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ASYMPTOTICALLY DOUBLE λ_2-STATISTICALLY EQUIVALENT SEQUENCES OF INTERVAL NUMBERS

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Abstract. In this paper we have introduced the concept of λ_2 - asymptotically double statistical equivalent of interval numbers and strong λ_2 - asymptotically double statistical equivalent of interval numbers. We have investigated the relations related to these spaces.

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Key words. Asymptotically, interval number, λ_2-statistically.

1. INTRODUCTION

Interval arithmetic was first suggested by Dwyer [8] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [26] in 1959 and Moore and Yang [27] in 1962. Furthermore, Moore and others [9, 20] have developed applications to differential equations.

Chiao in [4] introduced sequences of interval numbers and defined usual convergence of sequences of interval numbers. Sengonul and Eryilmaz in [32] introduced and studied bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete metric spaces.

Recently, Esi [10, 11] introduced and studied strongly almost λ-convergence and statistically almost λ-convergence of interval numbers and lacunary sequence spaces of interval numbers, respectively. For more information about interval numbers, one may refer to Esi [12–18], Debnath and Saha [7], Debnath et al. [5, 6].

The idea of statistical convergence for ordinary sequences was introduced by Fast [19] in 1951. Schoenberg [31] studied statistical convergence as a summability method and listed some elementary properties of statistical convergence. Both of these authors noted that if a bounded sequence is statistically convergent, then it is Cesaro summable. Existing work on statistical convergence appears to have been restricted to real or complex sequences, but several authors extended the idea to apply it to sequences of fuzzy numbers and also

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introduced and discussed the concept of statistical sequences of fuzzy numbers. For some very interesting investigations concerning statistical convergence, one may consult the papers of Cakalli [3], Miller [25] , Maddox [23] and many others [1, 2, 22], where more references on this important summability method can be found.

In 1993, Marouf [24] presented definitions for asymptotically equivalent sequences and asymptotic regular matrices. In 2003, Patterson [29] extended these concepts by presenting as an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for non-negative summability matrices.

2. PRELIMINARIES

We denote the set of all real valued closed intervals by $\mathbb{IR}$. Any element of $\mathbb{IR}$ is called interval number and denoted by $\overline{A} = [x_l, x_r]$. Let x_l and x_r be the first and last points of the $\overline{x}$ interval number, respectively. For $\overline{A}_1, \overline{A}_2 \in \mathbb{IR}$, we have $\overline{A}_1 = \overline{A}_2 \Leftrightarrow x_{1_l} = x_{2_l}, x_{1_r} = x_{2_r}$. $\overline{A}_1 + \overline{A}_2 = \{x \in \mathbb{R} : x_{1_l} + x_{2_l} \leq x \leq x_{1_r} + x_{2_r}\}$, and if $\alpha \geq 0$, then $\alpha\overline{A} = \{x \in \mathbb{R} : \alpha x_{1_l} \leq x \leq \alpha x_{1_r}\}$ and if $\alpha < 0$, then $\alpha\overline{A} = \{x \in \mathbb{R} : \alpha x_{1_r} \leq x \leq \alpha x_{1_l}\}$, $\overline{A}_1.\overline{A}_2 =$ set of real numbers x such that $\min\{x_{1_l}.x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l}, x_{1_r}.x_{2_r}\} \leq x \leq \max\{x_{1_l}.x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l}, x_{1_r}.x_{2_r}\}$.

The set of all interval numbers $\mathbb{IR}$ is a complete metric space defined by

$$d\left(\overline{A}_1, \overline{A}_2\right) = \max\left\{|x_{1_l} - x_{2_l}|, |x_{1_r} - x_{2_r}|\right\}.$$

In the special case $\overline{A}_1 = [a, a]$ and $\overline{A}_2 = [b, b]$, we obtain the usual metric of $\mathbb{R}$. Let us define the transformation $f : \mathbb{N} \to \mathbb{R}$ by $k \to f(k) = \overline{x}$, $\overline{x} = (\overline{x}_k)$. Then $\overline{x} = (\overline{x}_k)$ is called a sequence of interval numbers. The $\overline{x}_k$ is called the k^{th} term of the sequence $\overline{x} = (\overline{x}_k)$. w^i denotes the set of all interval numbers with real terms.

A sequence $x = (x_k)$ is said to be statistically convergent to the number L if, for every $\varepsilon > 0$, $\lim_n \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0$, where the vertical bars indicate the number of elements in the enclosed set.

