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PERIODICITY AND POSITIVITY IN NONLINEAR NEUTRAL INTEGRO-DYNAMIC EQUATIONS WITH VARIABLE DELAY

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Abstract. Let $\mathbb{T}$ be a periodic time scale. We use Krasnoselskii's fixed point theorem for a sum of two operators to show new results on the existence of periodic and positive periodic solutions of the nonlinear neutral integro-dynamic equation with variable delay of the form

$$x^{\triangle}(t) = -\int_{t-\tau(t)}^{t} a(t,s)\, x(s)\, \Delta s + Q(t, x(t-\tau(t)))^{\triangle} + G(t, x(t), x(t-\tau(t))),\ t \in \mathbb{T}.$$

We invert this equation to construct a sum of a contraction and a completely continuous map which is suitable for applying Krasnoselskii's theorem. The uniqueness results of this equation are studied by the contraction mapping principle. The results obtained here extend the work of Mesmouli, Ardjouni and Djoudi [15].

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Key words. Fixed points, integro-dynamic equations, periodic solutions, positive solutions, time scales.

1. INTRODUCTION

In 1988, Stephan Hilger [11] introduced the theory of time scales (measure chains) as a means of unifying discrete and continuum calculi. Since Hilger's initial work there has been significant growth in the theory of dynamic equations on time scales, covering a variety of different problems; see [8, 9, 14] and references therein.

Let $\mathbb{T}$ be a periodic time scale such that $0 \in \mathbb{T}$. In this article, we are interested in the analysis of qualitative theory of periodic and positive periodic solutions of neutral integro-dynamic equations. Motivated by the papers [1]-[7], [10], [12], [13], [15]-[17] and the references therein, we consider the following

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nonlinear neutral integro-dynamic equation

$$x^{\triangle}(t) = -\int_{t-\tau(t)}^{t} a(t,s)\, x(s)\, \Delta s + Q(t, x(t-\tau(t)))^{\triangle}$$
$$+ G(t, x(t), x(t-\tau(t))),\ t \in \mathbb{T}. \tag{1}$$

Throughout this paper we assume that a and τ are positive rd-continuous functions, $id - \tau : \mathbb{T} \to \mathbb{T}$ is increasing so that the function $x(t-\tau(t))$ is well defined over $\mathbb{T}$. The functions Q and G are continuous. To reach our desired end we have to transform (1) into an integral equation written as a sum of two mapping, one is a contraction and the other is continuous and compact. After that, we use Krasnoselskii's fixed point theorem for a sum of two operators, to show the existence of periodic and positive periodic solutions. We also obtain the existence of a unique periodic solution by employing the contraction mapping principle.

The organization of this paper is as follows. In Section 2, we introduce some notations and definitions, and state some preliminary material needed in later sections. We will state some facts about the exponential function on a time scale as well as the fixed point theorems. For details on fixed point theorems we refer the reader to [10, 18]. In Section 3, we establish the existence and uniqueness of periodic solutions. In Section 4, we give sufficient conditions to ensure the existence of positive periodic solutions. The results presented in this paper extend the main results in [15].

2. PRELIMINARIES

In this section, we consider some advanced topics in the theory of dynamic equations on a time scales. Again, we remind that for a review of this topic we direct the reader to the monographs of Bohner and Peterson [8] and [9].

A time scale $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$. For $t \in \mathbb{T}$ the forward jump operator σ, and the backward jump operator ρ, respectively, are defined as $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$. These operators allow elements in the time scale to be classified as follows. We say t is right scattered if $\sigma(t) > t$ and right dense if $\sigma(t) = t$. We say t is left scattered if $\rho(t) < t$ and left dense if $\rho(t) = t$. The graininess function $\mu : \mathbb{T} \to [0, \infty)$, is defined by $\mu(t) = \sigma(t) - t$ and gives the distance between an element and its successor. We set $\inf \varnothing = \sup \mathbb{T}$ and $\sup \varnothing = \inf \mathbb{T}$. If $\mathbb{T}$ has a left scattered maximum M, we define $\mathbb{T}^k = \mathbb{T} \backslash \{M\}$. Otherwise, we define $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right scattered minimum m, we define $\mathbb{T}^k = \mathbb{T} \backslash \{m\}$. Otherwise, we define $\mathbb{T}^k = \mathbb{T}$.

Let $t \in \mathbb{T}^k$ and let $f : \mathbb{T} \to \mathbb{R}$. The delta derivative of $f(t)$, denoted $f^{\Delta}(t)$, is defined to be the number (when it exists), with the property that, for each $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s] \right| \le \varepsilon \left| \sigma(t) - s \right|,$$

for all $s \in U$. If $\mathbb{T} = \mathbb{R}$ then $f^{\Delta}(t) = f'(t)$ is the usual derivative. If $\mathbb{T} = \mathbb{Z}$ then $f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t)$ is the forward difference of f at t.

A function is right dense continuous (rd-continuous), $f \in C_{rd} = C_{rd}(\mathbb{T}, \mathbb{R})$, if it is continuous at every right dense point $t \in \mathbb{T}$ and its left-hand limits exist at each left dense point $t \in \mathbb{T}$. The function $f : \mathbb{T} \to \mathbb{R}$ is differentiable on $\mathbb{T}^k$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^k$.

We are now ready to state some properties of the delta-derivative of f. Note $f^{\sigma}(t) = f(\sigma(t))$.

