MATHEMATICA, 62 (85), N° 1, 2020, pp. 89–103

ON THE POLYNOMIAL SOLUTIONS OF GENERAL POLYNOMIAL DIFFERENTIAL EQUATIONS

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Abstract. We deal with the ordinary differential equation of the form $y^m \mathrm{d}y/\mathrm{d}x = P(x,y)$ where $m \geq 2$ and $P(x,y)$ is a real polynomial in the variables x and y of degree n in the variable y. We study the maximum number of the polynomial solutions of this equation with respect to n. We also consider the multiplicity of polynomial limit cycles.

MSC 2010. 37D99.

Key words. Polynomial ordinary differential equations, polynomial solutions.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Consider the ordinary differential equation of the form

$$y^m \frac{\mathrm{d}y}{\mathrm{d}x} = A_0(x) + A_1(x)y + \ldots + A_n(x)y^n \tag{1}$$

where x, y are real variables, $A_0 A_i(x) \in \mathbb{R}[x]$ for $i = 0, 1, \ldots, n$ and n, m are integer numbers with $n \geq 1$ and $m \geq 2$. The case $m = 1$ was completely studied in [7]. We recall that $\mathbb{R}[x]$ is the set of polynomials with real coefficients. We also assume that $A_0 A_n \not\equiv 0$. We will denote the derivative of y with respect to x as $\mathrm{d}y/\mathrm{d}x$ or x'.

We are interested in the polynomial solutions $y = p(x)$ of this differential equation that is in the solutions of the form $y = p(x)$ where p is a polynomial. The computation of exact solutions (either polynomial or rational) of a nonlinear differential equation has a remarkable role in understanding the whole set of solutions and their dynamical properties. Rainville in [6] (1936) was the first author to determine all the Riccati differential equations of the form $y' = A_0(x) + A_1(x)y + y^2$, with $A_0(x)$ and $A_1(x)$ being polynomials that have polynomial solutions and he also provided an algorithm for the computation of these polynomial solutions. Campbell and Golomb in [2] provided an algorithm for finding all the polynomial solutions of the generalized Riccati differential equation of the form $B(x)y' = A_0(x) + A_1(x)y + A_2(x)y^2$ where B, A_i are polynomials in x for $i = 0, 1, 2$. Behloul and Cheng in [1] gave another algorithm to detect either polynomial or rational solutions of equations

Partially supported by FCT/Portugal through UID/MAT/04459/2013.

DOI: 10.24193/mathcluj.2020.1.10

of the form

$$B(x)y' = A_0(x) + A_1(x)y + A_2(x)y^2 + \cdots + A_n(x)y^n$$

where B, A_i are polynomials in x for $i = 0, \ldots, n$. Giné, Grau and Llibre in [3] study the maximum number and the multiplicity of the polynomial limit cycles of the form $y' = A_0(x) + A_1(x) + \ldots + A_n(x)$ where A_i are polynomials for $i = 0, \ldots, n$ and $A_n(x) \neq 0$. In [7] the author studies the maximum number and the multiplicity of the polynomial limit cycles of the form $yy' = A_0(x) + A_1(x) + \ldots + A_n(x)$ where A_i are polynomials for $i = 0, \ldots, n$ and $A_0 A_n(x) \neq 0$. The fact that $m \geq 2$ in (1) substantially difficult the computations in the paper and the results are more involved.

The following theorem, which is the first main theorem in the paper, establishes the maximum number of polynomial solutions that are coprime among each other and periodic polynomial solutions (also that are coprime among each other) that a differential equation (3) can have. It also establishes the solutions of that are the inverse of a polynomial. We recall that a periodic orbit is a solution $y = \varphi(x)$ such that $\varphi\colon [0, 1] \to \mathbb{R}$ is C^1 and $\varphi(0) = \varphi(1)$ (here the period is one but we could choose any period to define a periodic solution because by an affine change in the variable x there is no loss of generality in considering just period one).

Theorem 1.1.

(a) *Any differential equation* (1) *has at most n solutions that are constant, and there are examples of equations* (1) *with exactly n constant solutions.*

(b) *If $n = m$, there are infinite polynomial solutions of equation* (1).

(c) *If $n \geq m + 1$, the difference between two coprime polynomial solutions of equation* (1) *is a constant.*

(d) *If $n \geq m + 1$, there are examples of differential equations* (1) *with n polynomial periodic solutions that are coprime among each other.*

(e) *If $n \geq m + 1$, any differential equation* (1) *has no solutions that are the inverse of a non-constant polynomial.*

The proof of Theorem 1.1 is given in Section 2.

We are also interested in a somewhat similar problem which consists in knowing if there are polynomial equations of the form (1) that have a prescribed number of polynomial solutions. This is the content of the following theorem, the second main theorem of the paper.

Theorem 1.2.

(i) *If $n \geq m + 1$, given $n - m + 2$ non-constant polynomials that are coprime among each other, there are infinite equations of the form* (1) *with such $n - m + 2$ solutions.*

(ii) *If $n \geq m + 1$, given $n + 1$ non-constant polynomials that are coprime among each other, there are no equations of the form* (1) *with such $n + 1$ solutions for any $m \geq 2$.*

The proof of Theorem 1.2 is given in Section 3. In particular, Theorem 1.2 states that when $m = 2$ the maximum number of polynomial solutions is n.

We want also to study the multiplicity of the differential equations (1). In order to define the multiplicity we will consider the so-called *translation operator*. Let $\psi(x; y_0)$ be the solution of equation (1) defined for $x \in [0,1]$ such that $\psi(0; y_0) = y_0$. The *translation operator* associated to (1) is the function $\Psi\colon \mathbb{R} \to \mathbb{R}$ such that $\Psi(y_0) = \psi(1; y_0) - y_0$. Note that $\Psi(y_0) = 0$ if and only if equation (1) has a periodic solution starting at y_0 and so there is a one-to-one correspondence between the zeros of the translation operator and the periodic solutions of system (1). Following Lloyd [4, 5], the *multiplicity of a limit cycle* of (3) associated to the isolated zero y_0 of the translation operator is the multiplicity of y_0 as a zero of $\Psi(y_0)$. A periodic solution of multiplicity 1 is called *hyperbolic*.

