MATHEMATICA, 62 (85), N° 1, 2020, pp. 81–88

ON THE SUBCENTRAL AUTOMORPHISMS OF FINITE GROUPS

PARISA SEIFIZADEH and MOHAMMAD MEHDI NASRABADI*

Abstract. Let G be a group and let M be a characteristic subgroup of G. We denote by $\mathrm{Aut}_M^M(G)$ the set of all automorphisms of G which centralize G/M and M. In this paper, we give necessary and sufficient conditions for the equality of $\mathrm{Aut}_M^M(G)$ with $\mathrm{Aut}^M(G)$ and $C_{\mathrm{Aut}_M^M(G)}(Z(G))$.

MSC 2010. Primary: 20D45; Secondary: 20D15.

Key words. Characteristic subgroup, finite p-groups, Frattini subgroup, inner automorphism, subcentral automorphism.

1. INTRODUCTION

In this paper, p denotes a prime number. Let us denote by $\Phi(G)$, G', $Z(G)$, $\mathrm{Aut}(G)$ and $\mathrm{Inn}(G)$, respectively, the Frattini subgroup, commutator subgroup, the center, the full automorphism group and the inner automorphism group of G. An automorphism α of G is called a central automorphism if $x^{-1}\alpha(x) \in Z(G)$ for $x \in G$. All the elements of the central automorphism group of G, denoted by $\mathrm{Aut}^{Z(G)}(G)$, form a normal subgroup of $\mathrm{Aut}(G)$.

There has been a number of results on the central automorphisms of a group. Curran and McCaughan [5] proved that, for any non-abelian finite group G, $\mathrm{Aut}_{Z(G)}^{Z(G)}(G) \cong \mathrm{Hom}(G/G'Z(G), Z(G))$, where $\mathrm{Aut}_{Z(G)}^{Z(G)}(G)$ is the group of all those central automorphisms which preserve the center $Z(G)$ elementwise. Also, they showed that if G is a purely non-abelian finite p-group, of nilpotent class 2, then $|\mathrm{Aut}^{Z(G)}(G) : \mathrm{Inn}(G)| \geq p^{r(d-1)}$, where $r = \mathrm{rank}(G/Z(G))$ and $d = \mathrm{rank}(G')$, see [4]. Adney and Yen [1] proved that if a finite group G has no abelian direct factor, then there is a one-to-one and onto map between $\mathrm{Aut}^{Z(G)}(G)$ and $\mathrm{Hom}(G, Z(G))$. Ghumde and Ghate [6] proved that for a finite group G, $\mathrm{Aut}_M^M(G) \cong \mathrm{Hom}(G/KM, M)$. Also they proved that if G is a purely non-abelian finite group, then $|\mathrm{Aut}^M(G)| = |\mathrm{Hom}(G, M)|$. In [8] Shabani Attar characterized all finite p-groups G for which the equality $\mathrm{Aut}^{Z(G)}(G) = \mathrm{Aut}_{Z(G)}^{Z(G)}(G)$ holds. Kaboutari Farimani and Nasrabadi

The authors thank the referee for his helpful comments and suggestions.

*Corresponding author

DOI: 10.24193/mathcluj.2020.1.09

[7] showed necessary and sufficient conditions on finite p-groups such that $\mathrm{Aut}_l(G) = C_{\mathrm{Aut}_l(G)}(Z(G))$.

In this paper, we give necessary and sufficient conditions for G such that $\mathrm{Aut}_M^M(G) = \mathrm{Aut}^M(G)$ and $\mathrm{Aut}_M^M(G) = C_{\mathrm{Aut}_M^M(G)}(Z(G))$.

2. PRELIMINARY LEMMAS

Let M be a characteristic subgroup of G. By $\mathrm{Aut}^M(G)$ we mean the subgroup of $\mathrm{Aut}(G)$ consisting of all automorphisms which induce identity on G/M. By $\mathrm{Aut}_M(G)$ we mean the subgroup of $\mathrm{Aut}(G)$ consisting of all automorphisms which restrict to the identity on M. So we have $\mathrm{Aut}_M^M(G) = \mathrm{Aut}^M(G) \cap \mathrm{Aut}_M(G)$. From now on, M will be a characteristic central subgroup and the elements of $\mathrm{Aut}^M(G)$ will be called subcentral automorphisms of G (with respect to the subcentral subgroup M). It can be seen that $\mathrm{Aut}^M(G)$ is a normal subgroup of $\mathrm{Aut}^{Z(G)}(G)$.