Theorem 2.1 ([8, Theorem 1.20]). *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$ and let α be a scalar.*

(i) $(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t)$.

(ii) $(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t)$.

(iii) *The product rules*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)\, g(t) + f^{\sigma}(t)\, g^{\Delta}(t),$$
$$(fg)^{\Delta}(t) = f(t)\, g^{\Delta}(t) + f^{\Delta}(t)\, g^{\sigma}(t).$$

(iv) *If $g(t)\, g^{\sigma}(t) \neq 0$ then*

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)\, g(t) - f(t)\, g^{\Delta}(t)}{g(t)\, g^{\sigma}(t)}.$$

Definition 2.2 ([12]). We say that a time scale $\mathbb{T}$ is periodic if there exist a $w > 0$ such that if $t \in \mathbb{T}$ then $t \pm w \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive w is called the period of the time scale.

Definition 2.3 ([12]). Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scales with the period w. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is periodic with period T if there exists a natural number n such that $T = nw$, $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$ and T is the smallest number such that $f(t \pm T) = f(t)$. If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $T > 0$ if T is the smallest positive number such that $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$.

Remark 2.4 ([12]). If $\mathbb{T}$ is a periodic time scale with period w, then $\sigma(t \pm nw) = \sigma(t) \pm nw$. Consequently, the graininess function μ satisfies $\mu(t \pm nw) = \sigma(t \pm nw) - (t \pm nw) = \sigma(t) - t = \mu(t)$ and so, is a periodic function with period w.

The next theorem is the chain rule on time scales ([8, Theorem 1.93]).

Theorem 2.5 (Chain Rule). *Assume $v : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := v(\mathbb{T})$ is a time scale. Let $w : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $v^{\Delta}(t)$ and $w^{\widetilde{\Delta}}(v(t))$ exist for $t \in \mathbb{T}^k$, then $(w \circ v)^{\Delta} = (w^{\widetilde{\Delta}} \circ v) v^{\Delta}$.*

In the sequel we will need to differentiate and integrate functions of the form $f(t - \tau(t)) = f(v(t))$ where, $v(t) := t - \tau(t)$. Our next theorem is the substitution rule ([8, Theorem 1.98]).

THEOREM 2.6. *Assume $v : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := v(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous function and v is differentiable with rd-continuous derivative, then for $a, b \in T$,*

$$\int_a^b f(t)v^\Delta(t)\Delta t = \int_{v(a)}^{v(b)} \left(f \circ v^{-1}\right)(s)\, \widetilde{\Delta} s.$$

A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t)\, p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$. The set of all positively regressive functions $\mathcal{R}^+$, is given by $\mathcal{R}^+ = \{f \in \mathcal{R} : 1 + \mu(t)\, f(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Let $p \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The exponential function on $\mathbb{T}$ is defined by

$$e_p(t, s) = \exp\left(\int_s^t \frac{1}{\mu(z)} \log\left(1 + \mu(z)\, p(z)\right) \Delta z\right).$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t, s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t, s)$ is the solution to the initial value problem $y^\Delta = p(t)\, y$, $y(s) = 1$. Other properties of the exponential function are given by the following lemma.

LEMMA 2.7 ([8, Theorem 2.36]). *Let $p, q \in \mathcal{R}$. Then*

(i) $e_0(t, s) = 1$ *and* $e_p(t, t) = 1$,

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)\, p(t))\, e_p(t, s)$,

(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t, s)$, *where* $e_{\ominus p}(t, s) = -\frac{p(t)}{1+\mu(t)p(t)}$,

(iv) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(t, s)$,

(v) $e_p(t, s)\, e_p(s, r) = e_p(t, r)$,

(vi) $e_p^\Delta(., s) = p e_p(., s)$ *and* $\left(\frac{1}{e_p(.,s)}\right)^\Delta = -\frac{p(t)}{e_p^\sigma(.,s)}$.

THEOREM 2.8 ([7, Theorem 2.1]). *Let $\mathbb{T}$ be a periodic time scale with period $w > 0$. If $p \in C_{rd}(\mathbb{T})$ is a periodic function with the period $T = nw$, then*

$$\int_{a+T}^{b+T} p(u)\Delta u = \int_a^b p(u)\Delta u,\ \ e_p(b + T, a + T) = e_p(b, a) \text{ if } p \in \mathcal{R},$$

and $e_p(t + T, t)$ is independent of $t \in \mathbb{T}$ whenever $p \in \mathcal{R}$.

LEMMA 2.9 ([1]). *If $p \in \mathcal{R}^+$, then*

$$0 < e_p(t, s) \leq \exp\left(\int_s^t p(u)\, \Delta u\right),\ \forall t \in \mathbb{T}.$$

COROLLARY 2.10 ([1]). *If $p \in \mathcal{R}^+$ and $p(t) < 0$ for all $t \in \mathbb{T}$, then for all $s \in \mathbb{T}$ with $s \leq t$ we have*

$$0 < e_p(t, s) \leq \exp\left(\int_s^t p(u)\, \Delta u\right) < 1.$$

Lastly in this section, we state the fixed point theorems which enable us to prove the existence and uniqueness of solutions to equation (1). For its proof we refer the reader to [10, 18].

THEOREM 2.11 (Contraction mapping principle). *Let* (X, d) *be a complete metric space and* $P : X \to X$. *If there is a constant* $\alpha \in (0, 1)$ *such that for* $x, y \in X$ *we have* $d(Px, Py) \leq \alpha d(x, y)$, *then there is one and only one point* $z \in X$ *with* $Pz = z$.

