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NON-EXISTENCE OF $\mathcal{PR}$-PSEUDO-SLANT WARPED PRODUCT SUBMANIFOLDS OF PARACOSYMPLECTIC MANIFOLDS

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Abstract. The present article deals with the study of $\mathcal{PR}$-pseudo-slant warped product submanifolds of paracosymplectic manifols $\overline{M}$. Results of non-existence for non-trivial $\mathcal{PR}$-pseudo-slant warped product submanifolds with proper slant coefficient in $\overline{M}$ are shown. In addition to these results, we give an elementary illustration of non-trivial $\mathcal{PR}$-pseudo-slant warped product submanifold with improper slant coefficient in $\overline{M}$.

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1. INTRODUCTION

The warped product $B \times_f F$ of two pseudo-Riemannian manifolds (B, g_B) and (F, g_F) with a positive smooth function f on B is a product manifold of form $B \times_f F$ equipped with the metric tensor $g = g_B \oplus f^2 g_F$. For precise definitions of the described notations, we refer to Section 2 of this paper, as well as [5, 11]. The warped product has important contributions in pseudo-Riemannian geometry and has been successfully applied in the study of general theory of relativity and black holes (cf. [10, 16, 18]). In light of the physical applications of these warped product submanifolds, the important question of existence or non-existence of warped product submanifolds arises naturally. A number of significant submanifolds are exhibited as warped products submanifolds in Euclidean and complex space forms due to J.F. Nash's famous theorem, which state that every Riemannian manifold can be isometrically immersed in some Euclidean spaces with sufficiently high codimension (see, for details, [9, 10, 12]). Since the Riemannian geometric configuration may not be found suitable, where the metric is not necessarily positive definite, the geometry of warped product submanifolds with pseudo-Riemannaian metric became a topic of investigation. In light of this, many geometers have studied the *existence* and *non-existence* of such warped product submanifolds

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by setting of a Lorentzian metric (that is pseudo-Riemannian metric with index 1) (see [22, 24]). Recently, Chen-Munteanu [11] initiated the geometry of pseudo-Riemannian warped products submanifolds in para-Kähler manifolds and the authors [20] continued the study in paracontact manifolds. Furthermore, Aydin-Cöken [3] and Alegre-Carriazo [1] introduced the concept of slant submanifold in semi-Riemannian manifolds and in para-Hermitian manifods, respectively. Motivated by the above studies, in the present paper we investigate the existence or non-existence of $\mathcal{PR}$-pseudo-slant warped product submanifolds in paracosymplectic manifolds.

The structure of the manuscript is as follows. In Section 2 we recall some basic informations about paracontact manifolds, geometry of submanifolds, warped product submanifolds and give some existence and non-existence results for warped product submanifolds of a paracosymplectic manifold $\overline{M}$. In Section 3 we define $\mathcal{PR}$-pseudo-slant submanifolds and derive the necessary and sufficient conditions for foliation determined by distributions associated with this definition to be involutive and totally geodesic in $\overline{M}$. Section 4 deals with the non-existence results for non-trivial $\mathcal{PR}$-pseudo-slant warped product submanifolds of $\overline{M}$. To support the results in Section 4, we present an example for the existence of non-trivial $\mathcal{PR}$-pseudo-slant warped product submanifold with improper slant coefficient in $\overline{M}$.

2. PRELIMINARIES

A $(2m+1)$-dimensional C^∞ manifold has an almost paracontact structure (ϕ, ξ, η) if it admits a tensor field ϕ of $(1,1)$-type, a vector field ξ and a 1-form η satisfying the following conditions:

$$\phi^2 = I - \eta \otimes \xi,\ \eta(\xi) = 1, \tag{1}$$

where I is the identity transformation and the tensor field ϕ induces on the $2m$-dimensional horizontal distribution $D := \ker(\eta)$ an almost paracomplex structure J, that is $J^2 = I$ and the eigen subbundles $D^\pm$ corresponding to the eigenvalues ± 1 of J, respectively, have equal dimension m; hence, $D = D^+ \oplus D^-$. The direct consequence of (1) is that the structure endomorphism ϕ has rank $2m$, $\phi\xi = 0$ and $\eta \circ \phi = 0$. If a manifold $\overline{M}$ with (ϕ, ξ, η)-structure admits a pseudo-Riemannian metric $\overline{g}$ of signature $(m+1,\, m)$ such that

