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GROUP GRADED ENDOMORPHISM ALGEBRAS AND MORITA EQUIVALENCES

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Abstract. We prove a group graded Morita equivalences version of the "butterfly theorem" on character triples. This gives a method to construct an equivalence between block extensions from another related equivalence.

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Key words. Block extension, centralizer subalgebra, crossed product, group graded Morita equivalence.

1. INTRODUCTION AND PRELIMINARIES

The Butterfly theorem, as stated by B. Späth in [3, Theorem 2.16], gives the possibility to construct certain relations between character triples. The result is very useful in obtaining reduction methods for the local-global conjectures in modular representation theory of finite groups. In this paper, we consider group graded Morita equivalences between block extensions and we obtain an analogue of [3, Theorem 2.16]. Our main result, Theorem 4.2, shows how to construct a group graded Morita equivalence from a given one, under very similar assumptions to those in [3].

In general, our notations and assumptions are standard and follow [2]. To introduce our context, let G be a finite group, N a normal subgroup of G, and denote by $\bar{G}$ the factor group G/N. Let $A = \bigoplus_{\bar{g}\in\bar{G}} A_{\bar{g}}$ be a strongly $\bar{G}$-graded $\mathcal{O}$-algebra with the identity component $B := A_1$, where $(\mathcal{K}, \mathcal{O}, \Bbbk)$ is a p-modular system. For a subgroup $\bar{H}$ of $\bar{G}$, we denote by $A_{\bar{H}} := \bigoplus_{\bar{g}\in\bar{H}} A_{\bar{g}}$ the truncation of A from $\bar{G}$ to $\bar{H}$.

For the sake of simplicity, in this article we will mostly consider only crossed products, also because the generalization of the statements to the case of strongly graded algebras is a mere technicality. Recall that, if A is a crossed product, we can chose an invertible homogeneous element $u_{\bar{g}}$ in the component $A_{\bar{g}}$, for all $\bar{g} \in \bar{G}$.

Our main example for a $\bar{G}$-graded crossed product is obtained as follows: Regard $\mathcal{O}G$ as a $\bar{G}$-graded algebra with the 1-component $\mathcal{O}N$. Let $b \in Z(\mathcal{O}N)$ be a $\bar{G}$-invariant block idempotent. We denote $A := b\mathcal{O}G$ and $B := b\mathcal{O}N$. Then the block extension A is a $\bar{G}$-graded crossed product, with 1-component B.

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The paper is organized as follows. In Section 2, we recall from [2] the main facts on group graded Morita equivalences and we state a graded variant of the second Morita Theorem [1, Theorem 12.12]. In Section 3, we show that there is a natural map, compatible with Morita equivalences, from the centralizer $C_A(B)$ of B in A to the endomorphism algebra of a $\bar{G}$-graded A-module induced from a B-module. In the last section, we prove that a Morita equivalence between the 1-components of two block extensions always lifts to a graded equivalence between certain centralizer algebras. This is the main ingredient in the proof of our main result, Theorem 4.2.

2. GROUP GRADED MORITA EQUIVALENCES

Let $A = \bigoplus_{\bar{g}\in\bar{G}} A_{\bar{g}}$ and $A' = \bigoplus_{\bar{g}\in\bar{G}} A'_{\bar{g}}$ be strongly $\bar{G}$-graded algebras, with the 1-components B and B' respectively.

It is clear that $A\otimes_{\mathcal{O}}A'^{\mathrm{op}}$ is a $\bar{G}\times\bar{G}$-graded algebra. Let

$$\delta(\bar{G}) := \{(\bar{g},\bar{g}) \mid \bar{g}\in\bar{G}\}$$

be the diagonal subgroup of $\bar{G}\times\bar{G}$, and let Δ be the diagonal subalgebra of $A\otimes_{\mathcal{O}}A'^{\mathrm{op}}$

$$\Delta := (A\otimes_{\mathcal{O}}A'^{\mathrm{op}})_{\delta(\bar{G})} = \bigoplus_{\bar{g}\in\bar{G}} A_{\bar{g}}\otimes A'_{\bar{g}^{-1}}.$$

Then Δ is a $\bar{G}$-graded algebra, with 1-component $\Delta_1 = B\otimes_{\mathcal{O}}B'^{\mathrm{op}}$.