THEOREM 2.12 (Krasnoselskii). *Let* $\mathcal{M}$ *be a closed convex nonempty subset of a Banach space* $(X, \|.\|)$. *Suppose that* A *and* B *map* $\mathcal{M}$ *into* X *such that*
(i) $x, y \in \mathcal{M} \Longrightarrow Ax + By \in \mathcal{M}$,
(ii) A *is compact and continuous,*
(iii) B *is a contraction mapping.*
Then there exists $z \in \mathcal{M}$ *with* $z = Az + Bz$.

3. EXISTENCE AND UNIQUENESS OF PERIODIC SOLUTIONS

Let $T > 0$, $T \in \mathbb{T}$ be fixed and if $\mathbb{T} \neq \mathbb{R}$, $T = nw$ for some $n \in \mathbb{N}$. By the notation $[a, b]$ we mean $[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}$, unless otherwise specified. The intervals $[a, b)$, $(a, b]$ and (a, b) are defined similarly. Define

$$C_T = \{\phi \in C_{rd}(\mathbb{T}, \mathbb{R}) : \phi(t + T) = \phi(t)\}$$

where $C_{rd}(\mathbb{T}, \mathbb{R})$ is the space of rd-continuous functions on $\mathbb{T}$. Then $(C_T, \|.\|)$ is a Banach space when it is endowed with the supremum norm

$$\|x\| = \sup_{t \in [0,T]} |x(t)|.$$

We will need the following lemma whose proof can be found in [12].

LEMMA 3.1. *Let* $x \in C_T$. *Then* $\|x^\sigma\| = \|x \circ \sigma\|$ *exists and* $\|x^\sigma\| = \|x\|$.

In this paper we assume that

$$a(t - T, s - T) = a(t, s), \ \tau(t - T) = \tau(t), \ \tau(t) \geq \tau^* > 0, \tag{2}$$

with τ continuously delta-differentiable and τ^* is constant, a is positive rd-continuous function and

$$1 - e_h(t, t - T) \equiv \frac{1}{\eta} > 0, \tag{3}$$

where h is given as below. Functions $Q(.,.)$ and $G(.,.,.)$ are periodic in t of period T. That is

$$Q(t - T, x) = Q(t, x), \ G(t - T, x, y) = G(t, x, y). \tag{4}$$

The following lemmas are fundamental to our results.

Lemma 3.2. *The equation (1) is equivalent to*

$$\left[x(t) - \int_{t-\tau(t)}^{t} b(t,s)\,x(s)\,\Delta s - Q(t, x(t-\tau(t)))\right]^{\triangle} \tag{5}$$
$$= h(t)\,x(t-\tau(t)) + G(t, x(t), x(t-\tau(t))),$$

where

$$h(t) = b(\sigma(t), t-\tau(t))\left(1-\tau^{\triangle}(t)\right),$$

and

$$b(t,s) = \int_{t}^{\sigma(s)} a(u,s)\,\Delta u,\ \ b(t, t-\tau(t)) = \int_{t}^{\sigma(t-\tau(t))} a(u, t-\tau(t))\,\Delta u.$$

Proof. Differentiating the first integral term in (5), we obtain

$$\left(\int_{t-\tau(t)}^{t} b(t,s)\,x(s)\,\Delta s\right)^{\triangle}$$
$$= b(\sigma(t), t)\,x(t) - b(\sigma(t), t-\tau(t))\left(1-\tau^{\triangle}(t)\right)x(t-\tau(t))$$
$$+ \int_{t-\tau(t)}^{t} b^{\triangle}(t,s)\,x(s)\,\Delta s.$$

Substituting this into (5), it follows that (5) is equivalent to (1) provided b satisfies the following conditions

$$b(\sigma(t), t) = 0 \text{ and } b^{\triangle}(t,s) = -a(t,s). \tag{6}$$

This equality implies

$$b(t,s) = -\int_{0}^{t} a(u,s)\,\Delta u + \phi(s), \tag{7}$$

for some functions ϕ and $b(t,s)$ must satisfy

$$b(t,t) = -\int_{0}^{t} a(u,t)\,\Delta u + \phi(t) = \int_{t}^{\sigma(t)} a(u,t)\,\Delta u.$$

Consequently $\phi(t) = \int_0^{\sigma(t)} a(u,t)\,\Delta u$. Substituting this into (7), we obtain

$$b(t,s) = -\int_{0}^{t} a(u,s)\,\Delta u + \int_{0}^{\sigma(s)} a(u,s)\,\Delta u = \int_{t}^{\sigma(s)} a(u,s)\,\Delta u.$$

This definition of b satisfies (6). Consequently, (1) is equivalent to (5). $\square$