$$\overline{g} = -\overline{g}(\phi\cdot, \phi\cdot) + \eta \otimes \eta, \tag{2}$$

then $\overline{M}$ is said to have an almost paracontact metric structure $(\phi, \xi, \eta, \overline{g})$ and the manifold $\overline{M}$ equipped with $(\phi, \xi, \eta, \overline{g})$-structure is called an almost paracontact metric manifold, here $\overline{g}$ is known as compatible metric, see [25]. With respect to $\overline{g}$, η is metrically dual to the unitary vector field ξ, i.e. $\eta = \overline{g}(\cdot, \xi)$. By (1) and (2) we deduce that ϕ is a $\overline{g}$-skew-symmetric operator, that is

$$\overline{g}(\phi X, Y) = -\overline{g}(X, \phi Y), \tag{3}$$

for any $X, Y \in \Gamma(T\overline{M})$; $\Gamma(T\overline{M})$ denotes the sections of the tangent bundle $(T\overline{M})$ of $\overline{M}$. The fundamental 2-form $\Phi := \overline{g}(\cdot, \phi\cdot)$ is non-degenerate on the horizontal distribution D and $\eta \wedge \Phi^m \neq 0$. If $\Phi = d\eta$, then η is a paracontact form and the almost paracontact metric manifold $\overline{M}(\phi, \xi, \eta, \overline{g})$ is called paracontact metric manifold.

Definition 2.1. An almost paracontact metric manifold $\overline{M}(\phi, \xi, \eta, \overline{g})$ is said to be

(a) *almost paracosymplectic* if the forms η and Φ are closed, i.e. $d\eta = 0$ and $d\Phi = 0$, see [13].

(b) *paracosymplectic* if the forms η and Φ are parallel with respect to the Levi-Civita connection $\overline{\nabla}$ on $\overline{M}(\phi, \xi, \eta, \overline{g})$, i.e.

$$\overline{\nabla}\eta = 0 \quad \text{and} \quad \overline{\nabla}\Phi = 0, \tag{4}$$

see [13].

(c) *normal* if and only if the the eigendistributions $D^{\pm}$ of $\varphi|_D$ corresponding to the eigenvalues ± 1, respectively, are involutive and ξ is foliate with respect to both $D^{\pm}$, see [17].

2.1. GEOMETRY OF SUBMANIFOLDS

Let us consider that M is an isometrically immersed submanifold of a paracosymplectic manifold $\overline{M}$ in the sense of O'Neill [18] and g denotes the induced metric on M such that $g = \overline{g}|_M$ [14]. Let $\Gamma(TM^{\perp})$ indicate the set of vector fields normal to M and $\Gamma(TM)$ the sections of the tangent bundle TM of M. Then the Gauss-Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{5}$$

$$\overline{\nabla}_X \zeta = -A_\zeta X + \nabla^{\perp}_X \zeta, \tag{6}$$

for any $X, Y \in \Gamma(TM)$ and $\zeta \in \Gamma(TM^{\perp})$, where ∇ is the induced connection, $\nabla^{\perp}$ is the normal connection on the normal bundle $\Gamma(TM^{\perp})$, h is the second fundamental form and the shape operator A_ζ associated with the normal section ζ is given in [8] by

$$g\left(A_\zeta X, Y\right) = \overline{g}\left(h(X, Y), \zeta\right). \tag{7}$$

We write for all $X \in \Gamma(TM)$ and $\zeta \in \Gamma(TM^{\perp})$

$$\phi X = tX + nX, \tag{8}$$

$$\phi\zeta = t'\zeta + n'\zeta, \tag{9}$$

where tX (resp. nX) is the tangential (resp. normal) part of ϕX and $t'\zeta$ (resp. $n'\zeta$) is the tangential (resp. normal) part of $\phi\zeta$. Then the submanifold M is said to be *invariant* if n is identically zero and *anti-invariant* if t is identically zero. From (3) and (8), we obtain that

$$g(X, tY) = -g(tX, Y). \tag{10}$$

By the virtue of the Gauss formula and the fact that structure is paracosymplectic, we can give the following result for later use:

Lemma 2.2. *Let M be an isometrically immersed submanifold of a paracosymplectic manifold $\overline{M}(\phi, \xi, \eta, \overline{g})$ such that $\xi \in \Gamma(TM)$. Then for any $X \in \Gamma(TM)$ we have*

$$\nabla_X \xi = 0, \tag{11}$$

$$h(X, \xi) = 0. \tag{12}$$

2.2. WARPED PRODUCT SUBMANIFOLDS

Let (B, g_B) and (F, g_F) be two pseudo-Riemannian manifolds and f be a positive smooth function on B. Consider the product manifold $B \times F$ with canonical projections

$$\pi : B \times F \to B \quad \text{and} \quad \sigma : B \times F \to F. \tag{13}$$

Then the manifold $M = B \times_f F$ is said to be *warped product* if it is equipped with the following warped metric

$$g(X, Y) = g_B\left(\pi_*(X), \pi_*(Y)\right) + (f \circ \pi)^2 g_F\left(\sigma_*(X), \sigma_*(Y)\right) \tag{14}$$

for all $X, Y \in \Gamma(TM)$ and '$*$' stands for derivation map, or equivalently,

$$g = g_B \oplus f^2 g_F. \tag{15}$$

The function f is called *the warping function* and a warped product manifold M is said to be *trivial* if f is constant. For simplicity, we will determine a vector field X on B with its lift $\overline{X}$ and a vector field Z on F with its lift $\overline{Z}$ on $M = B \times_f F$ (see also [5, 11]).

Proposition 2.3 ([5]). *For $X, Y \in \Gamma(TB)$ and $Z, W \in \Gamma(TF)$, we obtain for the warped product manifold $M = B \times_f F$:*

(i) $\nabla_X Y \in \Gamma(TB)$,
(ii) $\nabla_X Z = \nabla_Z X = \left(\frac{Xf}{f}\right) Z$,
(iii) $\nabla_Z W = \frac{-g(Z,W)}{f} \nabla f$,

where ∇ denotes the Levi-Civita connection on M and ∇f is the gradient of f defined by $g(\nabla f, X) = Xf$.

Remark 2.4. It is also important to note that, for a warped product $M = B \times_f F$, B is totally geodesic and F is totally umbilical in M, see [5].

Now, by Lemma 2.2 and Proposition 2.3, we obtain for any non-degenerate vector fields $X \in \Gamma(TB)$ and $Z \in \Gamma(TF)$ that $X(\ln f)Z = 0$. This implies that f is a constant function, since X, Z are non-degenerate vector fields in M. This leads to the following result:

THEOREM 2.5. *Let $\overline{M}(\phi, \xi, \eta, \overline{g})$ be a paracosymplectic manifold. Then there doesn't exist a non-trivial warped product submanifold $M = B \times_f F$ of a paracosymplectic manifold for $\xi \in \Gamma(TF)$.*

Here, we recall the following important results from [20] for later use, when $\xi \in \Gamma(TB)$:

LEMMA 2.6. *Let $M = B \times_f F$ be a non-trivial warped product submanifold of a paracosymplectic manifold $\overline{M}(\phi, \xi, \eta, \overline{g})$ such that $\xi \in \Gamma(TB)$. Then we have*

$$\xi(\ln f) = 0, \tag{16}$$

$$A_{nZ}X = -t'h(X, Z), \tag{17}$$

$$g(A_{nZ}X, W) = g(A_{nW}X, Z) = -tX(\ln f)g(Z, W), \tag{18}$$

for any $X, Y \in \Gamma(TB)$ and $Z, W \in \Gamma(TF)$.