Let M be a (B,B')-bimodule, or, equivalently, M is a $B\otimes_{\mathcal{O}}B'^{\mathrm{op}}$-module, thus a Δ_1-module. Let $M^* := \mathrm{Hom}_B(M,B)$ be its B-dual. Note that if B is a symmetric algebra, then we have the isomorphism

$$M^* := \mathrm{Hom}_B(M,B) \simeq \mathrm{Hom}_{\mathcal{O}}(M,\mathcal{O}),$$

where $\mathrm{Hom}_{\mathcal{O}}(M,\mathcal{O})$, is the $\mathcal{O}$-dual of M.

Definition 2.1. We say that the $\bar{G}$-graded (A,A')-bimodule $\tilde{M}$ induces a $\bar{G}$-graded Morita equivalence between A and A', if $\tilde{M}\otimes_{A'}\tilde{M}^*\simeq A$ as $\bar{G}$-graded (A,A)-bimodules and that $\tilde{M}^*\otimes_A\tilde{M}\simeq A'$ as $\bar{G}$-graded (A',A')-bimodules, where the A-dual $\tilde{M}^* = \mathrm{Hom}_A(\tilde{M},A)$ of $\tilde{M}$ is a $\bar{G}$-graded (A',A)-bimodule.

By [2, Theorem 5.1.2], the following statements are equivalent:

(1) between B and B' we have a Morita equivalence given by the Δ_1-module M and M extends to a Δ-module;
(2) $\tilde{M} := A\otimes_B M$ is a $\bar{G}$-graded (A,A')-bimodule and $\tilde{M}^* := A'\otimes_{B'}M^*$ is a $\bar{G}$-graded (A',A)-bimodule, which induce a $\bar{G}$-graded Morita equivalence between A and A', given by the functors:

$$A \underset{{}_{A'}\tilde{M}^*_A\otimes_A -}{\overset{{}_A\tilde{M}_{A'}\otimes_{A'} -}{\rightleftarrows}} A'.$$

In this case, by [2, Lemma 1.6.3], we have the natural isomorphisms of $\bar{G}$-graded bimodules

$$\tilde{M} := A\otimes_B M \simeq M\otimes_{B'} A' \simeq ((A\otimes_{\mathcal{O}} A'^{\mathrm{op}})\otimes_\Delta M).$$

Assume that B and B' are Morita equivalent. Then, by the second Morita Theorem [1, Theorem 12.12], we can choose the bimodule isomorphisms

$$\varphi : M^*\otimes_B M \to B', \qquad \psi : M\otimes_{B'} M^* \to B.$$

such that

$$\psi(m\otimes m^*)n = m\varphi(m^*\otimes n), \quad \forall m, n \in M,\ m^* \in M^*$$

and that

$$\varphi(m^*\otimes m)n^* = m^*\psi(m\otimes n^*), \quad \forall m^*, n^* \in M^*,\ m \in M.$$

By the surjectivity of this functions, we may choose finite sets I and J and the elements m_j^*, $n_i^* \in M^*$ and m_j, $n_i \in M$, for all $i \in I,\ j \in J$ such that:

$$\varphi(\sum_{j\in J} m_j^*\otimes_B m_j) = 1_{B'}, \qquad \psi(\sum_{i\in I} n_i\otimes_B n_i^*) = 1_B.$$

Assume that $\tilde{M}$ and $\tilde{M}^*$ give a $\bar{G}$-graded Morita equivalence between A and A'. As above, by [1, Theorem 12.12], we can choose the isomorphisms

$$\tilde{\varphi} : \tilde{M}^*\otimes_A \tilde{M} \to A', \qquad \tilde{\psi} : \tilde{M}\otimes_{A'} \tilde{M}^* \to A$$

of $\bar{G}$-graded bimodules such that

$$\tilde{\psi}(\tilde{m} \otimes \tilde{m}^*)\tilde{n} = \tilde{m}\tilde{\varphi}(\tilde{m}^* \otimes \tilde{n}), \quad \forall \tilde{m}, \tilde{n} \in \tilde{M},\ \tilde{m}^* \in \tilde{M}^*$$

and that

$$\tilde{\varphi}(\tilde{m}^* \otimes \tilde{m})\tilde{n}^* = \tilde{m}^*\tilde{\psi}(\tilde{m} \otimes \tilde{n}^*), \quad \forall \tilde{m}^*, \tilde{n}^* \in \tilde{M}^*,\ \tilde{m} \in \tilde{M}.$$

