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APPLICATIONS OF HORADAM POLYNOMIALS TO GENERAL CLASSES OF BI-UNIVALENT FUNCTIONS INVOLVING THE q-DERIVATIVE OPERATOR

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Abstract. In this present investigation, by using the Horadam polynomials, we aim to build a bridge between the theory of geometric functions and that of special functions, which are usually considered very different fields. Thus, we introduce some new classes of bi-univalent functions defined by combining the q-derivative operator and the Horadam polynomials. Afterwards, we derive coefficient inequalities and consider the classical Fekete-Szegö problem.

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1. INTRODUCTION, PRELIMINARIES AND KNOWN RESULTS

Recently, Horzum and Gökçen Koçer [9] investigated the Horadam polynomials $h_n(x)$, which are given by the following recurrence relation

$$h_n(x) = pxh_{n-1}(x) + rh_{n-2}(x) \quad (n > 2), \tag{1}$$

with $h_1(x) = a,\ h_2(x) = bx$, for some real constants a, b, p and r.

The generating function of the Horadam polynomials $h_n(x)$ follows immediately:

$$\Pi(x, z) = \sum_{n=1}^{\infty} h_n(x)z^{n-1} = \frac{a + (b - ap)xz}{1 - pxz - rz^2}.$$

There are many classes of polynomials which are related to the Horadam polynomials such as (for example) the Fibonacci polynomials, the Lucas polynomials, the Chebychev polynomials, the Pell polynomials, the Lucas-Lehmer polynomials and the families of orthogonal polynomials and other special polynomials. These polynomials and their generalizations play an important role in mathematics, viscoelasticity, oscillating magnetic field, heat conduction, electromagnetism, biology, etc. (see, for example, [7, 8, 10, 13, 19, 20]). By properly choosing a, b, p and r, several classical polynomials can be obtained. In particular, we have:

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- the Fibonacci polynomials $F_n(x)$ when $a = b = p = r = 1$;
- the Lucas polynomials $L_n(x)$ when $a = 2$ and $b = p = r = 1$;
- the Pell polynomials $P_n(x)$ when $a = r = 1$ and $b = p = 2$;
- the Pell-Lucas polynomials $Q_n(x)$ when $a = b = p = 2$ and $r = 1$;
- the Chebyshev polynomials of the second kind $U_n(x)$ when $a = 1$, $b = p = 2$ and $r = 1$.

Let A denote the class of functions f of the form:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots , \tag{2}$$

which are analytic in the open unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and normalized under the conditions given by $f(0) = f'(0) - 1 = 0$. Let S be the subclass of A consisting of functions which are univalent in Δ. Further, let P be the class of functions with positive real part consisting of all analytic functions $\xi : \Delta \to \mathbb{C}$ satisfying $\xi(0) = 1$ and $\Re(\xi(z)) > 0$.

Next, we give the following lemma which is necessary to prove our investigations.

Lemma 1.1 ([15]). *If the function $\xi \in P$ is defined by*

$$\xi(z) = 1 + \xi_1 z + \xi_2 z^2 + \xi_3 z^3 + \cdots ,$$

then $|\xi_n| \leq 2$ $(n \in \mathbb{N} = \{1, 2, \ldots\})$.

In order to recall the principle of subordination between analytic functions, let the functions f, g be analytic in Δ. A function f is subordinate to g, denoted by $f \prec g$ (or $f(z) \prec g(z)$) $(z \in \Delta)$, if there exists a Schwarz function $\mathfrak{w} \in \Lambda$, where $\Lambda = \{\mathfrak{w} : \mathfrak{w}(0) = 0,\ |\mathfrak{w}(z)| < 1,\ z \in \Delta\}$, such that $f(z) = g(\mathfrak{w}(z))$ $(z \in \Delta)$.

According to the *Koebe-One Quarter Theorem* [5], the image of Δ under every univalent function $f \in A$ contains a disc of radius $1/4$. Thus, clearly, every such univalent function $f \in A$ has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$ and $f(f^{-1}(w)) = w$ $\left(|w| < r_0(f)\ ,\ r_0(f) \geq \frac{1}{4}\right)$, where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{3}$$

A function $f \in A$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ. Let Σ denote the class of bi-univalent functions in Δ given by (2). For a brief history and interesting examples of functions in the class Σ, see the pioneering work on this subject by Srivastava et al. [18], which has apparently revived the study of bi-univalent functions in recent years (see also [2, 3, 4, 12, 14, 17]).