In [20], we have defined $\mathcal{PR}$-semi-invariant submanifolds in paracontact manifold as follows:

DEFINITION 2.7. Let M is an isometrically immersed submanifold of an almost paracontact metric manifold $\overline{M}(\phi, \xi, \eta, \overline{g})$ such that the characteristic vector field $\xi \in \Gamma(TM)$. Then the submanifold M is called $\mathcal{PR}$-*semi-invariant* if it is furnished with a pair of non-degenerate orthogonal distributions $(\mathfrak{D}_T, \mathfrak{D}^\perp)$ which satisfies the following conditions:

(i) $TM = \mathfrak{D}_T \oplus \mathfrak{D}^\perp \oplus \{\xi\}$,
(ii) the distribution $\mathfrak{D}_T$ is invariant under ϕ, i.e. $\phi(\mathfrak{D}_T) = \mathfrak{D}_T$ and
(iii) the distribution $\mathfrak{D}^\perp$ is anti-invariant under ϕ, i.e. $\phi(\mathfrak{D}^\perp) \subset \Gamma(TM)^\perp$.

A $\mathcal{PR}$-*semi-invariant* submanifold is said to be *proper*, if $\mathfrak{D}_T \neq \{0\}$ and $\mathfrak{D}^\perp \neq \{0\}$ and $\mathcal{PR}$-*semi-invariant warped product* if it is a warped product of the form: $M_T \times_f M_\perp$ and $M_\perp \times_f M_T$, where M_T and $M_\perp$ are invariant and anti-invariant submanifolds of $\overline{M}$, resectively, and f is a non-constant positive smooth function on the first factor. If the warping function f is constant, then a $\mathcal{PR}$-semi-invariant warped product submanifold is said to be a $\mathcal{PR}$-*semi-invariant product* or *trivial product*.

3. $\mathcal{PR}$-PSEUDO-SLANT SUBMANIFOLDS

In this section, by following [1, 2, 19], we introduce $\mathcal{PR}$-pseudo-slant submanifolds in $\overline{M}$, which generalize the above defined $\mathcal{PR}$-*semi-invariant* submanifolds. Since submanifold M is treated to be a non-degenerate submanifold, this class of submanifolds can be reviewed as the generalization of submanifolds defined in [6, 7, 15], which includes the spacelike vector fields only.

Let M be a non-degenerate submanifold of an almost paracontact metric manifold $\overline{M}$ such that $t^2X = \lambda(X - \eta(X)\xi) = \lambda\phi^2 X, \quad g(tX, Y) = -g(X, tY)$

for any $X, Y \in \Gamma(TM- < \xi >)$, where λ is a constant. Then, with the help of (3), we have

$$\begin{aligned} \frac{g(\phi X, tY)}{|\phi X||tY|} &= -\frac{g(X, \phi tY)}{|\phi X||tY|} = -\frac{g(X, t^2 Y)}{|\phi X||tY|} \\ &= -\lambda \frac{g(X, \phi^2 Y)}{|\phi X||tY|} = \lambda \frac{g(\phi X, \phi Y)}{|\phi X||tY|}. \end{aligned} \tag{19}$$

On the other hand,

$$\frac{g(\phi X, tY)}{|\phi X||tY|} = \frac{g(tX, tY)}{|\phi X||tY|}. \tag{20}$$

In particular, from (19) and (20), we obtain for $X = Y$, that $\frac{g(\phi X, tX)}{|\phi X||tX|} = \sqrt{\lambda}$. Here we call $\lambda \geq 0$ a slant constant coefficient or simply *slant coefficient* and consequently M a slant submanifold. Conversely, assume that M is a slant submanifold. Then $\lambda \frac{|\phi X|}{|tX|} = \frac{|tX|}{|\varphi X|}$, where X is a non light like vector field. We obtain by the previous equation, for any non- light like vector fields $X, Y \in \Gamma(TM- < \xi >)$, that $-\lambda \frac{g(X, \phi^2 Y)}{|\phi X||tY|} = \frac{g(\phi X, tY)}{|\phi X||tY|}$, which yields $g(X, t^2 Y) = \lambda g(X, \phi^2 Y)$, $\quad g(tX, Y) = -g(X, tY)$. Hence, $t^2 = \lambda(I - \eta \otimes \xi)$, $\quad g(tX, Y) = -g(X, tY)$, by virtue of the fact that the structure is paracosymplectic and X is a nondegenerate vector field.