Actually, $\tilde{\varphi}_1$ and $\tilde{\psi}_1$ are the same with φ and ψ from before and are Δ-linear isomorphisms. Moreover, we have that $1_A = 1_B \in B$ and $1_{A'} = 1_{B'} \in B'$. Henceforth, we may choose the same finite sets I and J and the same elements m_j^*, $n_i^* \in M^*$ and m_j, $n_i \in M$, $\forall i \in I, j \in J$ such that:

$$\tilde{\varphi}(\sum_{j\in J} m_j^*\otimes_B m_j) = 1_{B'}, \qquad \tilde{\psi}(\sum_{i\in I} n_i\otimes_B n_i^*) = 1_B.$$

3. CENTRALIZERS AND GRADED ENDOMORPHISM ALGEBRAS

We will assume that A and A' are $\bar{G}$-graded crossed products, although the results of this section can be generalized to strongly graded algebras. Let $U \in B$-mod and $U' \in B'$-mod such that $U' = M^*\otimes_B U$. We denote

$$E(U) := \mathrm{End}(A\otimes_B U)^{\mathrm{op}}, \qquad E(U') := \mathrm{End}(A'\otimes_{B'} U')^{\mathrm{op}},$$

the $\bar{G}$-graded endomorphism algebras of the modules induced from U and U'.

We will prove that there exists a natural $\bar{G}$-graded algebra homomorphism between the centralizer of B in A and $E(U)$, compatible with $\bar{G}$-graded Morita equivalences.

Lemma 3.1. *The map*

$$\theta : C_A(B) \to E(U), \qquad \theta(c)(a\otimes u) = ac\otimes u,$$

where $c \in C_A(B)$, $a \in A$ and $u \in U$ is a homomorphism of $\bar{G}$-graded algebras.

Proof. We first need to show that the map is well-defined. For $c \in C_A(B)$, $a \in A$, $b \in B$ and $u \in U$, we have:

$$\theta(c)(ab\otimes_B u) = ab \cdot c\otimes_B u = acb\otimes_B u = ac\otimes_B bu = \theta(c)(a\otimes_B bu).$$

To show that $\theta(c)$ is A-linear, let $a' \in A$; we have:

$$\theta(c)(a'a\otimes_B u) = a'ac\otimes_B u = a'(ac\otimes_B u) = a'\theta(c)(a\otimes_B u).$$

To prove that the map is a ring homomorphism, let $c, c' \in C_A(B)$; we have:

$$\begin{aligned}(\theta(c)\cdot\theta(c'))(a\otimes_B u) &= (\theta(c')\circ\theta(c))(a\otimes_B u)\\ &= \theta(c')(\theta(c)(a\otimes_B u))\\ &= \theta(c')(ac\otimes_B u) = acc'\otimes_B u\\ &= \theta(cc')(a\otimes_B u).\end{aligned}$$

Finally, we check that θ is grade-preserving. Let $a_{\bar{g}}\otimes_B u \in A_{\bar{g}}\otimes_B U$ and $c \in C_A(B)_{\bar{h}}$, where $\bar{g}, \bar{h} \in \bar{G}$. Then the definition of θ says that

$$\theta(c)(a_{\bar{g}}\otimes_B u) = a_{\bar{g}} \cdot c\otimes_B u \in A_{\bar{g}\bar{h}}\otimes_B U.$$

If follows that $\theta(c)$ belongs to $E(U)_{\bar{h}}$. The other properties are obvious. □