Definition 2.1 ([10]). A sequence $\overline{x} = (\overline{x}_k)$ of interval numbers is said to be convergent to the interval number $\overline{x}_o$ if for each $\varepsilon > 0$ there exists a positive integer k_o such that $d(\overline{x}_k, \overline{x}_o) < \varepsilon$ for all $k \geq k_o$ and we denote it by $\lim_k \overline{x}_k = \overline{x}_o$. Thus, $\lim_k \overline{x}_k = \overline{x}_o \Leftrightarrow \lim_k x_{k_l} = x_{o_l}$ and $\lim_k x_{k_r} = x_{o_r}$.

Definition 2.2 ([24]). Two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if $\lim_k \frac{x_k}{y_k} = 1$ (denoted by $x \sim y$).

Definition 2.3 ([21]). The sequence $x = (x_k)$ has statistical limit L, denoted by $st - \lim x = L$ provided that for every $\varepsilon > 0$

$$\lim_n \frac{1}{n}\{\text{the number of } k \leq n : |x_k - L| \geq \varepsilon\} = 0.$$

Definition 2.4 ([29]). Two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistical equivalent of multiple L provided that for every $\varepsilon > 0$

$$\lim_n \frac{1}{n}\{\text{the number of } k \leq n : |\frac{x_k}{y_k} - L| \geq \varepsilon\} = 0$$

(denoted by $x \overset{S_L}{\backsim} y$) and simply asymptotically statistical equivalent if $L = 1$.

Definition 2.5 ([15]). Two non-negative sequences of interval numbers $\overline{x} = (\overline{x}_k)$ and $\overline{y} = (\overline{y}_k) \neq \overline{0} = [0,0]$ are said to be asymptotically $\overline{s}_\lambda$-statistical equivalent of multiple $\overline{L}$ provided that for every $\epsilon > 0$

$$\lim_n \frac{1}{\lambda_n} \left|\left\{k \in I_n :\ d(\frac{\overline{x}_k}{\overline{y}_k}, \overline{L})|) \geq \varepsilon\right\}\right| = 0$$

(denoted by $x \overset{\overline{s}_\lambda^{\overline{L}}}{\backsim} y$) and simply asymptotically $\overline{s}_\lambda$ statistical equivalent if L=1.

Definition 2.6 ([15]). Two non-negative sequences of interval numbers $\overline{x} = (\overline{x}_k)$ and $\overline{y} = (\overline{y}_k) \neq \overline{0} = [0,0]$ are said to be asymptotically statistical equivalent of multiple $\overline{L}$ provided that for every $\epsilon > 0$

$$\lim_n \frac{1}{n} \left|\left\{k \in n :\ d(\frac{\overline{x}_k}{\overline{y}_k}, \overline{L})|) \geq \varepsilon\right\}\right| = 0$$

(denoted by $x \overset{\overline{s}^{\overline{L}}}{\backsim} y$) and simply asymptotically $\overline{s}$ statistical equivalent if $L = 1$.

Definition 2.7 ([15]). Two non-negative sequences of interval numbers $\overline{x} = (\overline{x}_k)$ and $\overline{y} = (\overline{y}_k) \neq \overline{0} = [0,0]$ are said to be strongly asymptotically λ-equivalent of multiple $\overline{L}$ provided that for every $\epsilon > 0$

$$\lim_n \frac{1}{\lambda_n} \Sigma_{k \in I_n} : d(\frac{\overline{x}_k}{\overline{x}_k}, L) = 0$$

and simply strongly λ-asymptotically equivalent if $L = 1$.

A double sequence of real numbers is a function $x : N \times N \to R$. We shall use the notation $x = (x_{k,l})$.

A double sequence $x = (x_{k,l})$ has a Pringsheim limit L (denoted by $P - \lim x = L$) provided that, given an $\varepsilon > 0$, there exists an $n_0 \in N$ such that $|(x_{k,l}) - L,| < \varepsilon$, whenever $k, l > n_0$. We shall describe such an $x = (x_{k,l})$ briefly as "P-convergent". The double sequence $x = (x_{k,l})$ is bounded if there exists a positive number M such that $|(x_{k,l})| < M$ for all k and l and

$$\|x\| = \sup_{k,l} |x_{k,l}| < \infty.$$

Let $p = (p_{k,l})$ be a double sequence of positive real numbers. If $0 < h = \inf_{k,l}\ p_{k,l} \leq p_{k,l} \leq H = \sup_{k,l}\ p_{k,l} < \infty$ and $D = \max(1, 2^{H-1})$, then, for all $a_{k,l}, b_{k,l} \in C$ for all $k, l \in N$, we have

$$|a_{k,l} + b_{k,l}|^{p_{k,l}} \leq \mathrm{D}\ (|a_{k,l}|^{p_{k,l}} + |b_{k,l}|^{p_{k,l}}).$$

We should note that, in contrast to the case of single sequences, convergent double sequences need not to be bounded.