LEMMA 3.3. *Suppose (2)–(4) hold. If $x \in C_T$, then x is a solution of equation (1) if and only if*

$$\begin{aligned} x(t) &= Q(t, x(t-\tau(t))) + \int_{t-\tau(t)}^{t} (b(t,s) - h(s))\, x(s)\, \Delta s \\ &\quad + \eta \int_{t-T}^{t} k(t,s)\, h(s) \\ &\quad \times \left[\int_{s-\tau(s)}^{s} (b(s,u) - h(u))\, x(u)\, \Delta u + Q(s, x(s-\tau(s))) \right] \Delta s \\ &\quad + \int_{t-T}^{t} k(t,s) \left[c(s)\, x(s-\tau(s)) + G(s, x(s), x(s-\tau(s))) \right] \Delta s, \end{aligned} \tag{8}$$

where $c(t) = h(t) - \left(1 - \tau^{\triangle}(t)\right) h(t - \tau(t))$, *and*

$$k(t,s) = e_h(t, \sigma(s)). \tag{9}$$

Proof. Let $x \in C_T$ be a solution of (1). Rewrite the equation (5) as

$$\begin{aligned} &\left[x(t) - \int_{t-\tau(t)}^{t} b(t,s)\, x(s)\, \Delta s - Q(t, x(t-\tau(t))) \right]^{\triangle} \\ &= h(t)\, x(t) - h(t)\, x(t) + h(t)\, x(t-\tau(t)) + G(t, x(t), x(t-\tau(t))) \\ &= h(t)\, x(t) - \left(\int_{t-\tau(t)}^{t} h(u)\, x(u)\, \Delta u \right)^{\triangle} + G(t, x(t), x(t-\tau(t))) \\ &+ \left[h(t) - \left(1 - \tau^{\triangle}(t)\right) h(t-\tau(t)) \right] x(t-\tau(t)) + G(t, x(t), x(t-\tau(t))). \end{aligned}$$

We put $h(t) - \left(1 - \tau^{\triangle}(t)\right) h(t - \tau(t)) = c(t)$, we obtain

$$\begin{aligned} &\left[x(t) - \int_{t-\tau(t)}^{t} (b(t,s) - h(s))\, x(s)\, \Delta s - Q(t, x(t-\tau(t))) \right]^{\triangle} \\ &= h(t) \left[x(t) - \int_{t-\tau(t)}^{t} (b(t,s) - h(s))\, x(s)\, \Delta s - Q(t, x(t-\tau(t))) \right] \\ &+ h(t) \left[\int_{t-\tau(t)}^{t} (b(t,s) - h(s))\, x(s)\, \Delta s + Q(t, x(t-\tau(t))) \right] \\ &+ c(t)\, x(t-\tau(t)) + G(t, x(t), x(t-\tau(t))). \end{aligned}$$

Multiply both sides of the above equation by $e_{\ominus h}(\sigma(t),0)$ and then integrate from $t-T$ to t to obtain

$$\begin{aligned}&\int_{t-T}^{t}\left[\left(x(s)-\int_{s-\tau(s)}^{s}(b(s,u)-h(u))x(u)\Delta u-Q(s,x(s-\tau(s)))\right)\right.\\&\left.\times e_{\ominus h}(s,0)\right]^{\triangle}\Delta s\\&=\int_{t-T}^{t}h(s)\left[\int_{s-\tau(s)}^{s}(b(s,u)-h(u))x(u)\Delta u+Q(s,x(s-\tau(s)))\right]\\&\times e_{\ominus h}(\sigma(s),0)\Delta s\\&+\int_{t-T}^{t}[c(s)x(s-\tau(s))+G(s,x(s),x(s-\tau(s)))]e_{\ominus h}(\sigma(s),0)\Delta s.\end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned}&(e_{\ominus h}(t,0)-e_{\ominus h}(t-T,0))\\&\times\left(x(t)-\int_{t-\tau(t)}^{t}(b(t,u)-h(u))x(u)\Delta u-Q(t,x(t-\tau(t)))\right)\\&=\int_{t-T}^{t}h(s)\left[\int_{s-\tau(s)}^{s}(b(s,u)-h(u))x(u)\Delta u+Q(s,x(s-\tau(s)))\right]\\&\times e_{\ominus h}(\sigma(s),0)\Delta s\\&+\int_{t-T}^{t}[c(s)x(s-\tau(s))+G(s,x(s),x(s-\tau(s)))]e_{\ominus h}(\sigma(s),0)\Delta s.\end{aligned}$$

By dividing both sides of the above equation by $e_{\ominus h}(t,0)$ and using the fact that $x(t)=x(t-T)$, we obtain

$$\begin{aligned}&x(t)-\int_{t-\tau(t)}^{t}(b(t,u)-h(u))x(u)\Delta u-Q(t,x(t-\tau(t)))\\&=\eta\int_{t-T}^{t}h(s)\left[\int_{s-\tau(s)}^{s}(b(s,u)-h(u))x(u)\Delta u+Q(s,x(s-\tau(s)))\right]\\&\times e_h(t,\sigma(s))\Delta s\\&+\eta\int_{t-T}^{t}[c(s)x(s-\tau(s))+G(s,x(s),x(s-\tau(s)))]e_h(t,\sigma(s))\Delta s.\end{aligned}\tag{10}$$

Then (10) is equivalent to (8), where $1-e_h(t,t-T)\equiv\frac{1}{\eta}$. The converse implication is easily obtained and the proof is complete. □