By [2, Lemma 1.6.3], we have

$$A\otimes_B M \simeq M\otimes_{B'} A',$$

and we will need an explicit isomorphism between the two. We will choose invertible elements $u_{\bar{g}} \in U(A) \cap A_{\bar{g}}$ and $u'_{\bar{g}} \in U(A) \cap A'_{\bar{g}}$ of degree $\bar{g} \in \bar{G}$. We have that an arbitrary element $a'_{\bar{g}} \in A'_{\bar{g}}$ can be written uniquely in the form $a'_{\bar{g}} = u'_{\bar{g}}b'$, where $b' \in B'$. The desired $\bar{G}$-graded bimodule isomorphism is:

$$\varepsilon : M\otimes_{B'} A' \to A\otimes_B M \qquad m\otimes_{B'} a'_{\bar{g}} \mapsto u_{\bar{g}}\otimes_B u_{\bar{g}}^{-1} m a'_{\bar{g}}$$

for $m \in M$. We will also need the explicit isomorphism of $\bar{G}$-graded bimodules

$$\beta : A'\otimes_{B'} M^* \to M^*\otimes_B A \qquad a'_{\bar{g}}\otimes_{B'} m^* \mapsto a'_{\bar{g}} m^* u_{\bar{g}}^{-1} \otimes_B u_{\bar{g}}$$

for $m^* \in M^*$. Henceforth we consider the isomorphism of $\bar{G}$-graded A'-modules

$$\beta\otimes_B id_U : A'\otimes_{B'} M^*\otimes_B U \to M^*\otimes_B A\otimes_B U.$$

Proposition 3.2. *Assume that $\tilde{M}$ and $\tilde{M}^*$ give a $\bar{G}$-graded Morita equivalence between A and A'. Then the diagram*

$$\begin{array}{ccc} E(U) & \xrightarrow[\sim]{\varphi_1} & E(U') \\ {\scriptstyle\theta}\uparrow & & \uparrow{\scriptstyle\theta'} \\ C_A(B) & \xrightarrow[\sim]{\varphi_2} & C_{A'}(B') \end{array}$$

is commutative, where the maps are defined as follows:

$$\theta(c)(a\otimes u) = ac\otimes u,$$
$$\theta'(c')(a'\otimes u') = a'c'\otimes u'$$
$$\varphi_1(f) = (\beta\otimes_B id_U)^{-1}\circ(id_{\tilde{M}^*}\otimes f)\circ(\beta\otimes_B id_U),$$
$$\varphi_2(c) = \tilde{\varphi}(\sum_{j\in J} m_j^* c\otimes_B m_j).$$

for all $a\in A$, $a'\in A'$, $c\in C_A(B)$, $c'\in C_{A'}(B')$, $u\in U$, $u'\in U'$ and $f\in E(U)$.

Proof. According to Lemma 3.1, we have that θ, θ' are homomorphisms of $\bar{G}$-graded algebras. Moreover, φ_1 and φ_2 are the algebra isomorphisms induced by the $\bar{G}$-graded Morita equivalence.

To prove that the diagram is commutative, let $c\in C_A(B)_{\bar{h}}$, where $\bar{h}\in\bar{G}$. We consider arbitrary elements $a'_{\bar{g}}\in A'_{\bar{g}}$, where $\bar{g}\in\bar{G}$ and $u' = m^*\otimes_B u\in U' = M^*\otimes_B U$. By the above remarks, for all $f\in E(U)$, we have

$$\varphi_1(f)(a'_{\bar{g}}\otimes_{B'} m^*\otimes_B u) = a'_{\bar{g}} m^* u_{\bar{g}}^{-1}\otimes_B f(u_{\bar{g}}\otimes_B u),$$

hence, for $f = \theta(c)\in E(U)$ we get

$$\varphi_1(\theta(c))(a'_{\bar{g}}\otimes_{B'} m^*\otimes_B u) = a'_{\bar{g}} m^* u_{\bar{g}}^{-1}\otimes_B u_{\bar{g}} c\otimes_B u.$$