Later, Mursaleen and Edely [28] defined the statistical analogue for double sequence $x = (x_{k,l})$ as follows:

A real double sequence $x = (x_{k,l})$ is said to be P-statistical convergence to L provided that for each $\varepsilon > 0$

$$P - \lim_{m,n} \frac{1}{mn}\ |\{(k, l) : k\ <\ m,\ l\ <\ n;\ |x_{k,l} - L| \geq \varepsilon\}| = 0.$$

In this case, we write $St_2 - \lim_{k,l} x_{k,l} = L$ and we denote the set of all P-statistical convergent double sequences by St_2.

Let the double sequence $\lambda_2 = (\lambda_{rs})$ of positive real numbers be tending to infinity such that

$$\lambda_{(r+1)s} \leq \lambda_{rs} + 1, \ \lambda_{r(s+1)} \leq \lambda_{rs} + 1,$$

$$\lambda_{rs} - \lambda_{(r+1)s} \leq \lambda_{r(s+1)} - \lambda_{(r+1)(s+1)}, \lambda_{11} = 1$$

and

$$I_{r,s} = \{(k,l): \ r - \lambda_{rs} + 1 \leq k \leq r, \ s - \lambda_{rs} + 1 \leq l \leq s\}.$$

For double $\lambda_2 = (\lambda_{rs})$ sequence, the two double interval sequences $\overline{x} = (\overline{x}_{rs})$ and $\overline{y} = (\overline{y}_{rs}) \neq \overline{0} = [0,0]$ are said to be λ_2-asymptotically double statistical equivalent of interval number $\overline{x}_o$ provided that for every $\varepsilon > 0$

$$P - \lim_{r,s} \frac{1}{\lambda_{rs}} \left| \left\{ (k,l) \in I_{r,s} : \ d\left(\frac{\overline{x}_{rs}}{\overline{y}_{rs}}, \overline{x}_o \right) \geq \varepsilon \right\} \right| = 0$$

(denoted by $\overline{x} \overset{S_{\lambda^2}}{\backsim} \overline{y}$) and simply λ_2-asymptotically double statistical equivalent if $\overline{x}_o = \overline{1}$.

3. MAIN RESULT

Definition 3.1. Let $\lambda_2 = (\lambda_{rs})$ be a double sequence. Two non-negative double sequences of interval numbers $\overline{x} = (\overline{x}_{kl})$ and $\overline{y} = (\overline{y}_{kl}) \neq \overline{0} = [0,0]$ are said to be λ_2- asymptotically double statistical equivalent of interval number $\overline{x}_0$ provided that for every $\epsilon > 0$

$$P - \lim_{r,s} \frac{1}{\lambda_{r,s}} \left| \left\{ (k,l) \in I_{r,s} : \ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0 \right) \geq \varepsilon \right\} \right| = 0$$

$\left(\text{denoted by } \overline{x} \overset{S_{\lambda_2}}{\backsim} \overline{y}\right)$ and simply λ_2-asymptotically double lacunary statistical equivalent if $\overline{x}_0 = \overline{1}$.

If we take $\lambda_{rs} = rs$, the above definition reduces to the following definition:

Definition 3.2. Two non-negative double sequences of interval numbers $\overline{x} = (\overline{x}_{kl})$ and $\overline{y} = (\overline{y}_{kl}) \neq \overline{0} = [0,0]$ are said to be asymptotically double statistical equivalent of interval number $\overline{x}_0$ provided that for every $\epsilon > 0$

$$\lim_{r,s} \frac{1}{rs} \left| \left\{ (k,l) \in I_{r,s} : \ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0 \right) \geq \varepsilon \right\} \right| = 0$$

and we denote this by $\overline{x} \overset{S_2}{\backsim} \overline{y}$.

Definition 3.3. Let $\lambda_2 = (\lambda_{rs})$ be a double sequence and p $= (p_{k,l})$ be any double sequence of strictly positive real numbers. Two non-negative double sequences of interval numbers $\overline{x} = (\overline{x}_{kl})$ and $\overline{y} = (\overline{y}_{kl})$ are said to be strong λ_2- asymptotically double statistical equivalent of interval number $\overline{x}_0$ provided that for every $\epsilon > 0$

$$P - \lim_{rs} \frac{1}{\lambda_{rs}} \Sigma_{k \in I_{r,s}} [\ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0 \right)]^{p_{k,l}} = 0$$

$\left(\text{denoted by } \overline{x} \overset{2^{N^p_{\lambda_{r,s}}}}{\backsim} \overline{y}\right)$ and simply strong λ_2- asymptotically double lacunary statistical equivalent if $\overline{x}_0 = \overline{1}$.