THEOREM 1.3. *Consider system* (1) *and assume that* $y = \varphi(x)$ *is a periodic orbit of this equation. Assume also that* $A_j = \varphi^{m+3-j}\overline{A}_j$ *with* $\overline{A}_j \in \mathbb{R}[x]$ *for* $j = 0, \dots, m-1$. *Then* φ *is:*

(i) *a hyperbolic limit cycle if and only if* $\mathcal{A}_1(1) \neq 0$*;*
(ii) *a limit cycle of multiplicity two if and only if* $\mathcal{A}_1(1) = 0$ *and* $\mathcal{A}_2(1) \neq 0$*;*
(iii) *a limit cycle of multiplicity three if and only if* $\mathcal{A}_1(1) = \mathcal{A}_2(1) = 0$ *and* $\mathcal{A}_3(1) \neq 0$*;*
(iv) *a limit cycle of multiplicity greater than or equal to four, or it belongs to a continuum of periodic orbits if* $\mathcal{A}_1(1) = \mathcal{A}_2(1) = \mathcal{A}_3(1) = 0$,

where

$$\mathcal{A}_1(x) = -\int_0^x \Big(\varphi^3(\sigma)\sum_{j=0}^{m-1}\overline{A}_j(\sigma) - \frac{\partial \mathcal{F}}{\partial y}(\sigma, \varphi(\sigma))\Big)\,\mathrm{d}\sigma,$$

$$\mathcal{A}_2(x) = \int_0^x e^{\mathcal{A}_1(\sigma)}\Big(\varphi^2(\sigma)\sum_{j=0}^{m-1}\overline{A}_j(\sigma) + \frac{1}{2}\frac{\partial^2 \mathcal{F}}{\partial y^2}(\sigma, \varphi(\sigma))\Big)\,\mathrm{d}\sigma,$$

and

$$\mathcal{A}_3(x) = -\int_0^x e^{2\mathcal{A}_1(\sigma)}\Big(\varphi(\sigma)\sum_{j=0}^{m-1}\overline{A}_j(\sigma) - \frac{1}{6}\frac{\partial^3 \mathcal{F}}{\partial y^3}(\sigma, \varphi(\sigma)\Big)\,\mathrm{d}\sigma.$$

2. PROOF OF THEOREM 1.1

Let $y = p(x) = \xi$ be a polynomial solution of system (1) which is constant. Then the polynomial in y, $A_0(x) + A_1(x)y + \ldots + A_n(x)y^n = 0$ is divisible by $y - \xi$ and since, its degree in y is n we get that it has at most n different constant roots. So, there are at most n different constant solutions of system (1). The differential equation

$$y^m \frac{\mathrm{d}y}{\mathrm{d}x} = (y-1)(y-2)\ldots(y-n) \tag{2}$$

is of the form (1) and has exactly n constant solutions. This proves statement (a).

Now let $y = p(x)$ be a polynomial solution of system (1) which is not a constant, that is,

$$p^m(x)\frac{\mathrm{d}p}{\mathrm{d}x} = A_0(x) + A_1(x)p + \ldots + A_n(x)p^n.$$

Since p divides the left-hand side of the above equation, it must also divide the right-hand side and so $A_0(x) = \tilde{A}_0(x)p(x)$ for some polynomial $\tilde{A}_0(x)$. In this way equation (1) becomes

$$p^{m-1}(x)\frac{\mathrm{d}p}{\mathrm{d}x} = (\overline{A}_0(x) + A_1(x)) + A_2(x)p + \ldots + A_n(x)p^{n-1}. \tag{3}$$

Note that since $m \geq 2$, the polynomial p still divides the left-hand side of the above equation and so it must also divide the right-hand side, which means that $\tilde{A}_0(x) = A_0^*(x)p(x)$ and $A_1(x) = A_1^*(x)p(x)$ for some polynomials A_0^*, A_1^*.

Proceeding inductively we get that

$$A_i(x) = p^{m-i}\overline{A}_i(x), \quad i = 0, \ldots, m-1,$$

for some polynomials $\overline{A}_i(x)$ for $i = 0, \ldots, m-1$. So, if $n = m$, then $A_j(x) = 0$ for $j \geq m$ and system (1) becomes $\dfrac{\mathrm{d}p}{\mathrm{d}x} = \displaystyle\sum_{j=0}^{m-1} \overline{A}_j(x) + A_m(x)$. Hence,

$$p(x) = \int \Big(\sum_{j=0}^{m-1} \overline{A}_j(s) + A_m(s)\Big)\,\mathrm{d}s + c,$$

where c is any constant. This proves statement (b).

Now assume $n \geq m+1$ and let $q(x)$ be another solution of (1). Then we must have that $A_j(x) = q(x)\tilde{A}_j(x)$ for some polynomials $\tilde{A}_j(x)$ for $j = 0, \ldots, m-1$. In short, since $p(x)$ and $q(x)$ are coprime (i.e., $\gcd\{p(x), q(x)\} = 1$) we must have $A_j(x) = p(x)q(x)\bar{A}_j(x)$ for some polynomial $\bar{A}_j(x)$ and $j = 0, \ldots, m-1$. We consider the change $w = q - p$ and we transform equation (1) into the equation