On the other hand, $c' := \varphi_2(c)\in C_{A'}(B')_h$, hence, via the identification given by the isomorphism β, we have

$$\begin{aligned}\theta'(\varphi_2(c))(a'_{\bar{g}}\otimes_{B'} m^*\otimes_B u) &= a'_{\bar{g}} c' m^* u_{\bar{h}}^{-1} u_{\bar{g}}^{-1}\otimes_B u_{\bar{g}} u_{\bar{h}}\otimes_B u\\ &= a'_{\bar{g}}\tilde{\varphi}(\sum_j m_j^* c\otimes_B m_j) m^* u_{\bar{h}}^{-1} u_{\bar{g}}^{-1}\otimes_B u_{\bar{g}} u_{\bar{h}}\otimes_B u\\ &= a'_{\bar{g}}\sum_j m_j^* c\psi(m_j\otimes_{B'} m^*) u_{\bar{h}}^{-1} u_{\bar{g}}^{-1}\otimes_B u_{\bar{g}} u_{\bar{h}}\otimes_B u\\ &= a'_{\bar{g}}\sum_j m_j^*\psi(m_j\otimes_{B'} m^*) u_{\bar{g}}^{-1} u_{\bar{g}} c u_{\bar{h}}^{-1} u_{\bar{g}}^{-1}\otimes_B u_{\bar{g}} u_{\bar{h}}\otimes_B u\\ &= a'_{\bar{g}}\varphi(\sum_j m_j^*\otimes_B m_j) m^* u_{\bar{g}}^{-1}\otimes_B u_{\bar{g}} c u_{\bar{h}}^{-1} u_{\bar{g}}^{-1} u_{\bar{g}} u_{\bar{h}}\otimes_B u\\ &= a'_{\bar{g}} m^* u_{\bar{g}}^{-1}\otimes_B u_{\bar{g}} c\otimes_B u.\end{aligned}$$

Thus the statement is proved. □

4. THE BUTTERFLY THEOREM FOR $\bar{G}$-GRADED MORITA EQUIVALENCES

Let N be a normal subgroup of G, G' a subgroup of G, and N' a normal subgroup of G'. We assume that $N' = G' \cap N$ and $G = G'N$, hence $\bar{G} := G/N \simeq G'/N'$. Let $b \in Z(\mathcal{O}N)$ and $b' \in Z(\mathcal{O}N')$ be $\bar{G}$-invariant block idempotents. We denote

$$A := b\mathcal{O}G, \qquad A' := b'\mathcal{O}G', \qquad B := b\mathcal{O}N, \qquad B' := b'\mathcal{O}N'.$$

Then A and A' are strongly $\bar{G}$-graded algebras, with 1-components B and B' respectively.

Additionally, assume that $C_G(N) \subseteq G'$, and denote $\bar{C}_G(N) := NC_G(N)/N$. We consider the algebras

$$\begin{array}{ccc}
A := b\mathcal{O}G & & A' := b'\mathcal{O}G' \\
\Big| & & \Big| \\
C := b\mathcal{O}NC_G(N) & \overset{}{\underset{\sim}{\rule{6em}{0.4pt}}} & C' := b'\mathcal{O}N'C_G(N) \\
\Big| & & \Big| \\
B := b\mathcal{O}N & \overset{{}_BM_{B'}}{\underset{\sim}{\rule{6em}{0.4pt}}} & B' := b'\mathcal{O}N'.
\end{array}$$

If M induces a Morita equivalence between B and B', the question that arises is what can we deduce without the additional hypothesis that M extends to a Δ-module. One answer is given by the following proposition.

PROPOSITION 4.1. *Assume that:*

(1) $C_G(N) \subseteq G'$.
(2) *M induces a Morita equivalence between B and B'.*
(3) *$zm = mz$ for all $m \in M$ and $z \in Z(N)$.*

Then there is a $\bar{C}_G(N)$-graded Morita equivalence between C and C', induced by the $\bar{C}_G(N)$-graded (C, C')-bimodule

$$C \otimes_B M \simeq M \otimes_{B'} C' \simeq (C \otimes C'^{\mathrm{op}}) \otimes_{\Delta(C \otimes C'^{\mathrm{op}})} M.$$

Proof. Firstly, it is easy to see that our assumption implies that $NC_G(N)/N$ is isomorphic to $N'C_G(N)/N'$. Thus both C and C' are indeed strongly $\bar{C}_G(N)$-graded algebras.

Now, we prove that there is a $\bar{C}_G(N)$-graded Morita equivalence between C and C'. It suffices to prove that $C \otimes_B M$ is actually a $\bar{C}_G(N)$-graded (C, C')-bimodule.

