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# POSITIVE SOLUTIONS FOR DISCRETE ANISOTROPIC EQUATIONS

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**Abstract.** Using variational method, we study the existence of positive solutions for an anisotropic discrete Dirichlet problem with some functions $\alpha$,$\beta$ and a nonlinear term $f$.

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**Key words.** Discrete nonlinear boundary value problem, variational method, Ekeland's variational principle, mountain pass theorem, positive solution.

## 1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

In the recent mathematical literature a great deal of work has been devoted to the study of discrete boundary value problems. The studies of such kind of problems can be placed at the interface of certain mathematical fields, such as nonlinear differential equations and numerical analysis. More, they are strongly motivated by their applicability to various fields of research, such as computer science, mechanical engineering, control systems, artificial or biological neural networks, economics and many others. For this reasons, in these last years, there is a trend to study difference equations by using fixed point theory, lower and upper solutions method, variational methods and critical point theory, Morse theory and the mountain-pass theorem, and many interesting results have been obtained, see for instance [1], [3], [6], [7], [10], [11], [12], [13], [14], [16],[19].

Let $T$ be a positive integer, denote with $[1, T]$ the discrete interval $\{1, 2, \ldots, T\}$, $\lambda$ be a positive parameter and consider the following problem

(1)
$$\begin{cases} -\Delta\Big(\alpha(k)\varphi_{p_1(k-1)}(\Delta u(k-1)) + \beta(k)\varphi_{p_2(k-1)}(\Delta u(k-1))\Big) = \lambda f(k, u(k)), \\ \hfill k \in [1, T], \\ u(0) = u(T+1) = 0, \end{cases}$$

where $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator, $\varphi$ will stand for the homeomorphism defined by $\varphi_s(x) = |x|^{s-2}x$, $\alpha, \beta : [1, T+1] \to [0, \infty)$; $p_1, p_2 : [0, T+1] \to [2, \infty)$ and $f : [1, T] \times \mathbb{R} \to (0, \infty)$ is a continuous function, i.e. for any fixed $k \in [1, T]$ a function $f(k, .)$ is continuous.

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To the best of our knowledge, discrete problems like (1) involving anisotropic exponents have been discussed for the first time by Mihăilescu et al.[19] and for the second time by Koné and Ouaro [16], where known tools from the critical point theory are applied in order to get the existence of solutions.

In [4, 5], Ayoujil studied a parametric version of the problem (1) in the case $\alpha \equiv \beta \equiv 1$. Using variational arguments based on the direct method in the calculus of variation methods, the mountain pass lemma or Ekeland's variational principle, the author proves the existence of at least one nontrivial solution for the problems of type (1).

Galewski and Wieteska in [11] derived the intervals of the numerical parameter for which the parametric version of the problem (1) has at leas 1, exactly 1, or at least 2 positive solutions, and obtained the existence of infinitely many solutions for a parametric version of the problem (1) in case $\beta \equiv 0$

In [15], the author studied a parametric version of the problem (1) and, in the case $\alpha \equiv \beta \equiv 1$, studied the existence and the multiplicity of the solutions.

From now onwards, for all $k \in [0, T]$ and $i = 1, 2$, we will use the following notations:

$$p_{\min}(k) := \min_{i=1,2} p_i(k), \quad p_{\max}(k) := \max_{i=1,2} p_i(k),$$
$$p_{\min}^- = \min_{k\in[0,T]} p_{\min}(k), \quad p_{\max}^+ = \max_{k\in[0,T]} p_{\max}(k);$$
$$p_i^- = \min_{k\in[0,T]} p_i(k), \quad p_i^+ = \max_{k\in[0,T]} p_i(k), \quad \text{for } i = 1, 2;$$
$$\alpha^- = \min_{k\in[1,T+1]} \alpha(k), \quad \alpha^+ = \max_{k\in[1,T+1]} \alpha(k);$$
$$\beta^- = \min_{k\in[1,T+1]} \beta(k), \quad \beta^+ = \max_{k\in[1,T+1]} \beta(k);$$
$$m^+ = \max_{k\in[1,T]} m(k), \quad m^- = \max_{k\in[1,T]} m(k);$$
$$\psi_1^- = \min_{k\in[1,T]} \psi_1(k), \quad \psi_2^+ = \min_{k\in[1,T]} \psi_2(k);$$
$$\varphi_1^- = \min_{k\in[1,T]} \varphi_1(k), \quad \varphi_2^+ = \min_{k\in[1,T]} \varphi_2(k).$$