We further let C^* be the set $\{\alpha \in \mathrm{Aut}_M(G) : \alpha\beta = \beta\alpha,\ \forall\ \beta\ \in \mathrm{Aut}^M(G)\}$. Clearly, C^* is a normal subgroup of $\mathrm{Aut}(G)$. Every inner automorphism commutes with the elements of $\mathrm{Aut}^{Z(G)}(G)$, therefore $\mathrm{Inn}(G) \leq C^*$. Let

$$P = \langle [g, \alpha] : g \in G,\ \alpha \in C^* \rangle, \quad \text{where } [g, \alpha] = g^{-1}\alpha(g).$$

It is easy to check that P is a characteristic subgroup of G.

We call a group G purely non-abelian if it does not have an abelian direct factor. Now we state some results that will be used in the proof of the main theorems.

LEMMA 2.1 ([6]). *$\mathrm{Aut}^M(G)$ acts trivially on P.*

Let E^* be any normal subgroup of $\mathrm{Aut}(G)$ contained in C^* and

$$K = \langle [g, \alpha] :\ g \in G,\ \alpha \in E^* \rangle.$$

In particular, when $E^* = \mathrm{Inn}(G)$, we get $K = G'$. Since K is a subgroup of P, it is invariant under the action of $\mathrm{Aut}^M(G)$. It is easy to see that K is a characteristic subgroup of G and hence it is a normal subgroup of G.

LEMMA 2.2 ([6]). *If G is a purely non-abelian finite group, then*

$$|\mathrm{Aut}^M(G)| = |\mathrm{Hom}(G, M)|.$$

LEMMA 2.3 ([6]). *For a finite group G, $\mathrm{Aut}_M^M(G) \cong \mathrm{Hom}(G/KM, M)$.*

LEMMA 2.4 ([6]). *Let G be a purely non-abelian finite group. Then for each $\alpha \in \mathrm{Hom}(G, M)$ and each $x \in K$, we have $\alpha(x) = 1$. Furthermore,*

$$\mathrm{Hom}(G/K, M) \cong \mathrm{Hom}(G, M).$$

DEFINITION 2.5. A finite p-group G is special if G is elementary abelian or $Z(G) = G' = \Phi(G)$ and a non-abelian special p-group G is extraspecial if $Z(G) = G' = \Phi(G) \cong C_p$.

LEMMA 2.6 ([4]). *Suppose H is an abelian p-group of exponent p^c and K is a cyclic group of order divisible by p^c. Then $Hom(H, K)$ is isomorphic to H.*

Corollary 2.7 ([4]). *Let G be a purely non-abelian p-group of nilpotent class 2. Then*

$$|\mathrm{Hom}(G/Z(G), G')| \geq |G/Z(G)|p^{r(s-1)},$$

where $r = \mathrm{rank}(G/Z(G))$ and $s = \mathrm{rank}(G')$.

Lemma 2.8 ([2]). *Let G be a group, and let M and N be two normal subgroups of G such that $M \leq Z(G) \cap N$. Then $\mathrm{Aut}_N^M(G) \cong \mathrm{Hom}(G/N, M)$.*

3. MAIN RESULT

We note that in this section M is a central characteristic subgroup.

Theorem 3.1. *Let G be a finite group. Then G/M is abelian if and only if $\mathrm{Inn}(G) \leq \mathrm{Aut}^M(G)$.*

Proof. Suppose that G/M is abelian. Thus $G' \leq M$. Let $x \in G$. Then for the inner automorphism θ_x induced by x and every $g \in G$ we have, $g^{-1}\theta_x(g) = [g, x] \in G' \subseteq M$. So for every $\alpha \in \mathrm{Inn}(G)$, $g^{-1}\alpha(g) \in M$. This means $\mathrm{Inn}(G) \subseteq \mathrm{Aut}^M(G)$. Hence $\mathrm{Inn}(G) \leq \mathrm{Aut}^M(G)$.