In view of Lemma 3.1, there exists a $\bar{G}$-graded algebra homomorphism between $C_A(B)$ and $\mathrm{End}_A(A \otimes_B M)^{\mathrm{op}}$. Moreover, note that $A \otimes_B M$ is a $\bar{G}$-graded $(A, \mathrm{End}_A(A \otimes_B M)^{\mathrm{op}})$-bimodule, hence by restricting the scalars we obtain that $A \otimes_B M$ is a $\bar{G}$-graded $(A, C_A(B))$-bimodule. We truncate to the subgroup $\bar{C}_G(N)$ of $\bar{G}$, and we obtain that $A_{\bar{C}_G(N)}\otimes_B M$ is a $\bar{C}_G(N)$-graded $(A_{\bar{C}_G(N)}, C_A(B)_{\bar{C}_G(N)})$-bimodule, but $A_{\bar{C}_G(N)} = b\mathcal{O}NC_G(N) = C$, hence $\hat{M} := C\otimes_B M$ is a $\bar{C}_G(N)$-graded $(C, C_A(B)_{\bar{C}_G(N)})$-bimodule.

We have that $\mathcal{O}C_G(N)$ is $\bar{C}_G(N)$-graded with the 1-component $\mathcal{O}Z(N)$ and there is an algebra homomorphism from $\mathcal{O}C_G(N)$ to $C_A(B)$, whose image is evidently included in $C_A(B)_{\bar{C}_G(N)}$. Hence, by restricting the scalars, we obtain that $\hat{M}$ is a $\bar{C}_G(N)$-graded $(C, \mathcal{O}C_G(N))$-bimodule. Finally, since M is (B, B')-bimodule, where $B' = b'\mathcal{O}N'$, we may define on $\hat{M}$ a structure of a $\bar{C}_G(N)$-graded $(C, b'\mathcal{O}N'C_G(N))$-bimodule, as follows. Let $c \in C$, $m \in M$, $c' \in C_G(N) \subseteq C'$ and $n \in N$ and define $(c\otimes m)c'n = cc'\otimes mn$. To see that this is well-defined, let $z \in Z(N)$, so $c'n = (c'z)(z^{-1}n)$. Then, by assumption (3), we have

$$(c\otimes m)(c'z)(z^{-1}n) = cc'z\otimes mz^{-1}n = cc'\otimes zmz^{-1}n = cc'\otimes mn.$$

Consequently, $\hat{M}$ is a $\bar{C}_G(N)$-graded (C, C')-bimodule. $\square$

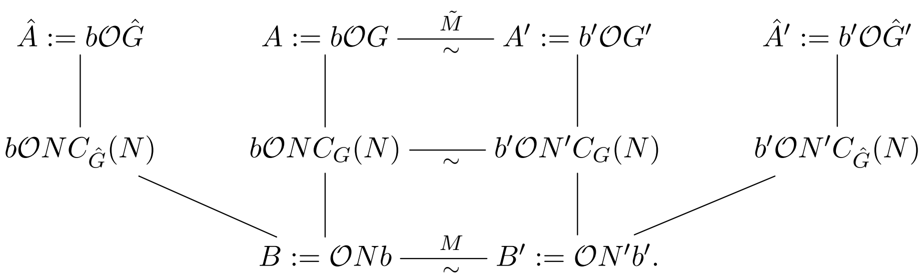
Our main result is a version for Morita equivalences of the so-called "butterfly theorem" [3, Theorem 2.16].

THEOREM 4.2. *Let $\hat{G}$ be another group with normal subgroup N such that the block b is also $\hat{G}$-invariant. Assume that:*

(1) *$C_G(N) \subseteq G'$;*
(2) *$\tilde{M}$ induces a $\bar{G}$-graded Morita equivalence between A and A';*
(3) *$zm = mz$ for all $m \in M$ and $z \in Z(N)$;*
(4) *the conjugation maps $\varepsilon : G \to Aut(N)$ and $\hat{\varepsilon} : \hat{G} \to \mathrm{Aut}(N)$ satisfy $\varepsilon(G) = \hat{\varepsilon}(\hat{G})$.*

Denote $\hat{G}' = \hat{\varepsilon}^{-1}(\varepsilon(G'))$. Then there is a $\hat{G}/N$-graded Morita equivalence between $\hat{A} := b\mathcal{O}\hat{G}$ and $\hat{A}' := b'\mathcal{O}\hat{G}'$.

