,u} = \{f \in L^1(J, \mathbb{R}^n) : f(t,x) \in F(t,x,u(t,x)) \ a.e.\ (t,x) \in J\}.$$

Let (X, d) be a metric space induced from the normed space $(X, \|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\right\},$$

where $d(A, b) = \inf_{a\in A} d(a, b)$, $d(a, B) = \inf_{b\in B} d(a, b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space (see [26]).

DEFINITION 4.1. A multivalued map $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is said to be Carathéodory if

(i) $(t, x) \longmapsto F(t, x, u)$ is measurable for each $u \in \mathbb{R}^n$.
(ii) $u \longmapsto F(t, x, u)$ is upper semicontinuous for almost all $(t, x) \in J$.

F is said to be L^1-Carathéodory if (i), (ii) and the following condition holds;

(iii) for each $c > 0$, there exists $\sigma_c \in L^1(J, \mathbb{R}_+)$ such that

$$\begin{aligned} \|F(t,x,u)\|_{\mathcal{P}} &= \sup\{\|f\| : f \in F(t,x,u)\} \\ &\leq \sigma_c(t,x) \ \text{ for all } \|u\| \leq c \text{ and for } a.e.\ (t,x) \in J. \end{aligned}$$

For more details on multivalued maps see the books of Aubin and Cellina [4], Aubin et al. [5], Deimling [13], Gorniewicz [16], Hu et al. [23] and Kisielewiecz [26].

5. SOME PROPERTIES IN FRÉCHET SPACES

Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies:

$$\|u\|_1 \leq \|u\|_2 \leq \ \|u\|_3 \leq \cdots \quad \text{for every } u \in X.$$

If $Y \subset X$, we say that Y is bounded if for every $n \in \mathbb{N}$, there exists $\overline{M}_n > 0$ such that $\|v\|_n \leq \overline{M}_n$, for all $v \in Y$. To X we associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ as follows: for every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by: $u \sim_n v$ if and only if $\|u - v\|_n = 0$ for $u, v \in X$. We denote $X^n = (X|_{\sim_n}, \|\cdot\|_n)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y \subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows: for every $u \in X$, we denote $[u]_n$ the equivalence class of u of subset X^n and we defined $Y^n = \{[u]_n : u \in Y\}$. We denote $\overline{Y^n}$, $int_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|_n$ in X^n. For more information about this subject see [15].

Definition 5.1. A multivalued map $F : X \longrightarrow \mathcal{P}(X)$ is called an admissible contraction with constant $\{k_n\}_{n\in\mathbb{N}}$ if for each $n \in \mathbb{N}$ there exists $k_n \in (0, 1)$ such that

(i) $H_d(F(u), F(v)) \leq k_n||u - v||_n$ for all $u, v \in X$.
(ii) For every $u \in X$ and every $\varepsilon \in (0, \infty)^n$, there exists $v \in F(u)$ such that $||u - v||_n \leq ||u - F(u)||_n + \varepsilon_n$ for every $n \in \mathbb{N}$.

Theorem 5.2 ([14, Nonlinear alternative of Frigon type]). *Let X be a Fréchet space and U an open neighborhood of the origin in X and let $N : \overline{U} \to \mathcal{P}(X)$ be an admissible multivalued contraction. Assume that N is bounded. Then one of the following statements is holds:*

$(C1)$ *N has at least one fixed point.*
$(C2)$ *There exist $\lambda \in [0, 1)$ and $u \in \partial U$ such that $u \in \lambda N(u)$.*

6. EXISTENCE OF SOLUTIONS

In this section, we give our main existence result for problem (1)–(3). Before starting and proving this result, we give what we mean by a solution of this problem. Let the space

$$\Omega := \{u : \mathbb{R}^2 \to \mathbb{R}^n : u_{(t,x)} \in \mathcal{B} \text{ for } (t,x) \in E \text{ and } u|_J \in C(J, \mathbb{R}^n)\}.$$

DEFINITION 6.1. A function $u \in \Omega$ is said to be a solution of (1)–(3) if there exists a function $f \in L^1(J, \mathbb{R}^n)$ with $f(t,x) \in F(t,x,u_{(t,x)})$ such that $({}^cD_0^r u)(t,x) = f(t,x)$ and u satisfies equations (3) on J and the condition (2) on $\tilde{J}$.

For the existence of solutions for the problem (1)–(3), we need the following lemma.