It may be of interest to recall that Srivastava used the basic (or q-) hypergeometric functions in a book chapter (see, for details, [16]). Thus, the theory of univalent functions was characterized by the concept of q-calculus. We first recall the definitions of fractional q-calculus operators of a complex valued function f.

Definition 1.2 ([11]). The q-derivative of a function f, defined on a subset of $\mathbb{C}$, is given by

$$(D_q f)(z) = \begin{cases} \dfrac{f(z)-f(qz)}{(1-q)z} & \text{for} \quad z \neq 0, \\[2ex] f'(0) & \text{for} \quad z = 0. \end{cases}$$

We note that $\lim\limits_{q \to 1^-} (D_q f)(z) = f'(z)$ if f is differentiable at z. From (2), we get

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q \, a_n z^{n-1},$$

where $[n]_q = \dfrac{1-q^n}{1-q}$ $(0 < q < 1, n \in \mathbb{N})$.

Additionally, from (3), we get

$$\begin{aligned} (D_q g)(w) = \frac{g(w)-g(qw)}{(1-q)w} &= 1 \; - [2]_q \, a_2 w \\ + [3]_q \left(2a_2^2 - a_3\right) w^2 - [4]_q &\left(5a_2^3 - 5a_2 a_3 + a_4\right) w^3 + \cdots . \end{aligned} \tag{4}$$

In this paper, we aim first at introducing some new classes of bi-univalent functions defined by means of the q-derivative operator and the Horadam polynomials. Furthermore, we derive coefficient inequalities and consider the classical Fekete-Szegö problem.

Definition 1.3. A function $f \in \Sigma$ is said to be in the class

$$\mathcal{H}_{\Sigma}^{q}(x) \quad (z, w \in \Delta)$$

if the following subordination conditions are satisfied:

$$D_q f(z) \prec \Pi(x, z) + 1 - a,$$

$$D_q g(w) \prec \Pi(x, w) + 1 - a,$$

where the function g is given by (3).

Remark 1.4. Upon setting $q \to 1^-$ it is readily seen that a function $f \in \Sigma$ is in the class $\mathcal{H}_{\Sigma}(x)$ $(z, w \in \Delta)$ if the following conditions are satisfied:

$$f'(z) \prec \Pi(x, z) + 1 - a,$$

$$g'(w) \prec \Pi(x, w) + 1 - a,$$

where $g = f^{-1}$.

Remark 1.5. Upon setting $a = 1, b = p = 2, r = -1$, the class $\mathcal{H}_{\Sigma}(x)$ reduce to the class $\mathcal{H}_{\Sigma}(t)$ studied by Altınkaya and Yalçın [1].

2. COEFFICIENT BOUND ESTIMATES

In this section, we obtain estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{H}_{\Sigma}^{q}(x)$.

THEOREM 2.1. *Let the function f given by (2) be in the class $\mathcal{H}_{\Sigma}^{q}(x)$. Then*

$$|a_2| \leq \frac{b|x|\sqrt{b|x|}}{\sqrt{\left|\left([3]_q b - [2]_q^2 p\right)bx^2 + [2]_q^2 bx - [2]_q^2 qa\right|}}$$

and

$$|a_3| \leq \frac{b^2x^2}{[2]_q^2} + \frac{b|x|}{[3]_q}.$$

Proof. Let $f \in \mathcal{H}_{\Sigma}^{q}(x)$. Then, by Definition 1.3, for two analytic functions Φ, Ψ such that

$$\Phi(0) = \Psi(0) = 0, \ |\Phi(z)| < 1, \ |\Psi(w)| < 1 \quad (\forall z, w \in \Delta),$$

we can write

$$D_q f(z) = \Pi(\Phi(z), x) + 1 - a$$

and

$$D_q g(w) = \Pi(\Psi(w), x) + 1 - a$$

or, equivalently,

$$D_q f(z) = 1 + h_1(x) - a + h_2(x)\Phi(z) + h_3(x)\Phi^2(z) + \cdots \tag{5}$$

and

$$D_q g(w) = 1 + h_1(x) - a + h_2(x)\Psi(w) + h_3(x)\Psi^2(w) + \cdots . \tag{6}$$

Next, define the functions $\xi, \tau \in P$ by

$$\xi(z) = \frac{1 + \Phi(z)}{1 - \Phi(z)} = 1 + \xi_1 z + \xi_2 z^2 + \cdots$$

and

$$q(w) = \frac{1 + \Psi(w)}{1 - \Psi(w)} = 1 + \tau_1 w + \tau_2 w^2 + \cdots .$$