Remark 3.1. The slant coefficient λ is sometimes $\cos^2 \theta$ or $\cosh^2 \theta$ or $\sinh^2 \theta$ depending on the nature of the vector fields, i.e. timelike or spacelike tangent to M, where θ is a slant angle [1].

Now, by the above discussion we have the following definitions:

Definition 3.2. Let M is an isometrically immersed submanifold of an almost paracontact metric manifold $\overline{M}(\phi, \xi, \eta, \overline{g})$ such that the characteristic vector field $\xi \in \Gamma(TM)$. Then the submanifold M is said to be a

(*a*) *slant submanifold* if for any nonzero vectors $X, Y \in \Gamma(\mathfrak{D}_\lambda)$ at $p \in M$, which are not proportional to ξ_p, there exists a constant $\lambda \geq 0$ satisfying:

$$t^2 = \lambda (I - \eta \otimes \xi) \quad \text{and} \quad g(tX, Y) = -g(X, tY),$$

where λ is a *slant coefficient* and the *slant distribution* $\mathfrak{D}_\lambda$ indicates the non-degenerate distribution on M.

Remark 3.3. It is important to note that the *invariant* and *anti-invariant* submanifolds are slant submanifolds with improper slant coefficents $\lambda = 1$ and $\lambda = 0$, respectively. Hence, a *slant submanifold*, which is neither invariant nor anti-invariant, is called a *proper slant submanifold.*

(*b*) *$\mathcal{PR}$-pseudo-slant* if it is furnished with a pair of non-degenerate orthogonal distribution $(\mathfrak{D}^\perp, \mathfrak{D}_\lambda)$ which satisfies the following conditions:

(i) $TM = \mathfrak{D}^\perp \oplus \mathfrak{D}_\lambda \oplus < \xi >$,
(ii) the distribution $\mathfrak{D}^\perp$ is anti-invariant under ϕ, i.e. $\phi(\mathfrak{D}^\perp) \subset \Gamma(TM)^\perp$ and
(iii) the distribution $\mathfrak{D}_\lambda$ is slant distribution with slant coefficient λ.

REMARK 3.4. Here, it is important to note that the $\mathcal{PR}$-pseudo-slant submanifold is a $\mathcal{PR}$-*semi-invariant* [20] if the slant coefficient $\lambda = 1$, $\mathfrak{D}^\perp \neq \{0\}$ and $\mathfrak{D}_\lambda \neq \{0\}$. We say a $\mathcal{PR}$-pseudo-slant submanifold is *proper* if $\lambda \neq 0, 1$, $\mathfrak{D}^\perp \neq \{0\}$ and $\mathfrak{D}_\lambda \neq \{0\}$.

Let M be a pseudo-slant submanifold of a paracosymplectic manifold $\overline{M}$ and set the projections on the distributions $\mathfrak{D}^\perp$ and $\mathfrak{D}_\lambda$ by $P^\perp$ and P_λ, respectively. Then, for any $V \in \Gamma(TM)$, we can write

$$V = P^\perp V + P_\lambda V + \eta(V)\xi. \tag{21}$$

Now, applying ϕ to (21) and using (8), we get

$$tV = tP_\lambda V,\ nV = nP^\perp V + nP_\lambda V. \tag{22}$$

From (1), (3) and (10), we obtain

$$g(tX, tY) = -g(X, t^2Y) = -\lambda g(X, \phi^2 Y) = \lambda g(\phi X, \phi Y).$$

Therefore, by (2) and Definition 3.2(a), we get the following proposition:

PROPOSITION 3.5. *Let M be a slant submanifold of an almost paracontact metric manifold $\overline{M}(\phi, \xi, \eta, \overline{g})$ with $\xi \in \Gamma(TM)$. Then*

$$g(tX, tY) = \lambda\{-g(X, Y) + \eta(X)\eta(Y)\} = \lambda g(\phi X, \phi Y), \tag{23}$$

$$g(nX, nY) = (1-\lambda)\{-g(X, Y) + \eta(X)\eta(Y)\} = (1-\lambda) g(\phi X, \phi Y), \tag{24}$$

for any $X, Y \in \Gamma(TM)$.