Proof. Consider the following diagram:



By the proof of [3, Theorem 2.16], we have that $C_{\hat{G}}(N) \leq \hat{G}'$, $\hat{G} = N\hat{G}'$ and $N' = N\cap\hat{G}'$. Note that $NC_G(N)$ is the kernel of the map $G \to \mathrm{Out}(N)$ induced by conjugation. Hence the hypothesis $\varepsilon(G) = \hat{\varepsilon}(\hat{G})$ implies that $G/NC_G(N) \simeq \hat{G}/NC_{\hat{G}}(N)$. It follows that $\bar{G}/\bar{C}_G(N) \simeq \bar{\hat{G}}/\bar{C}_{\hat{G}}(N)$.

Let C and C' be as in Proposition 4.1 and denote $\hat{C} = b\mathcal{O}NC_{\hat{G}}(N)$ and $\hat{C}' = b'\mathcal{O}N'C_{\hat{G}'}(N)$. By Proposition 4.1, we know that the Morita equivalence between B and B' induced by M extends to a $\bar{C}_{\hat{G}}(N)$-graded Morita equivalence between $\hat{C}$ and $\hat{C}'$, induced by $\hat{C}\otimes_B M$.

Let $\mathcal{T} \subseteq G'$ be a complete set of representatives for the cosets of $N'C_G(N)$ in G'. Because $G = NG'$, $\mathcal{T}$ is a complete set of representatives for the cosets of $NC_G(N)$ in G.

For any $t \in \mathcal{T}$, we choose $\hat{t} \in \hat{G}'$ such that $\varepsilon(t) = \hat{\varepsilon}(\hat{t})$. Thus, we obtain a complete set $\hat{\mathcal{T}}$ of representatives of $N'C_{\hat{G}}(N)$ in $\hat{G}'$, so $\hat{\mathcal{T}}$ is also a complete set of representatives for the cosets of $NC_{\hat{G}}(N)$ in $\hat{G}$.

We need to define a $\hat{\Delta} := \Delta(\hat{A} \otimes \hat{A}'^{\mathrm{op}})$-module structure on M, knowing that M is $\Delta(A \otimes A'^{\mathrm{op}})$-module and a $\Delta(\hat{A}_{\bar{C}_{\hat{G}}(N)} \otimes \hat{A}'^{\mathrm{op}}_{\bar{C}_{\hat{G}}(N)})$-module, where

$$\Delta(\hat{A}_{\bar{C}_{\hat{G}}(N)} \otimes \hat{A}'^{\mathrm{op}}_{\bar{C}_{\hat{G}}(N)}) \simeq \Delta(\hat{A} \otimes \hat{A}'^{\mathrm{op}})_{\bar{C}_{\hat{G}}(N)}.$$

We define $(\hat{t}\otimes\hat{t}^{\circ})\cdot m = (t\otimes t^{\circ})\cdot m$. It is a routine to verify that this definition does not depend on the choices we made and that it gives the required $\hat{\Delta}$-module structure on M.

Alternatively, one may argue as follows: The cohomology class $[\hat{\alpha}]$ from $H^2(\hat{G}/N, Z(B)^{\times})$ associated to the $\hat{\Delta}_1$-module M satisfies $\mathrm{Res}^{\hat{G}/N}_{\bar{C}_{\hat{G}}(N)}[\hat{\alpha}] = 1$, because M extends to a $\hat{\Delta}_{\bar{C}_{\hat{G}}(N)}$-module. It follows that $[\hat{\alpha}] \in \mathrm{ImInf}^{\hat{G}}_{NC_{\hat{G}}(N)}$. On the other hand, the class $[\alpha] \in H^2(\bar{G}, Z(B)^{\times})$ associated to the Δ_1-module M is trivial, since M extends to a Δ-module. It is easy to see that $(t\otimes t^{\circ})\otimes M \simeq (\hat{t} \otimes \hat{t}^{\circ}) \otimes M$ as (B, B)-bimodules, and, since $G/NC_G(N) \simeq \hat{G}/NC_{\hat{G}}(N)$, we deduce that $[\hat{\alpha}]$ is also trivial, hence M extends to a $\hat{\Delta}$-module. □

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