Conversely, suppose that $\mathrm{Inn}(G) \leq \mathrm{Aut}^M(G)$. Hence it is clear that $G' \subseteq M$, and so G/M is abelian. □

Corollary 3.2. *Let G be a finite p-group. If $M = \Phi(G)$, then $\mathrm{Inn}(G) \leq \mathrm{Aut}^M(G)$.*

Proof. Since G is a p-group and $M = \Phi(G) = G'G^p$, $G' \leq M$. Hence, by Theorem 3.1, we have $\mathrm{Inn}(G) \leq \mathrm{Aut}^M(G)$. □

Remark 3.3. If G is a purely non-abelian finite p-group, then $\Omega(M) \leq \Phi(G)$.

Proof. Suppose that $1 \neq m \in \Omega(M) \setminus \Phi(G)$. Therefore there exists a maximal subgroup D such that $m \notin D$, so we have $G \cong \langle m \rangle \times D$. Thus G is not purely non-abelian. □

Theorem 3.4. *Suppose that G is a purely non-abelian finite p-group for which G/M is abelian. Then*

$$|\mathrm{Aut}^M(G) : \mathrm{Inn}(G)| \geq p^{r(s-1)},$$

where $r = \mathrm{rank}(G/Z(G))$ and $s = \mathrm{rank}(G')$.

Proof. Using Lemma 2.2, $|\mathrm{Aut}^M(G)| = |\mathrm{Hom}(G, M)|$. Since G/M is abelian, $G' \leq M \leq Z(G)$, thus G is nilpotent of class 2. Now, by Corollary 2.7, we have

$$\begin{aligned} |\mathrm{Hom}(G, M)| &\geq |\mathrm{Hom}(G/Z(G), M)| \\ &\geq |\mathrm{Hom}(G/Z(G), G')| \\ &\geq |G/Z(G)|p^{r(s-1)}. \end{aligned}$$

Hence, $|\mathrm{Aut}^M(G)| \geq |G/Z(G)|p^{r(s-1)}$, and thus

$$|\mathrm{Aut}^M(G) : \mathrm{Inn}(G)| = |\mathrm{Aut}^M(G)| \geq |G/Z(G)| \geq p^{r(s-1)},$$

which finishes the proof. □

QUESTION 3.5. *Find necessary and sufficient conditions on a finite p-group G such that* $\mathrm{Aut}_M^M(G) = \mathrm{Aut}^M(G)$.

Let G be a non-abelian finite p-group. Let

$$G/K = C_{p^{a_1}} \times C_{p^{a_2}} \times \ldots \times C_{p^{a_k}},$$

where $C_{p^{a_i}}$ is a cyclic group of order p^{a_i}, $1 \leq i \leq k$, and $a_1 \geq a_2 \geq \ldots \geq a_k \geq 1$. Let

$$G/KM = C_{p^{b_1}} \times C_{p^{b_2}} \times \ldots \times C_{p^{b_l}}$$

and

$$M = C_{p^{c_1}} \times C_{p^{c_2}} \times \ldots \times C_{p^{c_m}},$$

where $b_1 \geq b_2 \geq \ldots \geq b_l \geq 1$ and $c_1 \geq c_2 \geq \ldots \geq c_m \geq 1$. Since G/KM is a quotient of G/K, we have $l \leq k$ and $b_i \leq a_i$ for all $1 \leq i \leq l$.

THEOREM 3.6. *Let G be a purely non-abelian finite p-group (p odd). Then* $\mathrm{Aut}_M^M(G) = \mathrm{Aut}^M(G)$ *if and only if* $M \leq K$ *or* $M \leq \Phi(G)$, $k = l$ *and* $c_1 \leq b_t$, *where t is the largest integer between* 1 *and k such that* $a_t > b_t$.