About the nonlinear term, we assume the following condition:

$(A)$ There exist a function $m : [1, T] \to [2, \infty)$ and functions $\psi_1$,$\psi_2$,$\phi_1$,$\phi_2$ : $[1, T] \to [0, \infty)$ such that

$$\psi_1(k) + \varphi_1(k)|u|^{m(k)-2}u \le f(k, u) \le \psi_2(k) + \varphi_2(k)|u|^{m(k)-2}u,$$

for all $u \ge 0$ and all $k \in [1, T]$.

Now, we will show an example of a function which satisfies condition $(A)$.

Example 1.1. Let $f : [1, T] \times \mathbb{R} \to (0, \infty)$ be given by

$$f(k, u) = ln(k + 1) + \frac{2 + arctg(u)}{T^2 k}|u|^{m(k)-2}u$$

for $(k, u) \in [1, T] \times \mathbb{R}$.

In the present paper, our goal is to use direct variational method, mountain pass geometry and Ekeland's variational principle in order to establish the existence of at least one positive solutions for the problem (1). Our results will depend on the relation between $p^-_{\min}$, $p^+_{\max}$ and $m^-$, $m^+$.

Solutions to (1) will be investigated in a space

$$H = \{u : [0, T + 1] \to \mathbb{R} : u(0) = u(T + 1) = 0\},$$

which is a T-dimensional Hilbert space, see [2], with the inner product

$$(u, v) = \sum_{k=1}^{T+1} \Delta u(k-1)\Delta v(k-1), \qquad \text{for all} \qquad u, v \in H.$$

The associated norm is defined by

$$||u|| = (\sum_{k=1}^{T+1} |\Delta u(k-1)|^2)^{\frac{1}{2}}.$$

For $u \in H$ let $u_+ = max\{u, 0\},\ u_- = max\{-u, 0\}$. Note that $u_+ \geq 0$, $u_- \geq 0$, $u = u_+ - u_-$, $u_+.u_- = 0$. Now, we can state our main results.

Theorem 1.2. *Let $m^+ < p^-_{min}$. Assume that the condition $(A)$ holds. Then for all $\lambda > 0$ the problem* (1) *has at least one positive solution.*

Theorem 1.3. *Assume that the condition $(A)$ is satisfied and $m^- > p^+_{max}$ or $m^- < p^-_{min}$ holds. Then there exists a positive constant $\lambda_0$ such that for any $\lambda \in (0, \lambda_0)$ the problem* (1) *has at least one positive solution.*

The structure of this paper is outlined as follows. In Section 2, some preliminary results and the statements of the main results are presented. In Section 3, the proofs of the main results are given.

## 2. PRELIMINARIES

The energy functional corresponding to problem (1) is defined as $J_\lambda : H \to \mathbb{R}$ by the formula

$$J_\lambda(u) = \sum_{k=1}^{T+1} (\frac{\alpha(k)}{p_1(k-1)} |\Delta u(k-1|^{p_1(k-1)} + \frac{\beta(k)}{p_2(k-1)} |\Delta u(k-1|^{p_2(k-1)}) - \lambda \sum_{k=1}^{T} F(k, u_+(k))$$

with

$$F(k, u) = \int_0^u f(k, s)ds \text{ for } u \in \mathbb{R} \text{ and } k \in [1, T] \tag{2}$$

with any fixed $\lambda > 0$. The functional $J_\lambda$ is continuously Gateaux differentiable and its Gateaux derivative $J_\lambda^{'}$ at $u$ reads

$$\begin{aligned}\langle J_\lambda^{'}(u), v\rangle := \sum_{k=1}^{T+1}(\alpha(k)|\Delta u(k-1|^{p_1(k-1)-2}\\ + \beta(k)|\Delta u(k-1|^{p_2(k-1)-2}) - \lambda\sum_{k=1}^{T} f(k, u_+(k))v(k)\end{aligned} \tag{3}$$

for all $v \in H$.