In the light of Proposition 3.5, (3), (8), (9) and by the definition of slant submanifold, we can easily derive the following lemma:

LEMMA 3.6. *Let M be a slant submanifold of an almost paracontact metric manifold $\overline{M}(\phi, \xi, \eta, \overline{g})$ with $\xi \in \Gamma(TM)$. Then for any $X \in \Gamma(TM)$ we have*

(i) $t'nX = (1-\lambda)(X - \eta(X)\xi)$ and (ii) $n'nX = -ntX$.

By virtue of (2), (4), (5) and (6), we can give the following result as a remark:

REMARK 3.7. On a pseudo-slant submanifold M of a paracosymplectic manifold $\overline{M}$ we have

$$A_{\phi X}Y = A_{\phi Y}X,\ \forall\ X, Y \in \Gamma(\mathfrak{D}^\perp).$$

Now, we obtain necessary and sufficient conditions for the foliation determined by distributions and the definition of $\mathcal{PR}$-pseudo-slant submanifolds of a paracosymplectic manifold $\overline{M}$ to be involutive and totally geodesic.

THEOREM 3.8. *Let M be a proper $\mathcal{PR}$-pseudo-slant submanifold M of a paracosymplectic manifold $\overline{M}$. Then the anti-invariant distribution $\mathfrak{D}^{\perp}$ is*

- *involutive if and only if shape operator satisfies $A_{nY}X = A_{nX}Y$ and*
- *totally geodesic foliation if and only if the shape operator satisfies $A_{nY}tZ = A_{ntZ}Y$,*

for any $X, Y \in \Gamma(\mathfrak{D}^{\perp} \oplus \{\xi\})$ and $Z \in \Gamma(\mathfrak{D}_{\lambda})$.

Proof. We have, from the Gauss formula, Lemma 2.2 and the facts that the manifold is almost paracosymplectic and the pair $(\mathfrak{D}^{\perp} \oplus \{\xi\}, \mathfrak{D}_{\lambda})$ is orthogonal, that

$$g([X,Y],Z) = -\overline{g}(\phi\overline{\nabla}_X Y, \phi Z) + \overline{g}(\phi\overline{\nabla}_Y X, \phi Z). \tag{25}$$

Furthermore, using (4), (8) and (9) in above equation, we can write that

$$-\overline{g}(\phi\overline{\nabla}_X Y, \phi Z) = -\overline{g}(\overline{\nabla}_X \phi Y, tZ) + \overline{g}(\overline{\nabla}_X Y, t'nZ) + \overline{g}(\overline{\nabla}_X Y, n'nZ). \tag{26}$$

Using the fact that h is symmetric, Lemma 3.6 and (6) in equation (26), we obtain that

$$-\overline{g}(\phi\overline{\nabla}_X Y, \phi Z) = g(A_{\phi Y}X, tZ) + (1-\lambda)\overline{g}(\overline{\nabla}_X Y, Z) - \overline{g}(\overline{\nabla}_X Y, ntZ). \tag{27}$$

On the other hand, by (4), (8) and (9), we arrive at

$$\overline{g}(\phi\overline{\nabla}_Y X, \phi Z) = -g(A_{\phi X}Y, tZ) - \overline{g}(\overline{\nabla}_Y X, t'nZ) - \overline{g}(\overline{\nabla}_Y X, n'nZ). \tag{28}$$

Therefore, (28), by Lemma 3.6, reduces to

$$\overline{g}(\phi\overline{\nabla}_Y X, \phi Z) = -g(A_{nX}Y, tZ) - (1-\lambda)\,\overline{g}(\overline{\nabla}_Y X, Z) + \overline{g}(\overline{\nabla}_Y X, ntZ). \tag{29}$$

Employing (27) and (29) in (25), we conclude that

$$\lambda g([X,Y],Z) = g(A_{nY}X, tZ) - g(A_{nX}Y, tZ). \tag{30}$$

On the other hand, we have, by the virtue of (2), (9) and the fact $\eta(Z) = 0$, that

$$g(\nabla_X Y, Z) = -\overline{g}(\overline{\nabla}_X \phi Y, tZ) + \overline{g}(\overline{\nabla}_X Y, t'nZ + n'nZ). \tag{31}$$