Proof. Let $M \leq K$. By Lemma 2.1 and since $K \leq P$, every $\alpha \in \mathrm{Aut}^M(G)$ fixes M, and so $\mathrm{Aut}^M(G) \leq \mathrm{Aut}_M^M(G)$, since $\mathrm{Aut}_M^M(G) \leq \mathrm{Aut}^M(G)$. Thus $\mathrm{Aut}_M^M(G) = \mathrm{Aut}^M(G)$. Now suppose that $M \leq \Phi(G)$, $k = l$ and $c_1 \leq b_t$. Since G is purely non-abelian, by Lemmas 2.2 and 2.4, we have

$$|\mathrm{Aut}^M(G)| = |\mathrm{Hom}(G, M)| = |\mathrm{Hom}(G/K, M)| = \prod_{1 \leq i \leq k, 1 \leq j \leq m} p^{min\{a_i, c_j\}}.$$

On the other hand, using Lemma 2.3, we have

$$|\mathrm{Aut}_M^M(G)| = |\mathrm{Hom}(G/KM, M)| = \prod_{1 \leq i \leq l, 1 \leq j \leq m} p^{min\{b_i, c_j\}}.$$

Since $b_t \geq c_1$, we have

$$b_1 \geq b_2 \geq \ldots \geq b_{t-1} \geq b_t \geq c_1 \geq c_2 \geq \ldots \geq c_m \geq 1.$$

Therefore, $c_j \leq b_i \leq a_i$ for all $1 \leq j \leq m$ and $1 \leq i \leq t$, whence min $\{a_i, c_j\}$ $= c_j =$ min $\{b_i, c_j\}$ for all $1 \leq j \leq m$ and $1 \leq i \leq t$. Since $a_i = b_i$ for all $i > t$, we have $\min\{a_i, c_j\} = \min\{b_i, c_j\}$ for all $1 \leq j \leq m$ and $t + 1 \leq i \leq k$. Thus $\min\{b_i, c_j\} =$ min $\{a_i, c_j\}$ for all $1 \leq j \leq m$ and $1 \leq i \leq k$. So we have $\mathrm{Aut}_M^M(G) = \mathrm{Aut}^M(G)$.

Conversely, let $\mathrm{Aut}_M^M(G) = \mathrm{Aut}^M(G)$ and $M \nleq K$. We claim that $M \leq \Phi(G)$. Assume contrarily that M is not contained in $\Phi(G)$. Then there exists a maximal subgroup D of G such that $M \nleq D$. The maximality of D implies that $G = DM$ and $D \leq G$. Hence we assume that $|G/D| = p$, where p is a prime number. Next, we consider the following two cases.

Case 1: $p||M \cap D|$. Choose $z \in M \cap D$ such that $o(z) = p$ and fix $g \in M \backslash D$. It is clear that $G = D < g >$. Consider the map α defined on G by $\alpha(dg^i) = dg^i z^i$ for every $d \in D$ and every $i \in \{0, 1, 2, \ldots, p-1\}$. Then $\alpha \in \mathrm{Aut}^M(G)$. By the given hypothesis, $g = \alpha(g) = gz$, whence $z = 1$, which is a contradiction. Hence $M \leq \Phi(G)$.

Case 2: $p \nmid |M \cap D|$. In this case, since

$$p = |G/D| = |DM/D| = |M/M \cap D|,$$

we see that p divides $|M|$ and we may choose $z \in M$ such that $o(z) = p$ and $z \notin D$. Hence $G = \langle D, z \rangle = D \times \langle z \rangle$. Consider the map $\alpha : G \to G$, where $\alpha(dz^i) = dz^{2i}$ for every $d \in D$ and every $i \in \{0, 1, 2, \ldots, p-1\}$. Then $\alpha \in \mathrm{Aut}^M(G)$. By the given hypothesis and since $z \in M$, it is clear that $z = \alpha(z) = \alpha(1.z^1) = z^2$, a contradiction. The proof of the theorem is now complete. □

Definition 3.7. If G is a group and M is a characteristic central subgroup of G, then G is called M-almost semicomplete if $\mathrm{Aut}_M^M(G) = \mathrm{Inn}(G)$.

Theorem 3.8. *Let G be a finite p-group such that G/M is abelian. Then the following are equivalent:*

(1) *G is M-almost semicomplete.*
(2) $\mathrm{Hom}(G/M, M) \cong G/Z(G)$.
(3) *M is cyclic and* $\mathrm{Hom}(G/M, M) \cong \mathrm{Hom}(G/Z(G), M)$.