Suppose that $u$ is a critical point to $J_\lambda$, i.e. $< J_\lambda^{'}(u), v >= 0$ for all $v \in H$. Summing by parts and taking boundary values into account, see [12], we observe that

$$\begin{aligned}\sum_{k=1}^{T+1}\Delta(\alpha(k)|\Delta u(k-1)|^{p_1(k-1)-2} + \beta(k)|\Delta u(k-1|^{p_2(k-1)-2})\\ - \lambda\sum_{k=1}^{T} f(k, u_+(k))v(k) = 0.\end{aligned}$$

Since $v \in H$ is arbitrary we see that $u$ satisfies (1).

Now, we list some inequalities that will be are used later. For $(a)$ see [19], for $(b)$ see [12], for $(c)$ see [20], for $(d) - (h)$ see [11].

Lemma 2.1. (a) *For every $u \in H$ with $\|u\| \leq 1$ we have has*

$$\sum_{k=1}^{T+1}|\Delta u(k-1)|^{p(k-1)} \geq T^{\frac{p^+-2}{2}}\|u\|^{p^+}.$$

(b) *For every $u \in H$ and for every $m \geq 2$ we have*

$$\sum_{k=1}^{T+1}|\Delta u(k-1)|^m \leq 2^m\sum_{k=1}^{T+1}|u(k)|^m.$$

(c) *For every $u \in H$ and for any $p, q > 1$ such that $\frac{1}{p} + \frac{q}{q} = 1$ we have*

$$||u||_C = \max_{k\in[1,T]}|u(k)| \leq (T+1)^{\frac{1}{p}}(\sum_{k=1}^{T+1}|\Delta u(k-1)|^p)^{\frac{1}{p}}.$$

(d) *For every $u \in H$ and for every $m > 1$ we have*

$$\sum_{k=1}^{T}|u(k)|^m < T(T+1)^{m-1}\sum_{k=1}^{T+1}|\Delta u(k-1)|^m.$$

(e) *For every $u \in H$ and for every $m \geq 1$ we have*

$$\sum_{k=1}^{T+1}|\Delta u(k-1)|^m \leq (T+1)||u||^m.$$

(f) *For every* $u \in H$ *and for every* $m \geq 2$ *we have*

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^m \geq T^{\frac{2-p^-}{2}} ||u||^m.$$

(g) *For every* $u \in H$ *with* $||u|| > 1$

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq T^{\frac{2-p^-}{2}} ||u||^{p^-} - (T+1).$$

(h) *For every* $u \in H$ *we have*

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \leq (T+1)||u||^{p^+} + (T+1).$$

(i) *For every* $u \in H$ *with* $\|u\| \geq 1$ *one has*

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq T^{\frac{2-p^-}{2}} \|u\|^{p^-} - T.$$

(j) *For any* $m \geq 2$ *there exists a positive constant* $c_m$ *such that*

$$\sum_{k=1}^{T} |u(k)|^m \leq c_m \sum_{k=1}^{T+1} |\Delta u(k-1)|^m, \ \ \forall\, u \in H.$$

Next, we provide some tools that are used throughout the paper.