In the following, one can derive

$$\Phi(z) = \frac{\xi(z) - 1}{\xi(z) + 1} = \frac{1}{2}\xi_1 z + \frac{1}{2}\left(\xi_2 - \frac{1}{2}\xi_1^2\right)z^2 + \cdots \tag{7}$$

and

$$\Psi(w) = \frac{\tau(w) - 1}{\tau(w) + 1} = \frac{1}{2}\tau_1 w + \frac{1}{2}\left(\tau_2 - \frac{1}{2}\tau_1^2\right)w^2 + \cdots . \tag{8}$$

Combining (5), (6), (7) and (8), we get

$$D_q f(z) = 1 + h_2(x)\xi_1 z + \left[h_2(x)\xi_2 + h_3(x)\xi_1^2\right]z^2 + \cdots , \tag{9}$$

and

$$D_q g(w) = 1 + h_2(x)\tau_1 w + \left[h_2(x)\tau_2 + h_3(x)\tau_1^2\right] w^2 + \cdots . \tag{10}$$

Thus, upon comparing the corresponding coefficients in (9) and (10), we have

$$[2]_q \, a_2 = \frac{h_2(x)}{2}\xi_1, \tag{11}$$

$$[3]_q \, a_3 = \frac{h_2(x)}{2}\left(\xi_2 - \frac{\xi_1^2}{2}\right) + \frac{h_3(x)}{4}\xi_1^2, \tag{12}$$

$$-[2]_q \, a_2 = \frac{h_2(x)}{2}\tau_1 \tag{13}$$

and

$$[3]_q \left(2a_2^2 - a_3\right) = \frac{h_2(x)}{2}\left(\tau_2 - \frac{\tau_1^2}{2}\right) + \frac{h_3(x)}{4}\tau_1^2. \tag{14}$$

From the equations (11) and (13), we can easily see that

$$\xi_1 = -\tau_1, \tag{15}$$

$$2\,[2]_q^2 = \frac{h_2^2(x)}{4}\left(\xi_1^2 + \tau_1^2\right). \tag{16}$$

If we add (12) to (14), we get

$$2\,[3]_q \, a_2^2 = \frac{h_2(x)}{2}\left(\xi_2 + \tau_2\right) + \frac{h_3(x) - h_2(x)}{4}\left(\xi_1^2 + \tau_1^2\right). \tag{17}$$

By using (16) in the equality (17), we have

$$2\left\{[3]_q \, h_2^2(x) - [2]_q^2 \left(h_3(x) - h_2(x)\right)\right\} a_2^2 = 2\mu^2 h_2^3(x)\left(\xi_2 + \tau_2\right) \tag{18}$$

which gives

$$|a_2| \le \frac{b\,|x|\sqrt{b\,|x|}}{\sqrt{\left|\left([3]_q \, b - [2]_q^2 \, p\right) bx^2 + [2]_q^2 \, bx - [2]_q^2 \, qa\right|}}.$$

Moreover, if we subtract (14) from (12), we obtain

$$2\,[3]_q \left(a_3 - a_2^2\right) = \frac{h_2(x)}{2}\left(\xi_2 - \tau_2\right) + \frac{h_3(x) - h_2(x)}{4}\left(\xi_1^2 - \tau_1^2\right). \tag{19}$$

Then, in view of (15) and (16), (19) becomes

$$a_3 = \frac{h_2^2(x)}{8\,[2]_q^2}\left(\xi_1^2 + \tau_1^2\right) + \frac{h_2(x)}{4\,[3]_q}\left(\xi_2 - \tau_2\right).$$

Then, with the help of (1), we deduce that $|a_3| \le \dfrac{b^2 x^2}{[2]_q^2} + \dfrac{b\,|x|}{[3]_q}$. □

COROLLARY 2.2. *Let $f \in \mathcal{H}_{\Sigma}^{q}(x)$. Then*

$$|a_2| \leq \frac{b|x|\sqrt{b|x|}}{\sqrt{|(3b-4p)\,bx^2+4bx-4qa|}} \quad \textit{and} \quad |a_3| \leq \frac{b^2x^2}{4}+\frac{b|x|}{3}.$$

COROLLARY 2.3 ([1]). *Let $f \in H_{\Sigma}(t)$. Then*

$$|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{1+2t-t^2}} \quad \textit{and} \quad |a_3| \leq t^2+\frac{2t}{3}.$$

3. FEKETE-SZEGÖ PROBLEM

The classical Fekete-Szegö inequality is investigated by Loewner's method, which for the coefficients of $f \in S$ is

$$\left|a_3-\vartheta a_2^2\right| \leq 1+2\exp(-2\vartheta/(1-\vartheta)) \quad \text{for } \vartheta \in [0,1)\,.$$

As $\vartheta \to 1^-$, we have the elementary inequality $\left|a_3-a_2^2\right| \leq 1$. Moreover, the problem of maximizing the modulus of the functional

$$\Psi_{\vartheta}(f)=a_3-\vartheta a_2^2$$

is called the Fekete-Szegö problem (see [6]).