By applying Theorems 2.11 and 2.12, we obtain in this section the existence and the uniqueness of periodic solution of (1). So, let a Banach space $(C_T,\|.\|)$, a closed bounded convex subset of C_T,

$$\mathcal{M}=\{\varphi\in C_T,\ \|\varphi\|\leq L\},\tag{11}$$

with $L > 0$, and by Lemma 3.3, let a mapping P given by

$$
\begin{aligned}
(P\varphi)(t) = {} & Q(t, \varphi(t - \tau(t))) + \int_{t-\tau(t)}^{t} (b(t,s) - h(s))\, \varphi(s)\, \Delta s \\
& + \eta \int_{t-T}^{t} k(t,s)\, h(s) \\
& \times \left[\int_{s-\tau(s)}^{s} (b(s,u) - h(u))\, \varphi(u)\, \Delta u + Q(s, \varphi(s - \tau(s))) \right] \Delta s \\
& + \eta \int_{t-T}^{t} k(t,s) \left[c(s)\, \varphi(s - \tau(s)) + G(s, \varphi(s), \varphi(s - \tau(s))) \right] \Delta s. \qquad (12)
\end{aligned}
$$

Therefore, we express equation (12) as $P\varphi = A\varphi + B\varphi$, where A and B are given by

$$
\begin{aligned}
(A\varphi)(t) = {} & \eta \int_{t-T}^{t} k(t,s)\, h(s) \\
& \times \left[\int_{s-\tau(s)}^{s} (b(s,u) - h(u))\, \varphi(u)\, \Delta u + Q(s, \varphi(s - \tau(s))) \right] \Delta s \\
& + \eta \int_{t-T}^{t} k(t,s) \left[c(s)\, \varphi(s - \tau(s)) + G(s, \varphi(s), \varphi(s - \tau(s))) \right] \Delta s, \qquad (13)
\end{aligned}
$$

and

$$(B\varphi)(t) = Q(t, \varphi(t - \tau(t))) + \int_{t-\tau(t)}^{t} (b(t,s) - h(s))\, \varphi(s)\, \Delta s. \qquad (14)$$

We assume that Q and G are also globally Lipschitz continuous in x and in x and y, respectively. So, there are positive constants k_1, k_2 and k_3 such that

$$|Q(t,x) - Q(t,y)| \le k_1 \|x - y\|, \qquad (15)$$

$$|G(t,x,y) - G(t,z,w)| \le k_2 \|x - z\| + k_3 \|y - w\|. \qquad (16)$$

By a series of steps we will prove the fulfillment of (i), (ii) and (iii) in Theorem 2.12. So that, since $\varphi \in C_T$ and (2)–(4) hold, we have for $\varphi \in \mathcal{M}$

$$(A\varphi)(t + T) = (A\varphi)(t) \text{ and } A\varphi \in C_{rd}(\mathbb{T}, \mathbb{R}) \Rightarrow A\mathcal{M} \subset C_T, \qquad (17)$$

and

$$(B\varphi)(t + T) = (B\varphi)(t) \text{ and } B\varphi \in C_{rd}(\mathbb{T}, \mathbb{R}) \Rightarrow B\mathcal{M} \subset C_T. \qquad (18)$$

Lemma 3.4. *Suppose* (2)–(4), (15) *and* (16) *hold. If A is defined by* (13), *then A is continuous and the image of A is contained in a compact set.*

Proof. Let $\varphi_n \in \mathcal{M}$ where n is a positive integer such that $\varphi_n \to \varphi$ as $n \to \infty$. Then

$$\begin{aligned}
&\left|\left(A\varphi_n\right)(t) - \left(A\varphi\right)(t)\right| \\
&\leq \eta \int_{t-T}^{t} k(t,s)\, h(s) \left[\int_{s-\tau(s)}^{s} \left(b(s,u) - h(u)\right) \left|\varphi_n(u) - \varphi(u)\right| \Delta u \right. \\
&\left. + \left|Q\left(s, \varphi_n\left(s-\tau(s)\right)\right) - Q\left(s, \varphi\left(s-\tau(s)\right)\right)\right|\right] \Delta s \\
&+ \eta \int_{t-T}^{t} k(t,s) \left[\left|c(s)\right| \left|\varphi_n\left(s-\tau(s)\right) - \varphi\left(s-\tau(s)\right)\right|\right. \\
&\left. + \left|G\left(s, \varphi_n(s), \varphi_n\left(s-\tau(s)\right)\right) - G\left(s, \varphi(s), \varphi\left(s-\tau(s)\right)\right)\right|\right] \Delta s.
\end{aligned}$$

Since Q and G are continuous, the Dominated Convergence Theorem implies,

$$\lim_{n\to\infty} \left|\left(A\varphi_n\right)(t) - \left(A\varphi\right)(t)\right| = 0,$$

then A is continuous. Next, we show that the image of A is contained in a compact set, by (15) and (16), we obtain

$$\left|Q(t,y)\right| \leq \left|Q(t,y) - Q(t,0) + Q(t,0)\right| \leq k_1 \|y\| + \left|Q(t,0)\right|,$$

and

$$\begin{aligned}
\left|G(t,x,y)\right| &\leq \left|G(t,x,y) - G(t,0,0) + G(t,0,0)\right| \\
&\leq k_2 \|x\| + k_3 \|y\| + \left|G(t,0,0)\right|.
\end{aligned}$$