LEMMA 6.2. *A function $u \in \Omega$ is a solution of problem* (1)–(3) *if and only if u satisfies the equation*

$$u(t,x) = z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau)\mathrm{d}\tau\mathrm{d}s$$

for all $(t,x) \in J$ and the condition (2) *on $\tilde{J}$, where $z(t,x) = \varphi(t)+\psi(x)-\varphi(0)$.*

For each $n \in \mathbb{N}$, we consider the following sets,

$$C_n = \{u : (-\infty, n] \times (-\infty, n] \to \mathbb{R}^n : u_{(t,x)} \in \mathcal{B},\ u_{(t,x)} = 0 \text{ for } (t,x) \in E \text{ and } u|_{J_0} \in C(J_0, \mathbb{R}^n)\},$$

and $C_0 = \{u \in \Omega : u_{(t,x)} = 0 \text{ for } (t,x) \in E\}$. On C_0 we define the semi-norms:

$$\|u\|_n = \sup_{(t,x)\in E} \|u_{(t,x)}\| + \sup_{(t,x)\in J_0} \|u(t,x)\| = \sup_{(t,x)\in J_0} \|u(t,x)\|, \quad u \in C_n.$$

Then C_0 is a Fréchet space with the family of semi-norms $\{\|u\|_n\}$.

Further, we present conditions for the existence and uniqueness of a solution of problem (1)–(3).

THEOREM 6.3. *Assume the following hypotheses hold:*

($H1$) *$F : J \times \mathcal{B} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is a L^1-Carathéodory map.*

($H2$) *For each $n \in \mathbb{N}$, there exists $\ell_n \in L^1(J_0, \mathbb{R}^+)$ such that*

$$H_d(F(t,x,u), F(t,x,v)) \le \ell_n(t,x)||u-v||_{\mathcal{B}}, \text{ for all } u, v \in \mathcal{B}.$$

If

$$\frac{k_n \ell_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} < 1, \tag{4}$$

where $\ell_n^* = \sup_{(t,x)\in J_0} \ell_n(t,x)$, then the problem (1)–(3) *has at least one solution on $(-\infty,\infty)\times(-\infty,\infty)$.*

Proof. Transform the problem (1)–(3) into a fixed point problem. Consider the operator $A : \Omega \to \mathcal{P}(\Omega)$ defined by $(Au)(t,x) = h \in \Omega$ such that

$$h(t,x) = \begin{cases} \phi(t,x), & (t,x) \in \tilde{J}, \\ z(t,x) \\ +\dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \\ \times f(s,\tau)\mathrm{d}\tau\mathrm{d}s,\ f \in S_{F,u} & (t,x) \in J. \end{cases}$$

Remark 6.4. For each $u \in \Omega$, the set $S_{F,u}$ is nonempty since by $(H1)$, F has a mesurable selection.

Let $v(\cdot,\cdot) : (-\infty,\infty) \times (-\infty,\infty) \to \mathbb{R}^n$ be a function defined by

$$v(t,x) = \begin{cases} z(t,x), & (t,x) \in J. \\ \phi(t,x), & (t,x) \in \tilde{J}, \end{cases}$$

Then $v_{(t,x)} = \phi$ for all $(t,x) \in E$. For each $w \in C(J,\mathbb{R}^n)$ with $w(t,x) = 0$ and for each $(t,x) \in E$ we denote by $\overline{w}$ the function defined by

$$\overline{w}(t,x) = \begin{cases} w(t,x) & (t,x) \in J, \\ 0, & (t,x) \in \tilde{J}. \end{cases}$$

If $u(\cdot,\cdot)$ satisfies the integral equation,

$$u(t,x) = z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau)\mathrm{d}\tau\mathrm{d}s,$$

we can decompose $u(\cdot,\cdot)$ as $u(t,x) = \overline{w}(t,x) + v(t,x)$; $(t,x) \in J$, which implies $u_{(t,x)} = \overline{w}_{(t,x)} + v_{(t,x)}$, for every $(t,x) \in J$, and the function $w(\cdot,\cdot)$ satisfies

$$w(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau)\mathrm{d}\tau\mathrm{d}s,$$

where $f \in S_{F,\overline{w}+v}$. Let the operators $A' : C_0 \to \mathcal{P}(C_0)$ defined by $(A'w)(t,x) = h \in C_0$, such that

$$h(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau)\mathrm{d}\tau\mathrm{d}s,$$

where $f \in S_{F,\overline{w}_{(t,x)}+v_{(t,x)}}$. Obviously, that the operator A has a fixed point is equivalent to A' has a fixed point, and so we turn to prove that A' has a fixed point. We shall show that the operator A' satisfies all conditions of Theorem 5.2 to prove that A' has a fixed point.