$$\begin{aligned}
\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{\mathrm{d}q}{\mathrm{d}x} - \frac{\mathrm{d}p}{\mathrm{d}x} &= \sum_{j=0}^{m-1} \bar{A}_j(x)p(x) + A_m(x) + \ldots + A_n(x)q(x)^{n-m} \\
&\quad - \sum_{j=0}^{m-1} \bar{A}_j(x)q(x) - A_m(x) - \ldots - A_n(x)p(x)^{n-m} \\
&= \Big(- \sum_{j=0}^{m-1} \bar{A}_0(x) + A_{m+1}(x)\Big)z + A_{m+2}(x)q^2(x) + \ldots + A_n(x)q(x)^{n-m} \\
&\quad - A_{m+2}(x)p^2 - \ldots - A_n(x)p(x)^{n-m} \\
&= A_0(x)^*w + A_{m+1}^*(x)w^2 + \ldots + A_n(x)w^{n-m}
\end{aligned}$$

where each A_i^* is a polynomial depending on $A_j(x)$ and $\bar{A}_j(x)$ for $j = i, \ldots, n$ and some powers of $p(x)$, for $i = m+1, \ldots, n-1$. Let $w = r(x)$ be another polynomial solution of equation (2) different than $w = 0$, that is,

$$r'(x) = \tilde{A}_{m+1}(x)r(x) + \tilde{A}_{m+2}(x)r(x)^2 + \ldots + A_n(x)r(x)^{n-m}.$$

Then, since $r(x)$ divides the right-hand side of the above equation, it must also divide the left-hand side. Therefore, $r|r'$ which implies that $r = \xi \in \mathbb{R}$ is a constant. So any polynomial solution of equation (2) must be a constant. In short, the difference of two coprime polynomial solutions of (3) must be a constant. Hence, statement (c) is proved.

The proof of statement (d) is a direct consequence of statement (c) and equation (2), as any constant solution is a periodic orbit.

Finally, we prove statement (e), that is, we study the solutions that are the inverse of a polynomial. Let $z = \frac{1}{y(x)}$ where y is the inverse of a polynomial, and so z is a polynomial. Then we have

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}x} &= -\frac{\mathrm{d}y}{\mathrm{d}x}\frac{1}{y} = -\frac{A_0(x)}{y^{m+1}} - \frac{A_1(x)}{y^{m+2}} - \cdots - \frac{A_n(x)}{y^{n+m+1}} \\ &= -A_0(x)z^{m+1} - A_1(x)z^{m+2} - \cdots - A_n(x)z^{n+m+1}.\end{aligned}$$

Hence,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = -A_2(x)z^{m+1} - A_1(x)z^{m+2} - \cdots - A_n(x)z^{n+m+1}. \tag{4}$$

Note that the constant solution $z = 0$ does not correspond to any solution which is the inverse of a polynomial. Let $z = q(x)$ be a polynomial solution different from zero. Then

$$q' = \frac{\mathrm{d}q}{\mathrm{d}x} = -q^{m+1}(A_2(x) + A_1(x)q + \cdots + A_n(x)q^n).$$

Then, since q divides the right-hand side of the latter expression, we have that it also must divide the left-hand side, that is, $q|q'$. Hence, q is a constant. So any polynomial solution of (4) needs to be a constant. This proves statement (e) and concludes the proof of Theorem 1.1.

3. PROOF OF THEOREM 1.2

We start proving statement (i). To this end, assume that system (3) has $n - m + 2$ non-constant polynomial solutions $p_j(x)$ for $j = 1, \ldots, n$, that are coprime among each other. In view of statement (c) in Theorem 1.1, since the difference among two coprime polynomial solutions is constant, we can write $p_k(x) = p_j(x) + c_{k,j}$ with $c_{k,j} \in \mathbb{R}$ for $k \neq j$ and $j = 1, \ldots, n-m+2$. We have

$$p_j(x)^m \frac{\mathrm{d}p_j}{\mathrm{d}x} = A_0 + A_1 p_j(x) + \cdots + A_n(x)p_j(x)^n.$$

Since $p_j(x)$ divides the left-hand side of the above equation, it must also divide $A_0(x)$ for any $p_j(x)$, $j = 1, \dots, n$. Hence, $A_i(x) = \prod_{j=1}^{n-m+2} p_j(x)^{m-i}\overline{A}_i(x)$, where $\overline{A}_i(x) \in \mathbb{R}[x]$ for $i = 0, \dots, m-1$. So, we have the system

$$\frac{dp_j}{dx} = \prod_{k=1,k\neq j}^{n-m+2} p_k(x)^m \overline{A}_0 + \prod_{k=1,k\neq j}^{n-m+2} p_k(x)^{m-1} A_1(x) + \dots + \prod_{k=1,k\neq j}^{n-m+2} p_k(x)\overline{A}_{m-1}(x)$$
$$+ A_m(x) + A_{m+1}(x)p_j + \dots + A_n(x)p_j^{n-m}.$$

Again, by Theorem 1.1 (c), we have that

$$\frac{\mathrm{d}p_j(x)}{\mathrm{d}x} = \frac{\mathrm{d}p_i(x)}{\mathrm{d}x} \quad \text{for any } j \neq i := B(x).$$

So we have a system of $n-m+2$ equations with the unknowns $\overline{A}_0, \dots, \overline{A}_{m-1}$, A_m, A_n. They satisfy

$$\Big(B(x) - \sum_{i=0}^{m-2} \prod_{k=1,k\neq j}^{n-m+2} p_k(x)^{m-i}\overline{A}_i(x)\Big) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = M \begin{pmatrix} \overline{A}_{m-1} \\ A_m \\ \vdots \\ A_n(x) \end{pmatrix}$$

where M is the matrix

$$M = \begin{pmatrix} \prod_{k=2}^{n-m+2} p_k(x) & 1 & p_1(x) & p_1(x)^2 & \cdots & p_1(x)^{n-m} \\ \prod_{k=1,k\neq 2}^{n-m+2} p_k(x) & 1 & p_2(x) & p_2(x)^2 & \cdots & p_2(x)^{n-m} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \prod_{k=1}^{n-m+1} p_k(x) & 1 & p_{n-m+2}(x) & p_{n-m+2}(x)^2 & \cdots & p_{n-m+2}(x)^{n-m} \end{pmatrix}.$$