Now, we derive Fekete-Szegö inequalities for the coefficients of $f \in \mathcal{H}_{\Sigma}^{q}(x)$.

THEOREM 3.1. *Let the function f given by (2) be in the class $\mathcal{H}_{\Sigma}^{q}(x)$. Suppose also that $\vartheta \in \mathbb{R}$. Then*

$$\left|a_3-\vartheta a_2^2\right| \leq \begin{cases} \dfrac{b|x|}{[3]_q}, \ \textit{if } |\vartheta-1| \leq \left|\dfrac{1}{4}-\dfrac{[2]_q^2}{[3]_q}\dfrac{(pbx^2-bx+ra)}{b^2x^2}\right| \\ \dfrac{b^3|1-\vartheta|\,|x|^3}{\left|\left([3]_q b-[2]_q^2 p\right)bx^2+[2]_q^2 bx-[2]_q^2 qa\right|}, \ \textit{otherwise.} \end{cases}$$

Proof. From (18) and (19), we find that

$$\begin{aligned} a_3-\vartheta a_2^2 &= \frac{h_2^3(x)\,(1-\vartheta)\,(\xi_2+\tau_2)}{4\left\{[3]_q h_2^2(x)-[2]_q^2\,(h_3(x)-h_2(x))\right\}}+\frac{h_2(x)\,(\xi_2-\tau_2)}{4\,[3]_q} \\ &= h_2(x)\left[\left(\Omega(\vartheta,x)+\frac{1}{4\,[3]_q}\right)\xi_2+\left(\Omega(\vartheta,x)-\frac{1}{4\,[3]_q}\right)\tau_2\right], \end{aligned}$$

where

$$\Omega(\vartheta,x)=\frac{h_2^2(x)\,(1-\vartheta)}{4\left\{[3]_q h_2^2(x)-[2]_q^2\,(h_3(x)-h_2(x))\right\}}.$$

Hence, in view of (1), we conclude that

$$\left|a_3-\vartheta a_2^2\right| \leq \begin{cases} \dfrac{\mu|h_2(x)|}{[3]_q}, & 0 \leq |\Omega(\vartheta,x)| \leq \dfrac{1}{4\,[3]_q} \\ 4\,|h_2(x)|\,|\Omega(\vartheta,x)|\,, & |\Omega(\vartheta,x)| \geq \dfrac{1}{4\,[3]_q}. \end{cases}$$

$\square$

Corollary 3.2. *Let $f \in \mathcal{H}_{\Sigma}(x)$ and $\vartheta \in \mathbb{R}$. Then*

$$\left|a_3 - \vartheta a_2^2\right| \begin{cases} \dfrac{b\,|x|}{3}, & \text{if } |\vartheta - 1| \leq \left|\dfrac{1}{4} - \dfrac{4(pbx^2 - bx + ra)}{3b^2x^2}\right| \\ \dfrac{b^3\,|1-\vartheta|\,|x|^3}{|(3b-4p)\,bx^2 + 4bx - 4qa|}, & \text{otherwise.} \end{cases}$$

Corollary 3.3 ([1]). *Let $f \in H_{\Sigma}(t)$ and $\eta \in \mathbb{R}$. Then*

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \dfrac{2t}{3}; & |\eta - 1| \leq \frac{1+2t-t^2}{3t^2}, \\ \dfrac{2\,|1-\eta|\,t^3}{1+2t-t^2}; & |\eta - 1| \geq \frac{1+2t-t^2}{3t^2}. \end{cases}$$

If we set $\vartheta = 1$, we get the following corollaries.

Corollary 3.4. *If $f \in \mathcal{H}_{\Sigma}^{q}(x)$, then $\left|a_3 - a_2^2\right| \leq \dfrac{b\,|x|}{[3]_q}$.*

Corollary 3.5 ([1]). *If $f \in H_{\Sigma}(t)$, then $\left|a_3 - a_2^2\right| \leq \dfrac{2t}{3}$.*

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