THEOREM 3.4. *Let $\lambda_2 = (\lambda_{rs})$ be a double sequence. Then:*

(i) *If $\overline{x} \overset{2^{N^p_{\lambda_{r,s}}}}{\backsim} \overline{y}$, then $\overline{x} \overset{S_{\lambda_2}}{\backsim} \overline{y}$.*

(ii) *If $\overline{x} \overset{S_{\lambda_2}}{\backsim} \overline{y}$, then $\overline{x} \overset{2^{N^p_{\lambda_{r,s}}}}{\backsim} \overline{y}$.*

(iii) *If $\overline{x} = (\overline{x}_{kl}) \in \overline{m}$, then $S_{\lambda_2} \bigcap \overline{m} = 2^{N^p_{\lambda_{r,s}}} \bigcap \overline{m}$, where $\overline{m}$ denote the set of bounded sequences.*

Proof. (i) Let $\epsilon > 0$ and $\overline{x} \overset{2^{N^p_{\lambda_{r,s}}}}{\backsim} \overline{y}$, then

$$\left|\left\{(k,l) \in I_{r,s}: \ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0\right) \geq \varepsilon\right\}\right| \geq \Sigma_{(k,l)\in I_{r,s} \text{ and } d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0\right) \geq \epsilon} \ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0\right)$$

and $P - \lim_{rs} \frac{1}{\lambda_{rs}} \Sigma_{k\in I_{r,s}}[\ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0\right)]^{p_{k,l}} = 0$. This implies that

$$P - \lim_{rs} \frac{1}{\lambda_{rs}} \left|\left\{(k,l) \in I_{r,s}: \ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0\right) \geq \varepsilon\right\}\right| = 0.$$

Therefore, $\overline{x} \overset{S_{\lambda_2}}{\backsim} \overline{y}$.

(ii) Suppose that $\overline{x} = (\overline{x}_{kl})$ and $\overline{y} = (\overline{y}_{kl})$ in $\overline{m}$ and $\overline{x} \overset{S_{\lambda_2}}{\backsim} \overline{y}$. Then there is a constant M > 0 such that $d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0\right) \leq M$. Let $\epsilon > 0$. We have

$$\frac{1}{\lambda_{rs}} \Sigma_{(k,l)\in I_{r,s}}[\ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0\right)]^{p_{k,l}} = \frac{1}{\lambda_{rs}} \Sigma_{(k,l)\in I_{r,s} \text{ and } d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0\right) \geq \epsilon}[\ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0\right)]^{p_{k,l}}$$

$$+ \frac{1}{\lambda_{rs}} \Sigma_{(k,l)\in I_{r,s} \text{ and } d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0\right) < \epsilon}[\ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0\right)]^{p_{k,l}}$$

$$\leq \frac{1}{\lambda_{rs}} \Sigma_{(k,l)\in I_{r,s} \text{ and } d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0\right) \geq \epsilon} \max(M^h, M^H)$$

$$+ \frac{1}{\lambda_{rs}} \Sigma_{(k,l)\in I_{r,s} \text{ and } d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0\right) < \epsilon} \epsilon^{p_{k,l}}$$

$$\leq \max(M^h, M^H) \frac{1}{\lambda_{rs}} \left|\left\{(k,l) \in I_{r,s}: \ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0\right) \geq \varepsilon\right\}\right| + \max(\epsilon^h, \epsilon^H)$$

Therefore, $\overline{x} \overset{2^{N^p_{\lambda_{r,s}}}}{\backsim} \overline{y}$.

(iii) It follows from (i) and (ii).

□

THEOREM 3.5. *Let $\lambda_2 = (\lambda_{rs})$ be a double sequence with $P - \liminf_{r,s} \frac{\lambda_{rs}}{rs} > 0$. Then $\overline{x} \overset{S_2}{\backsim} \overline{y}$ implies $\overline{x} \overset{S_{\lambda_2}}{\backsim} \overline{y}$.*

Proof. If $\overline{x} \overset{S_2}{\backsim} \overline{y}$, then, for every $\epsilon > 0$ and for sufficiently large r and s, we have

$$\begin{aligned}
&\frac{1}{\lambda_{rs}} \left| \left\{ (k,l) \in I_{r,s}; k \leq k_r \text{ and } l \leq l_s : \ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0 \right) \geq \varepsilon \right\} \right| \\
&\geq \frac{1}{\lambda_{rs}} \left| \left\{ (k,l) \in I_{r,s} : \ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0 \right) \geq \varepsilon \right\} \right| \\
&\geq \frac{\lambda_{rs}}{rs} \frac{1}{\lambda_{r,s}} \left| \left\{ (k,l) \in I_{r,s} : \ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0 \right) \geq \varepsilon \right\} \right|.
\end{aligned}$$

Hence, $\overline{x} \overset{S_{\lambda_2}}{\backsim} \overline{y}$.