Let w be a possible solution of the problem $w = \lambda A'(w)$ for some $0 < \lambda < 1$. This implies by $(H2)$ that, for each $(t,x) \in J_0$, we have

$$\begin{aligned} ||w(t,x)|| &\le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \ell_n(s,\tau) \\ &\quad \times (1 + ||\overline{w}_{(s,\tau)} + v_{(s,\tau)}||_{\mathcal{B}})\mathrm{d}\tau\mathrm{d}s \\ &\le \frac{\ell_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \\ &\quad + \frac{\ell_n^*}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} ||\overline{w}_{(s,\tau)} + v_{(s,\tau)}||_{\mathcal{B}}\mathrm{d}\tau\mathrm{d}s. \end{aligned}$$

But

$$\begin{aligned}\|\overline{w}_{(s,\tau)}+v_{(s,\tau)}\|_{\mathcal{B}} &\leq \|\overline{w}_{(s,\tau)}\|_{\mathcal{B}}+\|v_{(s,\tau)}\|_{\mathcal{B}}\\ &\leq K_n\sup\{w(\tilde{s},\tilde{\tau}):(\tilde{s},\tilde{\tau})\in[0,s]\times[0,\tau]\}\\ &\quad+M_n\|\phi\|_{\mathcal{B}}+K_n\|\phi(0,0)\|.\end{aligned}\tag{5}$$

If we name $y(s,\tau)$ the right hand side of (5), then we have $\|\overline{w}_{(s,\tau)}+v_{(s,\tau)}\|_{\mathcal{B}}\leq y(t,x)$, and therefore, for each $(t,x)\in J_0$, we obtain

$$\begin{aligned}\|w(t,x)\| &\leq \frac{\ell_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\\ &\quad+\frac{\ell_n^*}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}y(s,\tau)\mathrm{d}\tau\mathrm{d}s.\end{aligned}\tag{6}$$

Using the above inequality and the definition of y for each $(t,x)\in J_0$, we have

$$\begin{aligned}y(t,x) &\leq M_n\|\phi\|_{\mathcal{B}}+K_n\|\phi(0,0)\|+\frac{K_n\ell_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\\ &\quad+\frac{K_n\ell_n^*}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}y(s,t)\mathrm{d}\tau\mathrm{d}s.\end{aligned}$$

Then by Lemma 2.5, there exists $\delta=\delta(r_1,r_2)$ such that we have

$$\|y(t,x)\| \quad\leq\quad R_n+\delta\frac{K\ell_n^*}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}R_n\mathrm{d}\tau\mathrm{d}s,$$

where

$$R_n=M_n\|\phi\|_{\mathcal{B}}+K_n\|\phi(0,0)\|+\frac{K_n\ell_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}.$$

Hence

$$\|y\|_n\leq R_n+\frac{R_n\delta K_n\ell_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}:=\widetilde{R}_n.$$

Then (6) implies that

$$\|w\|_n\leq\frac{\ell_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}(1+\widetilde{R}_n):=R_n^*.$$

Set $U'=\{w\in C_0:\|w\|_n\leq \widetilde{R}_n^*+1 \text{ for all } n\in\mathbb{N}\}$. We shall show that $A':U'\to\mathcal{P}(U')$ is a contraction and an admissible operator.

First, we prove that A' is a contraction; that is, there exists $\gamma<1$, such that

$$H_d(A'(w)-A'(w^*))\leq\gamma\|\overline{w}-\overline{w}^*\|_n,\qquad \text{for } w,w^*\in U'.$$

Let $w,w^*\in U'$ and $h\in A'(w)$. Then there exists $f(t,x)\in F(t,x,\overline{w}_{(s,\tau)}+v_{(s,\tau)})$ such that for each $(t,x)\in J_0$,

$$h(t,x)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}f(s,\tau)\mathrm{d}\tau\mathrm{d}s.$$

From $(H2)$ it follows that

$$H_d(F(t,x,\overline{w}_{(t,x)}+v_{(t,x)})-F(t,x,\overline{w^*}_{(t,x)}+v_{(t,x)}))\leq \ell_n(t,x)||\overline{w}_{(t,x)}-\overline{w}^*_{(t,x)}||.$$

Hence there exists $f^*\in F(t,x,\overline{w^*}_{(t,x)}+v_{(t,x)})$ such that

$$|f(t,x)-f^*(t,x)|\leq \ell_n(t,x)||\overline{w}_{(t,x)}-\overline{w^*}_{(t,x)}||,\quad \forall (t,x)\in J_0.$$

Let us define $\forall (t,x)\in J_0$,

$$h^*(t,x)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-\tau)^{r_2-1}f^*(s,\tau)\mathrm{d}\tau\mathrm{d}s.$$