THEOREM 2.2 ([18]). *Let* $E$ *be a reflexive Banach space. If a functional* $J \in C^1(E, \mathbb{R})$ *is weakly lower semicontinous and coercive, i.e.* $\lim_{||u||\to\infty} J(x) = +\infty$*, then there exists* $\widetilde{x} \in E$ *such that* $\inf_{x\in E} J(x) = J(\widetilde{x})$ *and* $\widetilde{x}$ *is also a critical point of* $J$*, i.e.* $J^{'}(\widetilde{x}) = 0$*. Moreover, if* $J$ *is strictly convex, then the critical point is unique.*

THEOREM 2.3 ([8, Ekeland's principle]). *Let* $X$ *be a complete metric space and* $\Phi : X \to \mathbb{R}$ *a lower semicontinuous functon that is bounded below. Let* $\epsilon > 0$ *and* $\overline{u} \in X$ *be given such that* $\Phi(\overline{u}) \leq \inf_X \Phi + \frac{\epsilon}{2}$*. Then given* $\lambda > 0$ *there exists* $u_\lambda \in X$ *such that*
*(i)* $\Phi(u_\lambda) \leq \Phi(\overline{u})$,
*(ii)* $d(u_\lambda, \overline{u}) < \lambda$,
*(iii)* $\Phi(u_\lambda) < \Phi(u) + \frac{\epsilon}{\lambda} d(u, u_\lambda)$ *for all* $u \neq u_\lambda$*.*

DEFINITION 2.4. Let $E$ be a real Banach space. We say that a functional $J : E \to \mathbb{R}$ satisfies the Palais-Smale condition if every sequence $(u_n)$ such that $\{J(u_n)\}$ is bouned and $J^{'}(u_n) \to 0$ has a convergent subsequence.

Finally, we will provide some results that are used in the proof of the Main Theorem. The following lemma may be viewed as a kind of discrete maximum principle. These results follow as in [9].

LEMMA 2.5. *Let $\lambda > 0$. Assume that $u \in H$ is a solution of the equation*

$$\text{(4)} \qquad \begin{cases} -\Delta\Big(\alpha(k)\varphi_{p_1(k-1)}(\Delta u(k-1)) + \beta(k)\varphi_{p_2(k-1)}(\Delta u(k-1))\Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad = \lambda f(k, u_+(k)), k \in [1, T], \\ u(0) = u(T+1) = 0. \end{cases}$$

*Then $u(k) > 0$ for all $k \in [1, T]$ and, moreover, $u$ is a positive solution of* (1).

*Proof.* Note that $\Delta u(k-1)\Delta u_-(k-1) \leq 0$ for every $k \in [1, T+1]$. Assume that $u \in H$ is a solution to (4). Taking $v = u_-$ in (3) we obtain

$$\sum_{k=1}^{T+1}(\alpha(k)|\Delta u(k-1)|^{p_1(k-1)-2} + \beta(k)|\Delta u(k-1)|^{p_2(k-1)-2})$$
$$\times \Delta u(k-1)\Delta_- u(k-1) = \lambda f(k, u_+(k))u_-(k).$$

Since the term on the left is nonpositive and the one on the right is nonnegative, this equation holds true if both terms are equal to zero, which leads to $u_-(k) = 0$ for all $k \in [1, T]$. Then $u = u_+$. Moreover, $u(k) \neq$ for all $k \in [1, T]$. Indeed, assume that there exists $k \in [1, T]$ such that $u(k) = 0$. Then, by (4) we have

$$\alpha(k+1)u(k+1)^{p_1(k+1)-1} + \alpha(k)u(k-1)^{p_1(k-1)-1}$$
$$+\beta(k+1)u(k+1)^{p_2(k+1)-1} + \beta(k)u(k-1)^{p_2(k-1)-1} + \lambda f(k, 0) = 0;$$

Since $\lambda > 0$ and $f(k, 0) > 0$, we have a contradiction. Thus $u(k) \neq 0$ for all $k \in [1, T]$, and it follows $u$ is a positive solution of (1) We will prove that $J_\lambda$ satisfies the Palais-Smale condition. □