Proof. (1) ⇒ (2) By Lemma 2.8 and since G is M-almost semicomplete, we have

$$\mathrm{Hom}(G/M, M) \cong \mathrm{Inn}(G) \cong G/Z(G).$$

(2) ⇒ (1) By Lemma 2.8 and since $\mathrm{Hom}(G/M, M) \cong G/Z(G)$, we have $\mathrm{Aut}_M^M(G) \cong \mathrm{Inn}(G)$. Also, since G/M is abelian, we have $G' \leq M$, and so $\mathrm{Inn}(G) \leq \mathrm{Aut}^M(G)$. For every $\alpha \in \mathrm{Inn}(G)$ and $m \in M$, we have $\alpha(m) = m$. Therefore $\mathrm{Inn}(G) \leq \mathrm{Aut}_M^M(G)$, and so G is M-almost semicomplete.

(1) ⇒ (3) Since G is M-almost semicomplete, every $f \in \mathrm{Aut}_M^M(G)$ is an inner one, and so it fixes each element of $Z(G)$. Therefore, for every $f \in \mathrm{Aut}_M^M(G)$, the map $\sigma_f : G/Z(G) \to M$ defined by $\sigma_f(gZ(G)) = g^{-1}f(g)$ is well defined. Now, consider the map $\sigma : f \to \sigma_f$. It is easy to check that σ is an isomorphism from $\mathrm{Aut}_M^M(G)$ onto $\mathrm{Hom}(G/Z(G), M)$, thus

$$\mathrm{Hom}(G/Z(G), M) \cong G/Z(G).$$

Next, we show that M is cyclic. Assume contrarily that M is not cyclic and $\exp(M) = p^e$. Then $M = C_{p^e} \times N$, where C_{p^e} is cyclic subgroup of M and N

is a non-trivial proper subgroup of M. We have

$$\begin{aligned}|G/Z(G)| &= |\mathrm{Hom}(G/Z(G), M)| \\ &= |\mathrm{Hom}(G/Z(G), C_{p^e} \times N)| \\ &= |\mathrm{Hom}(G/Z(G), C_{p^e})||\mathrm{Hom}(G/Z(G), N)|.\end{aligned}$$

Since G/M is abelian, we have $G^{'} \leq M \leq Z(G)$, so G is nilpotent of class 2 and $\exp(G/Z(G)) = \exp(G^{'})$. Now, by Lemma 2.6, we have

$$|\mathrm{Hom}(G/Z(G), C_{p^e})| = |G/Z(G)|.$$

Therefore

$$|G/Z(G)| = |\mathrm{Hom}(G/Z(G), M)| = |G/Z(G)||\mathrm{Hom}(G/Z(G), N)|,$$

which is a contradiction. Hence M is cyclic.

(3) $\Rightarrow$ (1) Since M is cyclic, $G/Z(G)$ is an abelian p-group of exponent $|G^{'}|$ and $G^{'}$ is cyclic, by Lemma 2.6, it follows that $\mathrm{Hom}(G/Z(G), M) \cong G/Z(G)$. Using Lemma 2.8, we have $\mathrm{Aut}_M^M(G) \cong \mathrm{Hom}(G/M, M)$. Since G/M is abelian, by Theorem 3.1, we have $\mathrm{Inn}(G) \leq \mathrm{Aut}^M(G)$. On the other hand, $M \leq Z(G)$, so $\mathrm{Inn}(G) \leq \mathrm{Aut}_M(G)$. Thus $\mathrm{Inn}(G) \leq \mathrm{Aut}_M^M(G)$. Therefore, G is M-almost semicomplete. □

Example 3.9. Let G be an extraspecial p-group and $M = G^{'} \cong C_p$. Then $G^{'}$ is cyclic and

$$\mathrm{Hom}(G/G^{'}, G^{'}) \cong \mathrm{Hom}(G/Z(G), G^{'}).$$

So, by Theorem 3.8, G is a $G^{'}$-almost semicomplete.