Then we have

$$\begin{aligned}
&|h(t,x)-h^*(t,x)|\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-\tau)^{r_2-1}\\
&\times|f(s,\tau)-f^*(s,\tau)|\mathrm{d}\tau\mathrm{d}s\\
&\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-\tau)^{r_2-1}\ell_n(s,\tau)||\overline{w}_{(s,\tau)}-\overline{w^*}_{(s,\tau)}||\\
&\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-\tau)^{r_2-1}K\ell_n(s,\tau)\\
&\times\sup_{(s,\tau)\in[0,t]\times[0,x]}||\overline{w}_{(s,\tau)}-\overline{w^*}_{(s,\tau)}||\mathrm{d}\tau\mathrm{d}s\\
&\leq \frac{K\ell_n^*||\overline{w}-\overline{w^*}||_n}{\Gamma(r_1)\Gamma(r_2)}\int_0^n\int_0^n(t-s)^{r_1-1}(x-\tau)^{r_2-1}\mathrm{d}\tau\mathrm{d}s,
\end{aligned}$$

where $\ell_n^*=\sup_{(s,\tau)\in J_0}\ell_n(s,\tau)$. Therefore

$$\|h-h^*\|_n\leq\frac{K_n\ell_n^*n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\|\overline{w}-\overline{w^*}\|_n.$$

By an analogous relation, obtained by interchanging the roles of w and w^*, it follows that

$$H_d(A'(w)-A'(w^*))\leq\frac{K_n\ell_n^*n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}||\overline{w}-\overline{w^*}||_n.$$

So, A' is a contraction.

Now, $A':C_n\to\mathcal{P}_{cp}(C_n)$ is given by $(A'w)(t,x)=h\in C_n$ such that

$$h(t,x)=\begin{cases}\phi(t,x), & (t,x)\in\tilde{J},\\ \dfrac{1}{\Gamma(r_1)\Gamma(r_2)}\displaystyle\int_0^t\int_0^x(t-s)^{r_1-1}(x-\tau)^{r_2-1}f(s,\tau)\mathrm{d}\tau\mathrm{d}s, & (t,x)\in J_0,\end{cases}$$

where $f\in S^n_{F,u}=\{f\in L^1(J_0,\mathbb{R}^n):f(t,x)\in F(t,x,u_{(t,x)})\ a.e.\ (t,x)\in J_0\}$. From $(H2)$ and since F is compact valued, we can prove that for every $w\in C_n, A'(w)\in\mathcal{P}_{cp}(C_n)$ and there exists $w^*\in C_n$ such that $w^*\in A'(w^*)$ (for the proof see Benchohra *et al* [7]). Let $h\in C_n, w\in U'$ and $\varepsilon>0$. Now, if $\tilde{w}\in A'(w^*)$, then we have $||w^*-\tilde{w}||_n\leq||w^*-h||_n+||\tilde{w}-h||_n$. Since h

is arbitrary we may supose that $h \in B(\tilde{w}, \varepsilon) = \{k \in C_n : ||k - \tilde{w}||_n \le \varepsilon\}$. Therefore,

$$||w^* - \tilde{w}||_n \le ||w^* - A'(w^*)||_n + \varepsilon.$$

On the other hand, if $\tilde{w} \notin A'(w^*)$, then $||\tilde{w} - A'(w^*)||_n \neq 0$. Since $A'(w^*)$ is compact, there exists $v \in A'(w^*)$ such that $||\tilde{w} - A'(w^*)||_n = ||\tilde{w} - v||_n$. Then we have

$$||w^* - v||_n \le ||w^* - h||_n + ||v - h||_n.$$

Therefore,

$$||w^* - v||_n \le ||w^* - A'(w^*)||_n + \varepsilon.$$

So, A' is an admissible operator contraction. By our choice of U', there is no $w \in \partial U'$ such that $w \in \lambda A'(w)$, for $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of Frigon type, we deduce that A' has a fixed point which is a solution to problem (1)–(3). □

REFERENCES

[1] R.P. Agarwal, M. Belmekki and M. Benchohra, *A survey on semilinear differential equations and inclusions involving Riemann-Liouville fractional derivative*, Adv. Difference. Equ., **2009**, Article 981728, 1–47.

[2] R.P Agarwal, M. Benchohra and S. Hamani, *A survey on existence result for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta Appl. Math., **109** (2010), 973–1033.

[3] A. Arara, M. Benchohra, N. Hamidi and J.J. Nieto, *Fractional order differential equations on an unbounded domain*, Nonlinear Anal., **72** (2010), 580–586.