## 3. PROOFS

**Proof of Theorem 1.2**. Fix $\lambda > 0$. Since $H$ is finite dimensional and since $J_\lambda$ is Gateaux differentiable and continuous it suffices to show that it is coercive. By the condition $(A)$ and the inequalities $(c)$, $(d)$, $(e)$ and $(g)$ in Lemma(2.1), for sufficiently large $||u||$, we obtain

$$J_\lambda(u_n) \geq \frac{\alpha^- + \beta^-}{p^+_{\max}}\Big(T^{\frac{2-p^-_{\min}}{2}}||u||^{p^-_{\min}} - (T+1)\Big) -$$
$$\lambda(\frac{\varphi_2^+}{m^-}\sum_{k=1}^{T}|u_+(k)|^{m(k)} + \psi_1^+\sum_{k=1}^{T}|u_+(k)|) \geq$$
$$\frac{\alpha^- + \beta^-}{p^+_{\max}}||u||^{p^-_{\min}} - \frac{\alpha^- + \beta^-}{p^+_{\max}}(T+1)$$
$$-\lambda(\frac{\varphi_2^+}{m^-}T(T+1^{m^+})||u_+||^{m^+} - \lambda T\psi_1^+\max_{k\in[1,T]}|u_+(k)| \geq$$
$$\frac{\alpha^- + \beta^-}{p^+_{\max}}||u||^{p^-_{\min}} - \frac{\alpha^- + \beta^-}{p^+_{\max}}(T+1)$$

$$-\lambda(\frac{\varphi_2^+}{m^-}T(T+1^{m^+})||u_+||^{m^+}-\lambda T(T+1)^{\frac{1}{2}}\psi_1^+||u_+||.$$

Since $m^+ < p_{min}^-$, the functional $J_\lambda$ is coercive on $H$. The assumptions of Theorem(2.2) are satisfied and, by Lemma(2.5), the problem (1) has a positive solution.

**Proof of Theorem 1.3.** In order to use a mountain pass lemma, we start by proving that $J_\lambda$ satisfied the Palais-Smale condition.

Let $\{u_n\} \subset H$ be a sequence such that $\{J_\lambda(u_n)\}$ is bounded and $J_\lambda^{'}(u_n) \longrightarrow 0$. Since $H$ is finitely dimensional, it is enough to show $\{u_n\}$ is bounded. Let $\{u_{n_k}\}$ be such a subesquence of the sequence $\{u_n\}$ whose all elements are non-negative and $\{u_{n_l}\}$ be subesquence of $\{u_n\}$ whose all elements are non-positive. Either of this sequences must have an infinite number of elements. Assume that $\{u_n\}$ is unbounded. Note that either $\{u_{n_k}\}$ or $\{u_{n_l}\}$ is then unbounded, up to subsequence that we assume to be chosen. Suppose that $\{u_{n_k}\}$ is unbounded. Then by (2), by the condition $(A)$, and from inequality $(b)$,$(f)$ and $(h)$ in Lemma(2.1) we have

$$J_\lambda(u_{n_k}) \leq \frac{\alpha^+ + \beta^+}{p_{\min}^-}\left((T+1)||u_{n_k}||^{p_{\max}^+} + (T+1)\right)$$
$$-\lambda\left(\frac{\varphi_1^-}{m^+}2^{-m^-}(T+1)^{\frac{2-m^-}{2}}||u_{n_k}||^{m^-} + \psi_1^-\sum_{k=1}^{T}u_{n_k}(k)\right)$$

Since $m^- > p_{\max}^+$ we have $J_\lambda(u_{n_k}) \to -\infty$ as $||u_{n_k}|| \to +\infty$ which is a contradiction with the fact that $\{J_\lambda(u_n)\}$ is bounded since in this case also $\{J_\lambda(u_k)\}$ is bounded. Now suppose $\{u_{n_l}\}$ is bounded. Then from Lemma(2.1) $(g)$ we observe that

$$J_\lambda(u_{n_l}) \geq \frac{\alpha^- + \beta^-}{p_{\max}^+}\left(T^{\frac{2-p_{\min}^-}{2}}||u_{n_l}||^{p_{\min}^-} - (T+1)\right)$$

Since $m^- > p_{\max}^+$ we have $J_\lambda(u_{n_k}) \to -\infty$ as $||u_{n_k}|| \to +\infty$ which is a contradiction with the fact that $\{J_\lambda(u_n)\}$ is bounded. It follows that $\{u_{n_k}\}$ is bounded. Hence the sequence $\{u_n\}$ is bounded.