Let $\varphi_n \in \mathcal{M}$ where n is a positive integer, then

$$\begin{aligned}
&\left|\left(A\varphi_n\right)(t)\right| \leq \eta \int_{t-T}^{t} k(t,s) \left|h(s)\right| \\
&\times \left[\int_{s-\tau(s)}^{s} \left(\left|b(s,u)\right| + \left|h(u)\right|\right) \left|\varphi_n(u)\right| \Delta u + \left|Q\left(s, \varphi_n\left(s-\tau(s)\right)\right)\right|\right] \Delta s \\
&+ \eta \int_{t-T}^{t} k(t,s) \left[\left|c(s)\right| \left|\varphi_n\left(s-\tau(s)\right)\right| + \left|G\left(s, \varphi_n(s), \varphi_n\left(s-\tau(s)\right)\right)\right|\right] \Delta s \\
&\leq \eta\delta T \left[\theta\left(\alpha\left(\zeta+\theta\right) L + k_1 L + \beta\right) + \lambda L + k_2 L + k_3 L + \gamma\right] = E,
\end{aligned}$$

where

$$\begin{aligned}
&\alpha = \sup_{t\in[0,T]} \left|\tau(t)\right|,\ \beta = \sup_{t\in[0,T]} \left|Q(t,0)\right|,\ \gamma = \sup_{t\in[0,T]} \left|G(t,0,0)\right|, \\
&\lambda = \sup_{t\in[0,T]} \left|c(t)\right|,\ \delta = \sup_{t\in[t,t+T]} \left\{k(t,s)\right\}, \\
&\zeta = \sup_{t\in[t,t+T]} \left|b(t,s)\right|,\ \theta = \sup_{t\in[0,T]} \left|h(t)\right|.
\end{aligned}$$

Second, we calculate $(A\varphi_n)^{\triangle}(t)$ and show that it is uniformly bounded. By taking the derivative in (13) that

$$\begin{aligned}(A\varphi_n)^{\triangle}(t) &= h(t)(A\varphi_n)(t)\\ &\quad + h(t)\left[\int_{t-\tau(t)}^{t}(b(t,s)-h(s))\varphi_n(s)\Delta s + Q(t,\varphi_n(t-\tau(t)))\right]\\ &\quad + c(t)\varphi_n(t-\tau(t)) + G(t,\varphi_n(t),\varphi_n(t-\tau(t))).\end{aligned}$$

then

$$\left\|(A\varphi_n)^{\triangle}\right\| \leq \theta E + \frac{E}{\eta\delta T}.$$

Thus the sequence $(A\varphi_n)$ is uniformly bounded and equicontinuous. Hence by Ascoli-Arzela's theorem $A(\mathcal{M})$ is contained in a compact subset of C_T. $\square$

Lemma 3.5. *Suppose* (2)–(4), (15) *hold and*

$$k_1 + \alpha(\zeta+\theta) < 1. \tag{19}$$

If B is defined by (14)*, then B is a contraction.*

Proof. Let B be defined by (14). Then for $\varphi_1, \varphi_2 \in \mathcal{M}$ we have by (15)

$$\begin{aligned}|(B\varphi_1)(t) - (B\varphi_2)(t)| &\leq |Q(t,\varphi_1(t-\tau(t))) - Q(t,\varphi_2(t-\tau(t)))|\\ &\quad + \left|\int_{t-\tau(t)}^{t}(b(t,s)-h(s))(\varphi_1(s)-\varphi_2(s))\Delta s\right|\\ &\leq (k_1 + \alpha(\zeta+\theta))\|\varphi_1-\varphi_2\|.\end{aligned}$$

Hence B is contraction by (19). $\square$

Theorem 3.6. *Suppose the assumptions of Lemmas 3.4 and 3.5 hold. If there exists a constant $L>0$ defined in $\mathcal{M}$ such that*

$$\begin{aligned}&\eta\delta T\left[\theta(\alpha(\zeta+\theta)L + k_1L + \beta) + \lambda L + (k_2+k_3)L + \gamma\right]\\ &\quad + k_1L + \beta + \alpha(\zeta+\theta)L \ \leq\ L.\end{aligned}$$

Then (1) has a T-periodic solution.

Proof. By Lemma 3.4, $A:\mathcal{M}\to C_T$ is continuous and $A(\mathcal{M})$ is contained in a compact set. Also, from Lemma 3.5, the mapping $B:\mathcal{M}\to C_T$ is a contraction. Next, we show that if $\varphi,\phi\in\mathcal{M}$, we have$\|A\varphi + B\phi\| \leq L$. Let $\varphi,\phi\in\mathcal{M}$ with $\|\varphi\|,\|\phi\|\leq L$. Then

$$\begin{aligned}\|A\varphi + B\phi\| &\leq \eta\delta T\left[\theta(\alpha(\zeta+\theta)L + k_1L + \beta) + \lambda L + (k_2+k_3)L + \gamma\right]\\ &\quad + k_1L + \beta + \alpha(\zeta+\theta)L \leq L.\end{aligned}$$

Clearly, all the hypotheses of Krasnoselskii's theorem are satisfied. Thus there exists a fixed point $z\in\mathcal{M}$ such that $z = Az + Bz$. By Lemma 3.2 this fixed point is a solution of (1). Hence (1) has a T-periodic solution. $\square$