The solution is just

$$\begin{pmatrix} \overline{A}_{m-1} \\ A_m \\ \vdots \\ A_n(x) \end{pmatrix} = M^{-1}\Big(B(x) - \sum_{i=0}^{m-2} \prod_{k=1,k\neq j}^{n-m+2} p_k(x)^{m-i}\overline{A}_i(x)\Big) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \tag{5}$$

in case M^{-1} exists. We take the notation

$$P_j^\ell = \prod_{k=\ell,k\neq j}^{n-m+2} p_k(x) \quad \text{and} \quad P_{j_1,j_2}^\ell = \prod_{k=\ell,k\neq j_1,k\neq j_2}^{n-m+2} p_k(x).$$

Then we can write

$$M = \begin{pmatrix} P_1^1 & 1 & p_1 & p_1^2 & \cdots & p_1^{n-m} \\ P_2^1 & 1 & p_2 & p_2^2 & \cdots & p_2^{n-m} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{n-m+2}^1 & 1 & p_{n-m+2} & p_{n-m+2}^2 & \cdots & p_{n-m+2}^{n-m} \end{pmatrix},$$

where we have dropped the dependence in x to obtain a lighter notation. We claim that

$$
\begin{aligned}
\det M &= (-1)^{(n-m+1)(n-m+2)/2} \prod_{1\le i<j\le n-m+2} (p_i - p_j) \\
&= (-1)^{(n-m+1)(n-m+2)/2} \prod_{1\le i<j\le n-m+2} c_{i,j} \in \mathbb{R}.
\end{aligned}
\tag{6}
$$

We prove the claim by induction over the dimension $n \ge m+1$ of the matrix M. Assume $n = m+1$. Then M becomes the 3×3 matrix

$$
\begin{pmatrix} p_2p_3 & 1 & p_1 \\ p_1p_3 & 1 & p_2 \\ p_2p_3 & 1 & p_3 \end{pmatrix} = -(p_1-p_2)(p_1-p_3)(p_2-p_3) = -c_{1,2}c_{1,3}c_{2,3} \in \mathbb{R}.
$$

Next, we assume it is true for $n = 1, \ldots, m+\ell-2$ and we will prove it for $n = m+\ell-1$ (obtaining an $(\ell+1)\times(\ell+1)$-matrix M). For $i = 2, \ldots, \ell$, let r_i be the ith row. We substract to each row f_i the first row r_1 and we get that

$$
\begin{aligned}
M &= \begin{pmatrix} P_1^1 & 1 & p_1 & p_1^2 & \cdots & p_1^{\ell-1} \\ P_2^1 - P_1^1 & 0 & p_2 - p_1 & p_2^2 - p_1^2 & \cdots & p_2^{\ell-1} - p_1^{\ell-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{\ell+1}^1 - P_1^1 & 0 & p_{\ell+1} - p_1 & p_{\ell+1}^2 - p_1^2 & \cdots & p_{\ell+1}^{\ell-1} - p_1^{\ell-1} \end{pmatrix} \\
&= \begin{pmatrix} P_1^1 & 1 & p_1 & p_1^2 & \cdots & p_1^{\ell-1} \\ (p_1-p_2)P_{1,2}^1 & 0 & p_2 - p_1 & p_2^2 - p_1^2 & \cdots & p_2^{\ell-1} - p_1^{\ell-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (p_1-p_{\ell+1})P_{1,\ell+1}^1 & 0 & p_{\ell+1} - p_1 & p_{\ell+1}^2 - p_1^2 & \cdots & p_{\ell+1}^{\ell-1} - p_1^{\ell-1} \end{pmatrix}.
\end{aligned}
$$

Let c_i be the i-th column. For each $i = 3, \ldots, \ell+1$, we multiply the column c_i by $-p_1(x)$ and we add it to the column c_{i+1}. Then M is equal to

$$
\begin{pmatrix} P_1^1 & 1 & p_1 & 0 & \cdots & p_1^{\ell-1} \\ (p_1-p_2)P_{1,2}^1 & 0 & p_2 - p_1 & p_2(p_2-p_1) & \cdots & p_2^{\ell-2}(p_2-p_1) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (p_1-p_{\ell+1})P_{1,\ell+1}^1 & 0 & p_{\ell+1} - p_1 & p_{\ell+1}(p_{\ell+1}-p_1) & \cdots & p_{\ell+1}^{\ell-2}(p_{\ell+1}-p_1). \end{pmatrix}
$$

By the Laplace theorem for determinants we obtain that $\det M = -\det \tilde{M}$ where $\tilde{M}$ is the $\ell\times\ell$ matrix

$$
\begin{pmatrix} (p_1-p_2)P_{1,2}^1 & p_2 - p_1 & p_2(p_2-p_1) & \cdots & p_2^{\ell-2}(p_2-p_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (p_1-p_{\ell+1})P_{1,\ell+1}^1 & p_{\ell+1} - p_1 & p_{\ell+1}(p_{\ell+1}-p_1) & \cdots & p_{\ell+1}^{\ell-2}(p_{m+1}-p_1) \end{pmatrix}.
$$