Corollary 3.10. *Suppose that G is a finite p-group such that G/M is abelian. Then $C_{\mathrm{Aut}_M^M(G)}(Z(G)) = \mathrm{Inn}(G)$ if and only if M is cyclic.*

Proof. We first prove that $C_{\mathrm{Aut}_M^M(G)}(Z(G)) \cong \mathrm{Hom}(G/Z(G), M)$. Since every element of $C_{\mathrm{Aut}_M^M(G)}(Z(G))$ fixes each element of $Z(G)$, for each $f \in C_{\mathrm{Aut}_M^M(G)}(Z(G))$, the map $\sigma_f : G/Z(G) \to M$ defined by $\sigma_f(gZG)) = g^{-1}f(g)$ is well defined. Now, as in the proof of Theorem 3.8, it is easy to see that the map $f \mapsto \sigma_f$ is an isomorphism of $C_{\mathrm{Aut}_M^M(G)}(Z(G))$ onto $\mathrm{Hom}(G/Z(G), M)$. If $C_{\mathrm{Aut}_M^M(G)}(Z(G)) = \mathrm{Inn}(G)$, then $\mathrm{Hom}(G/Z(G), M) \cong G/Z(G)$, by the proof of Theorem 3.8. Since G/M is abelian, M is cyclic.

Conversely, assume that M is cyclic. Since G/M is abelian, we have

$$|C_{\mathrm{Aut}_M^M(G)}(Z(G))| = |\mathrm{Hom}(G/Z(G), M)| = |G/Z(G)| = |\mathrm{Inn}(G)|.$$

It follows from $\mathrm{Inn}(G) \leq C_{\mathrm{Aut}_M^M(G)}(Z(G))$ that $C_{\mathrm{Aut}_M^M(G)}(Z(G)) = \mathrm{Inn}(G)$. □

Remark 3.11. Let G be a finite p-group and let $\alpha \in \mathrm{Aut}_M^M(G)$ and $p^n = \exp(M)$. Since $g^{-1}\alpha(g) \in M$, $\alpha(g) = gm$ for $m \in M$, we have

$$\alpha(g^{p^n}) = g^{p^n} m^{p^n} [g, m]^{\binom{p^n}{2}}.$$

Now since $M \leq Z(G)$, $[g,m]=1$. Also $m^{p^n}=1$. Therefore, $\alpha(g^{p^n})=g^{p^n}$ for every $g \in G$.

THEOREM 3.12. *Let G be a non-abelian finite p-group. Then* $\mathrm{Aut}_M^M(G) = C_{\mathrm{Aut}_M^M(G)}(Z(G))$ *if and only if* $Z(G)G' \subseteq G'MG^{p^n}$, *where* $p^n = \exp(M)$.

Proof. Suppose that $Z(G)G' \subseteq G'MG^{p^n}$, where $p^n = \exp(M)$. We know that

$$C_{\mathrm{Aut}_M^M(G)}(Z(G)) \leq \mathrm{Aut}_M^M(G).$$

Now, assume that $\sigma \in \mathrm{Aut}_M^M(G)$ and $x \in Z(G)$. We can write $x = abg^{p^n}$ for some $a \in G'$, $b \in M$, and $g \in G$. According to Remark 3.11, $\sigma(g^{p^n}) = g^{p^n}$ and $\sigma(b) = b$. Also, $\mathrm{Aut}_M^M(G)$ acts trivially on G'. Hence, $\sigma(x) = x$, and so $\sigma \in C_{\mathrm{Aut}_M^M(G)}(Z(G))$. This shows that $\mathrm{Aut}_M^M(G) \leq C_{\mathrm{Aut}_M^M(G)}(Z(G))$, whence $\mathrm{Aut}_M^M(G) = C_{\mathrm{Aut}_M^M(G)}(Z(G))$.

To prove the converse, suppose that $\mathrm{Aut}_M^M(G) = C_{\mathrm{Aut}_M^M(G)}(Z(G))$ and $Z(G)G' \nsubseteq G'MG^{p^n}$. Thus there exists $x \in Z(G)$, which is not in $G'MG^{p^n}$. Let