[4] J.P. Aubin and A. Cellina, *Differential Inclusions*, Springer-Verlag, Berlin-Heidelberg, New York, 1984.

[5] J.P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston, 1990.

[6] A. Belarbi, M. Benchohra and A. Ouahab, *Uniqueness results for fractional functional differential equations with infinite delay in Fréchet spaces*, Appl. Anal., **85** (2006), 1459-1470.

[7] M. Benchohra, L. Górniewicz, S.K. Ntouyas and A. Ouahab, *Controllability results for impulsive functional differential inclusions*, Rep. Math. Phys. **54** (2004), 211–227.

[8] M. Benchohra and M. Hellal, *Perturbed partial functional fractional order differential equations with Infinite delay*, Journal of Advanced Research in Dynamical and Control System, **5** (2013), 1–15.

[9] M. Benchohra and M. Hellal, *Perturbed partial fractional order functional differential equations with Infinite delay in Fréchet spaces*, Nonlinear Dyn. Syst. Theory, **14** (2014), 244–257.

[10] M. Benchohra and M. Hellal, *A global uniqueness result for fractional partial hyperbolic differential equations with state-dependent delay*, Ann. Polon. Math., **110** (2014), 259–281.

[11] T. Czlapinski, *Existence of solutions of the Darboux problem for partial differential-functional equations with infinite delay in a Banach space*, Comment. Math. Prace Mat., **35** (1995), 111–122.

[12] T. Czlapinski, *On the Darboux problem for partial differential-functional equations with infinite delay at derivatives*, Nonlinear Anal., **44** (2001), 389–398.

[13] K. Deimling *Multivalued Differential Equations*, Walter de Gruyter, Berlin-New York, 1992.

[14] M. Frigon, *Fixed Point Results for Multivalued Contractions on Gauge Spaces*, in *Set Valued Mappings with Applications in Nonlinear Analysis*, Ser. Math. Anal. Appl., Vol. 4, Taylor and Francis, London, 2002, 175–181.

[15] M. Frigon and A. Granas, *Résultats de type Leray-Schauder pour des contractions sur des espaces de Fréchet*, Ann. Math. Qué., **22** (1998), 161–168.

[16] L. Gorniewicz, *Topological Fixed Point Theory of Multivalued Mappings, Mathematics and its Applications*, Vol. 495, Kluwer Academic Publishers, Dordrecht, 1999.

[17] J. Hale and J. Kato, *Phase space for retarded equationswith infinite delay*, Funkcial. Ekvac., **21** (1978), 11–41.

[18] J.K. Hale and S. Verduyn Lunel, *Introduction to Functional-Differential Equations*, Appl. Math. Sci., Vol. 99, Springer-Verlag, New York, 1993.

[19] M. Helal, *Fractional Partial Hyperbolic Differential Inclusions with State-Dependent Delay*, J. Fract. Calc. Appl., **10** (2019), 179–196.

[20] D. Henry, *Geometric Theory of Semilinear Parabolic Partial Differential Equations*, Springer-Verlag, Berlin-New York, 1989.

[21] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[22] Y. Hino, S. Murakami and T. Naito, *Functional Differential Equations with Infinite Delay*, Lecture Notes in Mathematics, Vol. 1473, Springer-Verlag, Berlin, 1991.

[23] Sh. Hu and N. Papageorgiou, *Handbook of Multivalued Analysis, Theory, I*, Kluwer, Dordrecht, 1997.

[24] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Vol. 204, Elsevier Science B.V., Amsterdam, 2006.

[25] A.A. Kilbas and S.A. Marzan, *Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions*, Differ. Equ., **41** (2005), 84–89.

[26] M. Kisielewicz, *Differential Inclusions and Optimal Control*, Kluwer, Dordrecht, 1991.

[27] V. Kolmanovskii, and A. Myshkis, *Introduction to the Theory and Applications of Functional-Differential Equations*, Kluwer Academic Publishers, Dordrecht, 1999.

[28] V. Lakshmikantham, L. Wen and B. Zhang, *Theory of Differential Equations with Unbounded Delay*, Kluwer Academic Publishers, Dordrecht, 1994.

[29] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.

[30] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[31] A.N. Vityuk, *Existence of Solutions of partial differential inclusions of fractional order*, Izv. Vyssh. Uchebn. Zaved. Mat., **8** (1997), 13–19.

[32] A.N. Vityuk and A.V. Golushkov, *Existence of solutions of systems of partial differential equations of fractional order*, Nonlinear Oscillations, **7** (2004), 318–325.

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