Clearly, $\det M = -\det \tilde{M} = \prod_{j=2}^{\ell+1}(p_j - p_1)\overline{M} = (-1)^\ell \prod_{j=2}^{\ell+1}(p_1 - p_j)\overline{M}$, where

$$\overline{M} = \begin{pmatrix} P_{1,2}^1 & 1 & p_2 & p_2^2 & \cdots & p_2^{\ell-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{1,\ell+1}^1 & 1 & p_{\ell+1} & p_{\ell+1}^2 & \cdots & p_{\ell+1}^{\ell-2} \end{pmatrix}$$
$$= \begin{pmatrix} P_2^2 & 1 & p_2 & p_2^2 & \cdots & p_2^{\ell-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{\ell+1}^2 & 1 & p_{\ell+1} & p_{\ell+1}^2 & \cdots & p_{\ell+1}^{\ell-2} \end{pmatrix}$$

where it is again a $\ell \times \ell$ matrix M (starting in p_2 instead of p_1). By the induction hypotheses we have that

$$\begin{aligned} \det M &= (-1)^\ell \prod_{j=2}^{\ell+1}(p_1 - p_j)\overline{M} \\ &= (-1)^\ell \prod_{j=2}^{\ell+1}(p_1 - p_j)(-1)^{(\ell-1)\ell/2} \prod_{2\le i<j\le \ell+1}(p_i - p_j) \\ &= (-1)^{\ell(\ell+1)/2} \prod_{1\le i<j\le \ell+1}(p_i - p_j) = (-1)^{\ell(\ell+1)/2} \prod_{1\le i<j\le \ell+1} c_{i,j}, \end{aligned}$$

which proves (6). Now it is clear that statement (i) follows directly from the definition of the inverse of M together with (5), (6) and the fact that any polynomial solution depends on $\overline{A}_i(x)$ for $i = 0, \dots, m-2$ and we have a polynomial solution in $(A_{m-1}(x), A_m(x), \dots, A_n(x))$ for any polynomials $\overline{A}_i(x)$ with $i = 0, \dots, m-2$.

Now, we prove statement (ii). We will prove only the case $m = 2$ and at the end we will say how to adapt it to the case $m \ge 3$. The fact is that the proof follows exactly the same lines. When $m = 2$, we consider the matrix

$$\begin{pmatrix} \overline{A}_0 \\ \overline{A}_1(x) \\ A_2 \\ \vdots \\ A_n(x) \end{pmatrix} = M_2^{-1} B(x) \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

where, with the notation

$$P_{j_1}^{\ell,(j_3)} = \prod_{k=\ell, k\ne j_1}^{n} p_k(x)^{j_3} \quad \text{and} \quad P_{j_1,j_2}^{\ell,(j_3)} = \prod_{k=\ell, k\ne j_1, k\ne j_2}^{n} p_k(x)^{j_3},$$

M_1 is equal to

$$\begin{pmatrix} P_1^{1,(2)} & P_1^{1,(1)} & 1 & p_1(x) & p_1(x)^2 & \cdots & p_1(x)^{n-3} \\ P_2^{1,(2)} & P_2^{1,(1)} & 1 & p_2(x) & p_2(x)^2 & \cdots & p_2(x)^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ P_n^{1,(2)} & P_n^{1,(1)} & 1 & p_n(x) & p_n(x)^2 & \cdots & p_n(x)^{n-3} \end{pmatrix}.$$

Here we have dropped the dependence in x to obtain a lighter notation. We claim that

$$\begin{aligned} \det M_1 &= (-1)^{n(n-1)/2}\prod_{i=1}^{n} p_i \prod_{1\le i<j\le n}(p_i-p_j) \\ &= (-1)^{n(n-1)/2}\prod_{i=1}^{n} p_i \prod_{1\le i<j\le n} c_{i,j}. \end{aligned} \tag{7}$$

We prove the claim by induction over the dimension $n \geq 3$ of the matrix M_1. Assume $n = 3$. Then M_1 becomes the 3×3 matrix

$$\begin{pmatrix} p_2^2p_3^2 & p_2p_3 & 1 \\ p_1^2p_3^2 & p_1p_3 & 1 \\ p_1^2p_2^2 & p_1p_2 & 1 \end{pmatrix} = -p_1p_2p_3(p_1-p_2)(p_1-p_3)(p_2-p_3) = -p_1p_2p_3c_{1,2}c_{1,3}c_{2,3}.$$

Next, we assume it is true for $n = 1, \ldots, \ell-1$ and we will prove it for $n = \ell$ (obtaining an $\ell \times \ell$-matrix M_1). For $i = 2, \ldots, \ell$, let r_i be the ith row. We substract to each row r_i the first row r_1 and we get that M_1 is equal to

$$\begin{pmatrix} P_1^{1,(2)} & P_1^{1,(1)} & 1 & p_1 & p_1^2 & \cdots & p_1^{\ell-3} \\ P_2^{1,(2)}-P_1^{1,(2)} & P_2^{1,(1)}-P_1^{1,(1)} & 0 & p_2-p_1 & p_2^2-p_1^2 & \cdots & p_2^{\ell-1}-p_1^{\ell-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ P_\ell^{1,(2)}-P_1^{1,(2)} & P_\ell^{1,(1)}-P_1^{1,(1)} & 0 & p_\ell-p_1 & p_\ell^2-p_1^2 & \cdots & p_\ell^{\ell-3}-p_1^{\ell-3} \end{pmatrix}$$

$$= \begin{pmatrix} P_1^{1,(2)} & P_1^{1,(1)} & 1 & p_1 & p_1^2 & \cdots & p_1^{\ell-3} \\ (p_1^2-p_2^2)P_{1,2}^{1,(2)} & (p_1-p_2)P_{1,2}^{1,(1)} & 0 & p_2-p_1 & p_2^2-p_1^2 & \cdots & p_2^{\ell-1}-p_1^{\ell-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (p_1^2-p_\ell^2)P_{1,\ell}^{1,(2)} & (p_1-p_\ell)P_{1,\ell}^{1,(1)} & 0 & p_\ell-p_1 & p_\ell^2-p_1^2 & \cdots & p_\ell^{\ell-3}-p_1^{\ell-3} \end{pmatrix}$$