THEOREM 3.7. *Suppose* (2)–(4), (15) *and* (16) *hold. If*

$$\eta\delta T\left[\theta\left(\alpha\left(\zeta+\theta\right)+k_1\right)+\lambda+k_2+k_3\right]+k_1+\alpha\left(\zeta+\theta\right)<1, \tag{20}$$

then equation (1) *has a unique T-periodic solution.*

Proof. Let the mapping P be given by (12). For $\varphi_1,\varphi_2\in C_T$, we have

$$\begin{aligned}
&\left|\left(P\varphi_1\right)(t)-\left(P\varphi_2\right)(t)\right|\\
&\le\left|Q\left(t,\varphi_1\left(t-\tau\left(t\right)\right)\right)-Q\left(t,\varphi_2\left(t-\tau\left(t\right)\right)\right)\right|\\
&+\int_{t-\tau(t)}^{t}\left(\left|b\left(t,s\right)\right|+\left|h\left(s\right)\right|\right)\left|\varphi_1\left(s\right)-\varphi_2\left(s\right)\right|\Delta s\\
&+\eta\int_{t-T}^{t}k\left(t,s\right)h\left(s\right)\left[\int_{s-\tau(s)}^{s}\left(\left|b\left(s,u\right)\right|+\left|h\left(u\right)\right|\right)\left|\varphi_1\left(u\right)-\varphi_2\left(u\right)\right|\Delta u\right.\\
&\left.+\left|Q\left(s,\varphi_1\left(s-\tau\left(s\right)\right)\right)-Q\left(s,\varphi_2\left(s-\tau\left(s\right)\right)\right)\right|\right]\Delta s\\
&+\eta\int_{t-T}^{t}k\left(t,s\right)\left[\left|c\left(s\right)\right|\left|\varphi_1\left(s-\tau\left(s\right)\right)-\varphi_2\left(s-\tau\left(s\right)\right)\right|\right.\\
&\left.+\left|G\left(s,\varphi_1\left(s\right),\varphi_1\left(s-\tau\left(s\right)\right)\right)-G\left(s,\varphi_2\left(s\right),\varphi_2\left(s-\tau\left(s\right)\right)\right)\right|\right]\Delta s,\\
&=\left[\eta\delta T\left[\theta\left(\alpha\left(\zeta+\theta\right)+k_1\right)+\lambda+k_2+k_3\right]+k_1+\alpha\left(\zeta+\theta\right)\right]\left\|\varphi_1-\varphi_2\right\|,
\end{aligned}$$

Since (20) hold, the contraction mapping principle completes the proof. □

COROLLARY 3.8. *Suppose* (2)–(4) *hold. Let* $\mathcal{M}$ *defined by* (11). *Suppose there are positive constants* k_1^*, k_2^* *and* k_3^* *such that for* $x,y,z,w\in\mathcal{M}$, *we have*

$$\left|Q\left(t,x\right)-Q\left(t,y\right)\right|\le k_1^*\left\|x-y\right\|\ \text{ and } k_1^*<1, \tag{21}$$

$$\left|G\left(t,x,y\right)-G\left(t,z,w\right)\right|\le k_2^*\left\|x-z\right\|+k_3^*\left\|y-w\right\|, \tag{22}$$

and

$$\begin{aligned}
&\eta\delta T\left[\theta\left(\alpha\left(\zeta+\theta\right)L+k_1^*L+\beta\right)+\lambda L+\left(k_2^*+k_3^*\right)L+\gamma\right]\\
&+k_1^*L+\beta+\alpha\left(\zeta+\theta\right)L\ \le\ L.
\end{aligned} \tag{23}$$

If $\left\|P\varphi\right\|\le L$ *for* $\varphi\in\mathcal{M}$ *then* (1) *has a T-periodic solution in* $\mathcal{M}$. *Moreover, if*

$$\eta\delta T\left[\theta\left(\alpha\left(\zeta+\theta\right)+k_1^*\right)+\lambda+k_2^*+k_3^*\right]+k_1^*+\alpha\left(\zeta+\theta\right)<1, \tag{24}$$

then (1) *has a unique solution in* $\mathcal{M}$.

Proof. Let the mapping P defined by (12). Then the proof follow immediately from Theorem 3.6 and Theorem 3.7. □

Notice that the constants k_1^*, k_2^* and k_3^* may depend on L.

4. EXISTENCE OF POSITIVE PERIODIC SOLUTIONS

In this section we obtain the existence of a positive periodic solution of (1). By applying Theorem 2.12, we need to define a closed, convex, and bounded subset $\mathbb{M}$ of C_T. So, let

$$\mathbb{M} = \{\phi \in C_T : R \leq \phi \leq K\}, \tag{25}$$

where R and K are positive constants. To simplify notations and calculations, we let

$$F(t, x(t)) = \int_{t-\tau(t)}^{t} (b(t,u) - h(u))\, x(u)\, \Delta u, \tag{26}$$

such that there exist a non-negative constant b_1 and positive constant b_2 satisfy

$$b_1 x \leq F(t, x) \leq b_2 x, \tag{27}$$

and

$$m = \exp\left(-\int_0^{2T} |h(s)| \triangle s\right), \; M = \exp\left(\int_0^{2T} |h(s)| \triangle s\right). \tag{28}$$

It is easy to see that for all $(t,s) \in [0,2T]^2$,

$$m \leq k(t,s) \leq M. \tag{29}$$

Then we obtain the existence of a positive periodic solution of (1) by considering the two cases,

(1) $Q(t,y) \geq 0$ $\forall t \in [0,T]$, $y \in \mathbb{M}$.

(2) $Q(t,y) \leq 0$ $\forall t \in [0,T]$, $y \in \mathbb{M}$.