Now, we will verify the other asuumptions. Put

$$\Omega := \left\{u \in H : ||u|| \leq (T+1)^{-\frac{1}{2}}\right\}.$$

Then, by Lemma (2.1) $(c)$, it follows that

$$|u(k)| \leq \max_{s\in[1,T]}|u(s)| \leq (T+1)^{\frac{1}{2}}||u|| \leq 1,\ \forall u \in \Omega,\ \forall k \in [1,T].$$

Next we see that for all

$$\sum_{k=1}^{T}F(k,u_+(k)) \leq \sum_{k=1}^{T}\frac{\varphi_2(k)}{m(k)} + \psi_2(k),\ \forall u \in \Omega.$$

Therefore, in view Lemma (2.1) $(a)$, we deduce

$$J_\lambda(u) \geq \frac{\alpha^- + \beta^-}{p^+_{\max}}\left(T^{\frac{p^+_{\max}-2}{2}}(T+1)^{\frac{-p^+_{\max}}{2}}\right) - \lambda\sum_{k=1}^{T}\left(\frac{\varphi_2(k)}{m(k)} + \psi_2(k)\right),\ \forall u \in \partial\Omega.$$

Consequently, if we set

$$\lambda_0 = \frac{(\alpha^- + \beta^-)T^{\frac{p^+_{\max}-2}{2}}(T+1)^{\frac{-p^+_{\max}}{2}}}{p^+_{\max}\sum_{k=1}^{T}(\frac{\varphi_2(k)}{m(k)} + \psi_2(k))}, \tag{5}$$

then for $\lambda \in (0, \lambda_0)$, we have

$$J_\lambda(u) > 0,\ \forall u \in \partial\Omega. \tag{6}$$

Next, let $u_\zeta \in H$ be defined as follows: $\begin{cases} u_\zeta = \zeta \text{ for } k = 1, ..., T \\ u_\zeta(0) = u_\zeta(T+1) = 0. \end{cases}$ We will verify that there exists $\zeta$ such that

$$u_{\zeta_0} \in H\backslash\Omega \text{ and } J_\lambda(u_{\zeta_0}) < \min_{u\in\partial\Omega} J_\lambda(u). \tag{7}$$

Then for $\zeta > 1$ we have

$$\begin{aligned} J(u_\zeta) &\leq (\alpha^+ + \beta^+)\left(\frac{\zeta^{p_{\max}(0)}}{p_{\min}(0)} + \frac{\zeta^{p_{\max}(T)}}{p_{\min}(T)}\right) - \lambda\sum_{k=1}^{T}\left(\frac{\varphi_1(k)\zeta^{m(k)}}{m(k)} + \psi_1(k)\zeta\right) \\ &\leq 2(\alpha^+ + \beta^+)\frac{\zeta^{p^+_{\max}}}{p^-_{\min}} - \lambda T\left(\frac{\varphi_1^-}{m^+} + \psi_1^-\zeta^{1-m^-}\right)\zeta^{m^-}. \end{aligned}$$

Since $m^- > p^+_{max}$, $\lim_{\zeta\to\infty} J_\lambda(u_\zeta) = -\infty$. So, the assertion (7) holds true. Applying a mountain pass lemma, thus, by Lemma 2.5, the problem (1) has at least one positive solution.

**Case**: $m^- < p^-_{min}$.

In order to use Ekland's variational principle, let $\lambda \in (0, \lambda_0)$ be fixed, where $\lambda_0$ is given by (5). From (6) and using the Weierstrass theorem, we obtain $\inf_{x\in\partial\Omega} J_\lambda(u) > 0$.