Let c_i be the i-th column. We multiply the column c_i by $-p_1(x)$ and we add it to the column c_{i+1} for $i = 3, \ldots, \ell$. On the other hand, we multiply the second column by $-P_1^{1,(1)}$ and we add it to the first column. Then M_1 is equal to

$$\begin{pmatrix} P_1^{1,(2)} & P_1^{1,(1)} & 1 & 0 & 0 & \cdots & 0 \\ p_1(p_1-p_2)P_{1,2}^{1,(2)} & (p_1-p_2)P_{1,2}^{1,(1)} & 0 & p_2-p_1 & p_2(p_2-p_1) & \cdots & p_2^{\ell-3}(p_2-p_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ p_1(p_1-p_\ell)P_{1,\ell}^{1,(2)} & (p_1-p_\ell)P_{1,\ell}^{1,(1)} & 0 & p_\ell-p_1 & p_\ell(p_\ell-p_1) & \cdots & p_\ell^{\ell-3}(p_\ell-p_1) \end{pmatrix}$$

So, by the Laplace theorem for determinants we obtain that $\det M_1 = \det \tilde{M}_1$ where $\tilde{M}_1$ is the $\ell \times \ell$ matrix

$$\begin{pmatrix} p_1(p_1-p_2)P_{1,2}^{1,(2)} & (p_1-p_2)P_{1,2}^{1,(1)} & p_2-p_1 & \cdots & p_2^{\ell-3}(p_2-p_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_1(p_1-p_\ell)P_{1,\ell}^{1,(2)} & (p_1-p_\ell)P_{1,\ell}^{1,(1)} & p_\ell-p_1 & \cdots & p_\ell^{\ell-3}(p_\ell-p_1) \end{pmatrix}.$$

Clearly,

$$\det M_1 = \det \tilde{M}_1 = p_1 \prod_{j=2}^{\ell}(p_j-p_1)\overline{M}_1 = (-1)^{\ell-1}p_1\prod_{j=2}^{\ell}(p_1-p_j)\overline{M}_1$$

where

$$\begin{aligned} \overline{M}_1 &= \begin{pmatrix} P_{1,2}^{1,(2)} & P_{1,2}^{1,(1)} & 1 & p_2 & p_2^2 & \cdots & p_2^{\ell-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{1,\ell}^{1,(2)} & P_{1,\ell}^{1,(1)} & 1 & p_\ell & p_\ell^2 & \cdots & p_\ell^{\ell-3} \end{pmatrix} \\ &= \begin{pmatrix} P_{2}^{2,(2)} & P_{2}^{2,(1)} & 1 & p_2 & p_2^2 & \cdots & p_2^{\ell-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{\ell}^{2,(2)} & P_{\ell}^{2,(1)} & 1 & p_\ell & p_\ell^2 & \cdots & p_\ell^{\ell-3} \end{pmatrix} \end{aligned}$$

where it is again a $(\ell-1)\times(\ell-1)$ matrix M_1 (starting in p_2 instead of p_1). By the induction hypotheses, we conclude that

$$\begin{aligned} \det M_1 &= (-1)^{\ell-1}p_1\prod_{j=2}^{\ell}(p_1-p_j)\overline{M}_1 \\ &= (-1)^{\ell-1}p_1\prod_{j=2}^{\ell}(p_1-p_j)(-1)^{(\ell-1)(\ell-2)/2}\prod_{i=2}^{\ell}p_i\prod_{2\le i<j\le \ell}(p_i-p_j) \\ &= (-1)^{(\ell-1)\ell/2}\prod_{i=1}^{\ell}p_i\prod_{1\le i<j\le \ell}(p_i-p_j) \\ &= (-1)^{(\ell-1)\ell/2}\prod_{i=1}^{\ell}p_i\prod_{1\le i<j\le \ell}c_{i,j}, \end{aligned}$$

which proves (7).

Now we want to compute the inverse of M_1. We compute the first row of the matrix $M_1^{-1} = (m_{i,j}^{-1})_{1\le i\le j\le n}$, that is, $(m_{1,j}^{-1})_{1\le j\le n}$, and we will show that it is equal to

$$(-1)^{n-1}\frac{1}{\prod_{1\le k\le n, k\ne j} p_k(p_j-p_k)}. \tag{8}$$

Indeed, note that $m_{1,j}^{-1} = \frac{1}{\det M_1}(-1)^{j+1} n_{1,j}$, where

$$n_{1,1} = \det N_1 = \det \begin{pmatrix} P_2^{1,(1)} & 1 & p_2 & p_2^2 & \cdots & p_2^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ P_n^{1,(1)} & 1 & p_n & p_n^2 & \cdots & p_n^{n-3} \end{pmatrix},$$

$$n_{1,n} = \det N_n = \det \begin{pmatrix} P_1^{1,(1)} & 1 & p_1 & p_1^2 & \cdots & p_1^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{n-1}^{1,(1)} & 1 & p_{n-1} & p_{n-1}^2 & \cdots & p_{n-1}^{n-3} \end{pmatrix},$$

and, for $j \notin \{1, n\}$ we have

$$n_{1,j} = \det N_j = \det \begin{pmatrix} P_1^{1,(1)} & 1 & p_1 & p_1^2 & \cdots & p_1^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{j-1}^{1,(1)} & 1 & p_{j-1} & p_{j-1}^2 & \cdots & p_{j-1}^{n-3} \\ P_{j+1}^{1,(1)} & 1 & p_{j+1} & p_{j+1}^2 & \cdots & p_{j+1}^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ P_n^{1,(1)} & 1 & p_n & p_n^2 & \cdots & p_n^{n-3} \end{pmatrix}.$$