Using (6) and Lemma 3.6 in equation (31), we get

$$g(\nabla_X Y, Z) = g(A_{\phi Y}X, tZ) + \overline{g}(\overline{\nabla}_X Y, Z) - \lambda\overline{g}(\overline{\nabla}_X Y, Z) - g(A_{ntZ}X, Y). \tag{32}$$

Now, employing (5) in (32), we arrive at

$$\lambda g(\nabla_X Y, Z) = g(A_{nY}X, tZ) - g(A_{ntZ}X, Y). \tag{33}$$

Since $\lambda \neq 0$ and X, Y and Z are all non-degenerate vector fields, from Eqs. (30) and (33), we can conclude the desired necessary and sufficient conditions for the anti-invariant distribution $\mathfrak{D}^{\perp}$ to be involutive and totally geodesic foliation, respectively. This completes the proof of the theorem. □

THEOREM 3.9. *Let M be a proper $\mathcal{PR}$-pseudo-slant submanifold M of a paracosymplectic manifold $\overline{M}$. Then the slant distribution ($\mathfrak{D}_{\lambda}$) is*

- *involutive if and only if*

$$g(A_{ntZ}X - A_{nX}tZ, W) = g(A_{ntW}X - A_{nX}tW, Z),$$

- *totally geodesic foliation if and only if the shape operator satisfies* $A_{ntW}X = A_{nX}tW$,

for any $Z, W \in \Gamma(\mathfrak{D}_\lambda \oplus \{\xi\})$ *and* $X, Y \in \Gamma(\mathfrak{D}^\perp)$.

Proof. The proof can be achieved easily by following the proof of Theorem 3.8. □

4. $\mathcal{PR}$-PSEUDO-SLANT WARPED PRODUCT SUBMANIFOLDS

In this section, we investigate the existence of non-trivial $\mathcal{PR}$-pseudo-slant warped product submanifolds of the form $M_\perp \times_f M_\lambda$, $M_\lambda \times_f M_\perp$ in paracosymplectic manifolds $\overline{M}$, weather the structure vector field ξ is tangent to the first factor or second factor, where $M_\perp$ and M_λ are anti-invariant and proper slant submanifolds of $\overline{M}$, respectively.

Now, we give the following important results when ξ is tangent to the first factor or second factor:

THEOREM 4.1. *There doesn't exist any non-trivial $\mathcal{PR}$-pseudo-slant warped product submanifold of the form $M = M_\perp \times_f M_\lambda$ in paracosymplectic manifolds $\overline{M}(\phi, \xi, \eta, \overline{g})$, when ξ is tangent to*

(*a*) *a proper slant submanifold M_λ,*
(*b*) *an anti-invariant submanifold $M_\perp$.*

Proof. The proof of part-(a), when $\xi \in \Gamma(TM_\lambda)$, can be directly derived with the help of Theorem 2.5. For part-(b), let us assume that $M = M_\perp \times_f M_\lambda$ is a non-trivial $\mathcal{PR}$-pseudo-slant warped product submanifold of a paracosymplectic manifold $\overline{M}$ such that $\xi \in \Gamma(TM_\perp)$. Then, by applying Proposition 2.3, (2), (8) and the Gauss-Weingarten formulas, we obtain

$$g(A_{nX}Z, tZ) = X(\ln f)g(tZ, tZ) + g(A_{nZ}X, tZ), \tag{34}$$

for any non-degenerate vector fields $X \in \Gamma(TM_\perp)$ and $Z \in \Gamma(TM_\lambda)$. Employing Proposition 3.5 and the fact that $\eta(Z) = 0$ in (34), we deduce that

$$g(A_{nX}Z, tZ) = -\lambda X(\ln f)g(Z, Z) + g(A_{nZ}X, tZ). \tag{35}$$

Now, interchanging Z by tZ in the above equation, we achieve that

$$g(A_{nX}tZ, t^2Z) = -\lambda X(\ln f)g(tZ, tZ) + g(A_{ntZ}X, t^2Z). \tag{36}$$

Using the definition of slant submanifold and (23) in (36), we get

$$g(A_{nX}tZ, Z) = \lambda X(\ln f)g(Z, Z) + g(A_{ntZ}X, Z). \tag{37}$$

On the other hand, we have, from (3) and (4)-(8), that

$$g(A_{nZ}X, tZ) = g(\nabla_X Z, t^2Z) + g(A_{ntZ}X, Z) + g(\nabla_X tZ, tZ). \tag{38}$$