□

THEOREM 3.6. *Let $\lambda_2 = (\lambda_{rs})$ be a double sequence with $P - \limsup_{r,s} \frac{\lambda_{rs}}{rs} < \infty$. Then $\overline{x} \overset{S_{\lambda_2}}{\backsim} \overline{y}$ implies $\overline{x} \overset{S_2}{\backsim} \overline{y}$.*

Proof. Since $P - \limsup_{r,s} \frac{\lambda_{rs}}{rs} < \infty$, there exists D > 0 such that $\lambda_{rs} < D$, for all $r, s \geq 1$. Let $\overline{x} \overset{S_{\lambda_2}}{\backsim} \overline{y}$ and $\epsilon > 0$. Then there exist $r_0 < 0$ and $s_0 > 0$ such that, for every $i \geq r_0$ and $j \geq s_0$

$$C_{i,j} = \frac{1}{h_{i,j}} \left| \left\{ (k,l) \in I_{i,j} : \ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0 \right) \geq \varepsilon \right\} \right| < \varepsilon.$$

Let $M = \max\{C_{i,j} : 1 \leq i \leq r_0 \text{ and } 1 \leq j \leq s_0\}$ and m, n be such that $k_{r-1} < m \leq k_r$ and $l_{s-1} < n \leq l_s$. Thus, we obtain the following

$$\begin{aligned}
&\frac{1}{mn} \left| \left\{ (k,l) \in I_{i,j}; k \leq m\text{and } l \leq n : \ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0 \right) \geq \varepsilon \right\} \right| \\
&\leq \frac{1}{\lambda_{(r-1)(s-1)}} \left| \left\{ (k,l) \in I_{i,j}; k \leq k_r \text{ and } l \leq l_s : \ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0 \right) \geq \varepsilon \right\} \right| \\
&\leq \frac{1}{\lambda_{(r-1)(s-1)}} \Sigma_{(1 \leq t \leq r_0) \bigcup (1 \leq u \leq s_0)} h_{t,u} C_{t,u} + \frac{1}{k_{r-1} l_{s-1}} \Sigma_{(r_0 < t \leq r) \bigcup (s_0 < u \leq s)} h_{t,u} C_{t,u} \\
&\leq \frac{M}{\lambda_{(r-1)(s-1)}} \Sigma_{(1 \leq t \leq r_0) \bigcup (1 \leq u \leq s_0)} h_{t,u} + \frac{1}{k_{r-1} l_{s-1}} \Sigma_{(r_0 < t \leq r) \bigcup (s_0 < u \leq s)} h_{t,u} C_{t,u} \\
&\leq \frac{M k_{r_0} l_{s_0} r_0 s_0}{\lambda_{(r-1)(s-1)}} + \frac{1}{k_{r-1} l_{s-1}} \Sigma_{(r_0 < t \leq r) \bigcup (s_0 < u \leq s)} h_{t,u} C_{t,u} \\
&\leq \frac{M k_{r_0} l_{s_0} r_0 s_0}{\lambda_{(r-1)(s-1)}} + \left(\sup_{t \geq r_0 \bigcup u \geq s_0} C_{t,u} \right) \frac{1}{k_{r-1} l_{s-1}} \Sigma_{(r_0 < t \leq r) \bigcup (s_0 < u \leq s)} h_{t,u}
\end{aligned}$$

$$\leq \frac{Mk_{r_0}l_{s_0}r_0s_0}{\lambda_{(r-1)(s-1)}} + \frac{\varepsilon}{\lambda_{(r-1)(s-1)}}\Sigma_{(r_0<t\leq r)\bigcup(s_0<u\leq s)}h_{t,u}$$

$$\leq \frac{Mk_{r_0}l_{s_0}r_0s_0}{\lambda_{(r-1)(s-1)}} + \varepsilon D^2.$$

Since λ_{rs} approach infinity as both m and n approach infinity it follows that

$$\frac{1}{mn}\left|\left\{(k,l) \in I_{i,j}; k \leq m \text{ and } l \leq n : \ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0\right) \geq \varepsilon\right\}\right| \to 0.$$

This completes the proof. □

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