Using (6) we obtain

$$\det N_j = (-1)^{(n-1)(n-2)/2} p_j \prod_{1 \le i < l \le n, i \ne j, l \ne j} (p_i - p_l)$$

and so

$$\begin{aligned} m_{1,j}^{-1} &= (-1)^{j+1} \frac{(-1)^{(n-1)(n-2)/2} p_j \prod_{1 \le i < l \le n, i \ne j, l \ne j} (p_i - p_l)}{(-1)^{n(n-1)/2} \prod_{i=1}^n p_i \prod_{1 \le i < l \le n} (p_i - p_l)} \\ &= \frac{(-1)^{j+1}(-1)^{n-1}}{(-1)^{j-1} \prod_{1 \le k \le n, k \ne j} p_k (p_j - p_k)} \\ &= (-1)^{n-1} \frac{1}{\prod_{1 \le k \le n, k \ne j} p_k (p_j - p_k)}, \end{aligned}$$

which proves (8). Hence, it follows from it that

$$\begin{aligned} \overline{A}_0 &= (m_{1,1}^{-1}, \ldots, m_{1,n}^{-1}) \cdot (1, 1, \ldots, 1)^* \\ &= (-1)^{n-1} \sum_{j=1}^n \frac{1}{\prod_{1 \le k \le n, k \ne j} p_k (p_j - p_k)} \end{aligned} \tag{9}$$

We claim that

$$\sum_{j=1}^n \frac{1}{\prod_{1 \le k \le n, k \ne j} p_k (p_j - p_k)} = 0. \tag{10}$$

We prove the claim by induction over the number of elements n. If $n = 2$ it is clear. Now assume it is true for $n-1$ and we will show it for n. We write the left hand side of (10) as

$$
\begin{aligned}
&\sum_{j=1}^{n} \frac{1}{\prod_{1\le k\le n, k\ne j} p_k(p_j - p_k)} \\
(11) \qquad &= \frac{1}{p_n - p_{n-1}} \Bigg(\sum_{j=1}^{n} \frac{p_j - p_{n-1}}{\prod_{1\le k\le n, k\ne j} p_k(p_j - p_k)} \\
&\qquad - \sum_{j=1}^{n} \frac{p_j - p_n}{\prod_{1\le k\le n, k\ne j} p_k(p_j - p_k)} \Bigg).
\end{aligned}
$$

Each of the sums in (11) now has the form of the original sum in (10), except on $n-1$ elements, and the values turn out nicely by induction. Indeed,

$$\sum_{j=1}^{n} \frac{p_j - p_n}{\prod_{1\le k\le n, k\ne j} p_k(p_j - p_k)} = \frac{1}{p_n} \sum_{j=1}^{n-1} \frac{1}{\prod_{1\le k\le n-1, k\ne j} p_k(p_j - p_k)}$$

and

$$\sum_{j=1}^{n} \frac{p_j - p_{n-1}}{\prod_{1\le k\le n, k\ne j} p_k(p_j - p_k)} = \frac{1}{p_{n-1}} \sum_{j=1, j\ne n-1}^{n} \frac{1}{\prod_{1\le k\le n, k\notin\{j,n-1\}} (p_j - p_k)}$$

which are both zero by induction hypotheses.

Hence, in view of equations (9) and (10) we have that $\overline{A}_0 = 0$. But then $A_0(x) = 0$, which is not possible. This proves statement (ii) when $m = 2$. When $m \ge 3$ the proof is the same. First, using the same arguments as in case $m = 2$, we can prove that

$$\det M_1 = (-1)^{n(n-1)/2} \prod_{i=1}^{n} p_i^{m(m-1)/2} \prod_{1\le i<j\le n} (p_i - p_j),$$

where M_1 is the corresponding matrix. Denoting again $M_1^{-1} = (m_{i,j}^{-1})_{1\le i\le j\le n}$, we compute the first row of the inverse of the matrix M_1, that is, $m_{1,j}^{-1}$ for $j = 1, \dots, n$. Proceeding as we did for the case $m = 2$, we get that

$$m_{1,j}^{-1} = (-1)^{n-1} \frac{1}{\prod_{1\le k\le n, k\ne j} p_k^{m(m-1)/2} (p_j - p_k)}.$$

So,

$$
\begin{aligned}
\overline{A}_0 &= (m_{1,1}^{-1}, \dots, m_{1,n}^{-1}) \cdot (1, 1, \dots, 1)^* \\
&= (-1)^{n-1} \sum_{j=1}^{n} \frac{1}{\prod_{1\le k\le n, k\ne j} p_k^{m(m-1)/2} (p_j - p_k)}.
\end{aligned}
$$

Now, analog arguments as the ones used for the case $m = 2$ guarantee that

$$\sum_{j=1}^{n} \frac{1}{\prod_{1\le k\le n, k\ne j} p_k^{m(m-1)/2}(p_j - p_k)} = 0.$$

So $\overline{A}_0 = 0$, i.e., $A_0 = 0$, which is not possible. This concludes the proof of statement (ii) and in its turn the proof of Theorem 1.2.

4. PROOF OF THEOREM 1.3

We consider the differential equation $y^m \mathrm{d}y/\mathrm{d}x = \varphi^{m+3}\overline{A}_0(x) + A_1(x)y + \ldots + A_n(x)y^n := F(x, y)$ and we assume that $y = \varphi(x)$ is a periodic solution of this equation. We take the following change of the independent variable $z = y - \varphi(x)$ and we obtain the differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{F(x, z + \varphi(x))}{(z + \varphi(x))^m} - \frac{F(x, \varphi(x))}{\varphi(x)^m}.$$

The periodic orbit $y = \varphi(x)$ is transformed into the constant periodic solution $z = 0$. Note that

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}x} = {} & \sum_{j=0}^{m-1} \frac{\varphi^{m+3-j}\overline{A}_j(x)}{(z+\varphi)^{m-j}} - \varphi^3 \sum_{j=0}^{m-1} \overline{A}_j(x) + A_{m+1}(x)z \\ & + A_{m+2}(x)[(z+\varphi)^2 - \varphi^2] + \ldots + A_n(x)[(z+\varphi)^{n-m} - \varphi^{n-m}].\end{aligned}$$

Note that the right-hand side of the above equation is 0 when $z = 0$. We denote by $\psi(x; z_0)$ the solution of the above equation with initial condition z_0. Note that $\psi(x; 0) = 0$. We want to study the behavior of the solutions of this equation around $z = 0$. We define the translation operator $\Psi(z_0) = \psi(1; z_0) - \psi(0; z_0) = \psi(1; z_0) - z_0$ and we study it near $z_0 = 0$. If $\Psi(z_0)$ is identically zero, then the periodic orbit belongs to a continuum of periodic orbits. If $\Psi(z_0)$ is not identically zero, then the multiplicity of $z_0 = 0$ as a zero of $\Psi(z_0)$ is the multiplicity of $z = 0$ as a limit cycle of the differential equation and by the change of variables this multiplicity is the multiplicity of $y = \varphi(x)$ for the initial differential equation (1).