Now take $t \in [0, 1]$ and define $u_0 \in H$ a function such that

$$\begin{cases} u_0(k_0) = t & \text{for } k \in [1, T] \setminus \{k_0\} \\ u_0(k_0) = 0 \end{cases}$$

with $k_0 \in [1, T]$ is given such that $m(k_0) = m^-$. Then,

$$J_\lambda(u_0) \leq \frac{\alpha(k_0) + \beta(k_0)}{p_{\min}(k_0 - 1)}t^{p_{\min}(k_0-1)} + \frac{\alpha(k_0+1) + \beta(k_0+1)}{p_{\min}(k_0)}t^{p_{\min}(k_0)}$$

$$-\lambda\bigg(\frac{\varphi_1(k_0)}{m(k_0)}t^{m(k_0)}+\psi_1(k_0)t\bigg)\leq$$

$$\frac{2(\alpha^++\beta^+)}{p^-_{\min}}t^{p^-_{\min}}-\lambda\bigg(\frac{\varphi^+_1}{m^+}+\psi^-_1\bigg)t^{m^-}.$$

Hence, for $0<t<\left(\frac{\lambda p^-_{\min}(\frac{\varphi_1^+}{m^+}+\psi_1^-)}{2(\alpha^++\beta^+)}\right)^{\frac{1}{p^-_{\min}-m^-}}$, we have $J_\lambda(u_0)<0$. As $u_0\in$ Int$\Omega$, we write $\inf_{u\in \mathrm{Int}\Omega}J_\lambda(u)<0<\inf_{u\in\partial\Omega}J_\lambda(u)$. Let us choose $\epsilon>0$ such that

$$0<\epsilon<\inf_{u\in\partial\Omega}J_\lambda(u)-\inf_{u\in\mathrm{Int}\Omega}J_\lambda(u). \tag{8}$$

Therefore, by applying the Ekeland's variational principle (Theorem 2.3) to the functional $J_\lambda:\Omega\to\mathbb{R}$, we find $u_\epsilon\in\Omega$ such that

$$J_\lambda(u_\epsilon)<\inf_{u\in\Omega}J_\lambda(u)+\epsilon \text{ and } J_\lambda(u_\epsilon)<J_\lambda(u)+||u-u_\epsilon||, \text{ for } u\neq u_\epsilon.$$

Hence, by (6), $J_\lambda(u_\epsilon)<\inf_{u\in\Omega}J_\lambda(u)+\epsilon\leq\inf_{u\in Int\Omega}J_\lambda(u)+\epsilon<\inf_{u\in\partial\Omega}J_\lambda(u)$ and so, $u_\epsilon\in\mathrm{Int}\Omega$.

Now, let us define $\Phi_\lambda:\to\mathbb{R}$ by $\Phi_\lambda(u)=J_\lambda(u)+\epsilon||u-u_\epsilon||$ for $u\neq u_\epsilon$. It is easy to see that $u_\epsilon$ is a minimum point of $\Phi$, and thus

$$\frac{\Phi_\lambda(u)(u_\epsilon+hv)-\Phi_\lambda(u_\epsilon)}{h}\geq 0, \tag{9}$$

for $h>0$ small enough and any $v\in\Omega$. Note that formula (9) reduces to

$$\frac{J_\lambda(u)(u_\epsilon+hv)-J_\lambda(u_\epsilon)}{h}+\epsilon||v||\geq 0.$$

Letting $h\to0$, we deduce that $<J'_\lambda(u_\epsilon),v>+\epsilon||v||>0$, that is, $||J'_\lambda(u_\epsilon)||\leq\epsilon$. Therefore, there exists a sequence $\{u_n\}\subset\mathrm{Int}\Omega$ such that

$$J_\lambda(u_n)\to\inf_{u\in\Omega}J_\lambda(u) \text{ and } J'_\lambda(u_n)\to 0.$$

Since thre sequence $\{u_n\}$ is bounded in $H$, there exists $v_0\in H$ such that, up to a subsequence, $\{u_n\}$ converges to $v_0$ in $H$. Thus

$$J_\lambda(v_0)=\inf_{u\in\Omega}J_\lambda(u) \text{ and } J'_\lambda(v_0)=0.$$

The above relations imply that $v_0$ is a solution of problem (1).

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