In the case one, we assume for all $t \in [0,T]$, $x, y \in \mathbb{M}$, that there exist a non-negative constant q_1 and positive constant q_2 such that

$$q_1 y \leq Q(t,y) \leq q_2 y, \tag{30}$$

$$q_2 + b_2 < 1, \tag{31}$$

$$\begin{aligned} \frac{R(1-q_1-b_1)}{m\eta T} &\leq h(t)\left[F(t,x) + Q(t,y)\right] + c(t)\, y + G(t,x,y) \\ &\leq \frac{K(1-q_2-b_2)}{M\eta T}. \end{aligned} \tag{32}$$

THEOREM 4.1. *Suppose* (27)–(32) *and the hypothesis of Lemmas* 3.4, 3.5 *hold. Then equation* (1) *has a positive T-periodic solution x in the subset* $\mathbb{M}$.

Proof. By Lemma 3.4, $A : \mathbb{M} \to C_T$ is continuous and $A(\mathbb{M})$ is contained in a compact set. Also, from the Lemma 3.5, the mapping $B : \mathbb{M} \to C_T$ is a contraction. Next, we show that if $\varphi, \psi \in \mathbb{M}$, we have $R \leq A\psi + B\varphi \leq K$. Let $\varphi, \psi \in \mathbb{M}$ with $R \leq \psi, \varphi \leq K$. By (27)–(32)

$$\begin{aligned}
&(A\psi)(t) + (B\varphi)(t) \\
&= Q(t, \varphi(t - \tau(t))) + F(t, \varphi(t)) \\
&+ \eta \int_{t-T}^{t} k(t,s) h(s) [F(s, \psi(s)) + Q(s, \psi(s - \tau(s)))] \Delta s \\
&+ \eta \int_{t-T}^{t} k(t,s) [c(s) \psi(s - \tau(s)) + G(s, \psi(s), \psi(s - \tau(s)))] \Delta s \\
&\leq \eta \int_{t-T}^{t} k(t,s) \frac{K(1 - q_2 - b_2)}{M\eta T} \Delta s + q_2 K + b_2 K \\
&\leq \eta \int_{t-T}^{t} M \frac{K(1 - q_2 - b_2)}{M\eta T} \Delta s + q_2 K + b_2 K = K.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(A\psi)(t) + (B\varphi)(t) \\
&\geq \eta \int_{t-T}^{t} k(t,s) \frac{R(1 - q_1 - b_1)}{m\eta T} \Delta s + q_1 R + b_1 R \\
&\geq \eta \int_{t-T}^{t} m \frac{R(1 - q_1 - b_1)}{m\eta T} \Delta s + q_1 R + b_1 R = R.
\end{aligned}$$

Clearly, all the hypotheses of Krasnoselskii's theorem are satisfied. Thus there exists a fixed point $z \in \mathbb{M}$ such that $z = Az + Bz$. By Lemma 3.2 this fixed point is a solution of (1) and the proof is complete. □

In the case two, we substitute conditions (30)–(32) with the following conditions respectively. We assume that there exist negative constant q_3 and a non-positive constant q_4 such that

$$q_3 y \leq Q(t, y) \leq q_4 y, \tag{33}$$

$$b_2 \leq 1, \tag{34}$$

$$\begin{aligned}
\frac{R(1 - b_1) - q_3 K}{m\eta T} &\leq h(t)[F(t,x) + Q(t,y)] + c(t) y + G(t,x,y) \\
&\leq \frac{K(1 - b_2) - q_4 R}{M\eta T}.
\end{aligned} \tag{35}$$

Theorem 4.2. *Assume* (27), (33)–(35) *and the hypothesis of Lemmas* 3.4, 3.5. *Then equation* (1) *has a positive T-periodic solution x in the subset* $\mathbb{M}$.

Proof. By Lemma 3.4, $A : \mathbb{M} \to C_T$ is continuous and $A(\mathbb{M})$ is contained in a compact set. Also, from Lemma 3.5, the mapping $A : \mathbb{M} \to C_T$ is a contraction. Next, we show that if $\varphi, \psi \in \mathbb{M}$, we have $R \leq A\psi + B\varphi \leq K$. Let $\varphi, \psi \in \mathbb{M}$ with $R \leq \varphi, \psi \leq K$. By (27) and (33)–(35)

$$\begin{aligned}
&(A\psi)(t) + (B\varphi)(t) \\
&= Q(t, \varphi(t - \tau(t))) + F(t, \varphi(t)) \\
&+ \eta \int_{t-T}^{t} k(t,s) h(s) \left[F(s, \psi(s)) + Q(s, \psi(s - \tau(s)))\right] \Delta s \\
&+ \eta \int_{t-T}^{t} k(t,s) \left[c(s) \psi(s - \tau(s)) + G(s, \psi(s), \psi(s - \tau(s)))\right] \Delta s \\
&\leq \eta \int_{t-T}^{t} k(t,s) \frac{K(1-b_2) - q_4 R}{M\eta T} \Delta s + q_4 R + b_2 K \\
&\leq \eta \int_{t-T}^{t} M \frac{K(1-b_2) - q_4 R}{M\eta T} \Delta s + q_4 R + b_2 K = K.
\end{aligned}$$

On the other hand,

$$(A\psi)(t) + (B\varphi)(t) \geq \eta \int_{t-T}^{t} m \frac{R(1-b_1) - q_3 K}{m\eta T} \Delta s + q_3 K + b_1 R = R.$$

Clearly, all the hypotheses of Krasnoselskii's theorem are satisfied. Thus there exists a fixed point $z \in \mathbb{M}$ such that $z = Az + Bz$. By Lemma 3.2 this fixed point is a solution of (1) and the proof is complete. $\square$

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