We expand $\psi(x; z_0)$ in Taylor series in a neighborhood of $z_0 = 0$ and we get

$$\psi(x; z_0) = h_1(x)z_0 + h_2(x)z_0^2 + h_3(x)z_0^3 + O(z_0^4),$$

where $h_i(x)$ are differentiable functions for $i = 1, 2, 3$ and $h_1(0) = 1$ while $h_2(0) = h_3(0) = 0$, because $\psi(0; z_0) = z_0$. We have that $\psi(x; z_0)$ satisfies the equality

$$\begin{aligned}\frac{\partial\psi(x;z_0)}{\partial x} &= \sum_{j=0}^{m-1}\frac{\varphi^{m+3-j}\overline{A}_j(x)}{(\psi(x;z_0)+\varphi)^m} - \varphi^3\sum_{j=0}^{m-1}\overline{A}_j(x)\\ &\quad + A_{m+1}(x)\psi(x;z_0) + A_{m+2}(x)[(\psi(x;z_0)+\varphi)^2-\varphi^2]\\ &\quad + \ldots + A_n(x)[(\psi(x;z_0)+\varphi)^{n-m}-\varphi^{n-m}]\\ &= \sum_{j=0}^{m-1}\frac{\varphi^{m+3-j}\overline{A}_j(x)}{(\psi(x;z_0)+\varphi)^m} - \varphi^3\sum_{j=0}^{m-1}\overline{A}_j(x)\\ &\quad + \mathcal{F}(x,\psi(x;z_0)+\varphi) - \mathcal{F}(x,\varphi),\end{aligned}$$

where

$$\mathcal{F}(x,y) = A_{m+1}(x)y + A_{m+2}(x)y^2 + \cdots + A_n(x)y^{n-m}.$$

Now we expand this equation in Taylor series in a neighborhood of $z_0 = 0$. Since this identity needs to be satisfied for any value of z_0 in the considered neighborhood, we can equate the coefficients of the same powers of z_0. Thus we obtain the following system of differential equations for the functions $h_i(x)$ for $i = 1, 2, 3$ (we dropped the dependency of $h_i, \varphi, \overline{A}_0$ in x for simplicity of the notation):

$$\begin{aligned}h_1' &= -\Big(\varphi^3\sum_{j=0}^{m-1}\overline{A}_j - \frac{\partial\mathcal{F}}{\partial y}(x,\varphi)\Big)h_1,\\ h_2' &= -\Big(\varphi^3\sum_{j=0}^{m-1}\overline{A}_j - \frac{\partial\mathcal{F}}{\partial y}(x,\varphi)\Big)h_2 + \Big(\varphi^2\sum_{j=0}^{m-1}\overline{A}_j + \frac{1}{2}\frac{\partial^2\mathcal{F}}{\partial y^2}(x,\varphi)\Big)h_1^2,\\ h_3' &= -\Big(\varphi^3\sum_{j=0}^{m-1}\overline{A}_j - \frac{\partial\mathcal{F}}{\partial y}(x,\varphi)\Big)h_3 + \Big(2\varphi^2\sum_{j=0}^{m-1}\overline{A}_j + \frac{\partial^2\mathcal{F}}{\partial y^2}(x,\varphi)\Big)h_1h_2\\ &\quad - \Big(\varphi\sum_{j=0}^{m-1}\overline{A}_j - \frac{1}{6}\frac{\partial^3\mathcal{F}}{\partial y^3}(x,\varphi)\Big)h_1^3.\end{aligned}$$

The solution of this system of differential equations satisfies

$$\begin{aligned}h_1(x) &= e^{\mathcal{A}_1(x)},\\ h_2(x) &= h_1(x)\mathcal{A}_2(x),\\ h_3(x) &= h_1(x)\mathcal{A}_3(x) + \frac{h_2(x)^2}{h_1(x)},\end{aligned}$$

where the functions $\mathcal{A}_i(x)$ for $i = 1, \ldots, 3$ are the ones in the statement of the theorem. We observe that $h_1(x)$ is strictly positive for any value of x and so the former expressions are well-defined for any value of x. These expressions imply that $h_1(1) = 1$ if and only if $\mathcal{A}_1(1) = 0$. Moreover, for fixed $k \in \{2, 3\}$

we have that $h_1(1) = 1$ and $h_i(1) = 0$ for all i such that $2 \leq i \leq k$ if and only if, $\mathcal{A}_i(1) = 0$ for all i such that $1 \leq i \leq k$.

We note that the translation operator reads as

$$\Psi(z_0) = (h_1(1) - 1)z_0 + h_2(1)z_0^2 + h_3(1)z_0^3 + O(z_0^4),$$

and so $z_0 = 0$ is a limit cycle of multiplicity $m \in \{1, 2\}$ if and only if $\mathcal{A}_i(1) = 0$ for all i such that $1 \leq i < m$ and $\mathcal{A}_m(1) \neq 0$.

Finally, if $\mathcal{A}_1(1) = \mathcal{A}_2(1) = \mathcal{A}_3(1) = 0$, we have that either $z = 0$ is a limit cycle of multiplicity greater than or equal to 4 or it belongs to a continuum of periodic orbits. This concludes the proof of the theorem.

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Received November 18, 2018
Accepted March 3, 2019

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