$$G/G'M = \langle x_1G'M\rangle \times \ldots \times \langle x_kG'M\rangle,$$

where $x_1, x_2, \ldots, x_k \in G$. Therefore, $xG'M = x_1^{p^{t_1}}G'M \ldots x_k^{p^{t_k}}G'M$ for some $t_1, t_2, \ldots, t_k$. Since $x \notin G'MG^{p^n}$, we have $x_i^{p^{t_i}} \notin G^{p^n}$, and so $p^{t_i} < p^n$ for some i. Next, select $m \in M$, where $o(m) = min(p^n, o(x_i)G'M)$, and define $f : G/G'M \to M$ by $x_iG'M \mapsto m$ and $x_jG'M \mapsto 1$, for $i \neq j$. Then f can be considered as a homomorphism. Now, consider the map $\sigma_f : G \to G$ defined by $\sigma_f(a) = af(aG'M)$. Clearly, σ_f is an endomorphism of G. Next, suppose that $x \in Ker(\sigma_f)$. Then $f(xG'M) = x^{-1}$. Also, σ_f acts trivially on the elements of M, so we can write $x^{-1} = \sigma_f(x^{-1}) = x^{-1}f(x^{-1}G'M) = x^{-1}x = 1$. Therefore, $x = 1$. This shows that σ_f is one-to-one, and, since G is finite, one can see that the homomorphism σ_f is a bijection. Hence $\sigma_f \in \mathrm{Aut}_M^M(G)$. Moreover, $f(xG'M) = f(x_1^{p^{t_1}}G'M \ldots x_k^{p^{t_k}}G'M)$, and so $f(xG'M) = f(x_i^{p^{t_i}}G'M) = m^{p^{t_i}}$. We have $p^{t_i} < p^n$, and therefore $m^{p^{t_i}}$ is a non-trivial element of M. Hence $\sigma_f \notin C_{\mathrm{Aut}_M^M(G)}(Z(G))$, which is a contradiction. □

COROLLARY 3.13. *Let G be a non-abelian finite p-group such that G/M is abelian. Then* $\mathrm{Aut}_M^M(G) = C_{\mathrm{Aut}_M^M(G)}(Z(G))$ *if and only if* $Z(G) = MG^{p^n}$, *where* $p^n = \exp(M)$.

Proof. Suppose that $\mathrm{Aut}_M^M(G) = C_{\mathrm{Aut}_M^M(G)}(Z(G))$. Using Theorem 3.12 and since G/M is abelian, $Z(G) \subseteq MG^{p^n}$. Also, since $G' \leq M$, for every $a, b \in G$, we have $[a,b]^{p^n} = 1$, whence $[a^{p^n}, b] = 1$. This means that for every $a \in G$, $a^{p^n} \in Z(G)$ and $G^{p^n} \leq Z(G)$. Therefore, $MG^{p^n} \leq Z(G)$, and so $Z(G) = MG^{p^n}$. The converse holds by Theorem 3.12. □

REFERENCES

[1] J.E. Adney and T. Yen, *Automorphisms of a p-group*, Illinois J. Math., **9** (1965), 137–143.
[2] Z. Azhdari and M. Akhavan-Malayeri, *On automorphisms fixing certain groups*, J. Algebra Appl., **12** (2013), 1250163-1–17.
[3] R.D. Carmichael, *Groups of finite order*, Dover Publications, New York, 1965.
[4] M.J. Curran and D.J. McCaughan, *Central automorphisms that are almost inner*, Comm. Algebra, **29** (2001), 2081–2087.
[5] M.J. Curran and D.J. McCaughan, *Finite groups with central automorphism group of minimal order*, Math. Proc. R. Ir. Acad., **104A** (2004), 223–229.
[6] R.G. Ghumde and S.H. Ghate, *Group of automorphisms preserving cosets of a central charactristic subgroup and related results*, Acta Math. Univ. Comenian., **2** (2016), 181–189.
[7] Z. Kaboutari Farimani and M.M. Nasrabadi, *On absolute central automorphisms fixing the center elementwise*, J. Algebr. Syst., **2** (2016), 127–131.
[8] M. Shabani Attar, *Finite p-groups in which each central automorphism fixes centre elementwise*, Comm. Algebra, **40** (2012), 1096–1102.

Received March 17, 2019
Accepted July 21, 2019

University of Birjand
Department of Mathematics
Birjand, Iran
E-mail: paris.seifizadeh@birjand.ac.ir
E-mail: jparis.seifizade@gmail.com
E-mail: mnasrabadi@birjand.ac.ir