Using again the definition of slant submanifold and Proposition 2.3 in (38), we deduce that

$$g(A_{nZ}X, tZ) = \lambda X(\ln f)g(Z, Z) + g(A_{ntZ}X, Z) + X(\ln f)g(tZ, tZ). \tag{39}$$

The above equation, in view of Proposition 3.5 and the fact that $g(Z, \xi) = 0$, yields

$$g(A_{nZ}X, tZ) = g(A_{ntZ}X, Z). \tag{40}$$

From (35), (37) and (40), we conclude that

$$g(A_{nX}Z, tZ) + \lambda X(\ln f)g(Z, Z) = g(A_{nX}Z, tZ) - \lambda X(\ln f)g(Z, Z). \tag{41}$$

Thus we get

$$2\lambda X(\ln f)g(Z, Z) = 0, \tag{42}$$

by virtue of (7) and the symmetry of h. Now, (42) implies that either $\lambda = 0$ or f is a constant function on $M_\perp$, for any non-degenerate vector field $X \in \Gamma(TM_\perp)$ and $Z \in \Gamma(TM_\lambda)$. Since M_λ is a proper slant submanifold, f must be constant and this contradicts our supposition. This completes the proof of the theorem. □

From above theorem, we conclude the following consequence as a remark.

Remark 4.2. If in Theorem 4.1 $\lambda = 1$, then the submanifold reduces to the form $M_\perp \times_f M_T$. Thus, from (42), we can also say that there doesn't exist non-trivial warped product submanifolds of the form $M_\perp \times_f M_T$ with ξ either tangent to the first or the second factor. Hence, our result relates to the results obtained by several authors in [4, 15, 23] for contact settings.

Theorem 4.3. *There doesn't exist any non-trivial $\mathcal{PR}$-pseudo-slant warped product submanifold of the form $M = M_\lambda \times_f M_\perp$ in paracosymplectic manifolds $\overline{M}(\phi, \xi, \eta, \overline{g})$, when ξ is tangent to*

(*a*) *an anti-invariant submanifold $M_\perp$,*

(*b*) *a proper slant submanifold M_λ.*

Proof. The proof of the first part, when $\xi \in \Gamma(TM_\lambda)$, can be easily achieved by the use of Theorem 2.5. For the second part, we consider that $M = M_\lambda \times_f M_\perp$ is a non-trivial $\mathcal{PR}$-pseudo-slant warped product submanifold of a paracosymplectic manifold $\overline{M}$ such that $\xi \in \Gamma(TM_\lambda)$. We have from (4), for any non-degenerate vector fields $X \in \Gamma(TM_\perp)$ and $Z \in \Gamma(TM_\lambda)$, that $\overline{\nabla}_X \phi Z = \phi \overline{\nabla}_X Z$. The above expression, by virtue of (5), (6), (8) and Proposition (2.3), becomes

$$\begin{aligned} tZ(\ln f)X + h(X, tZ) - A_{nZ}X + \nabla^\perp_X nZ =& t(\nabla_X Z) + n(\nabla_X Z) \\ &+ t'h(X, Z) + n'h(X, Z). \end{aligned} \tag{43}$$

Equating the normal parts from (43) and then taking the inner product with nX, we get

$$g(h(X,tZ),nX)+g(\nabla^{\perp}_{X}nZ,nX)=g(Z(\ln f)nX,nX)+g(n'h(X,Z),nX). \tag{44}$$

Since, $M_{\perp}$ is an anti-invariant submanifold, using (2) and (3) in (44), we derive that

$$g(\nabla^{\perp}_{X}nZ,nX)=Z(\ln f)g(nX,nX)-\overline{g}(h(X,tZ),nX). \tag{45}$$

On the other hand, analogous to (43), we derive, by evaluating the expression $\overline{\nabla}_{Z}\phi X=\phi\overline{\nabla}_{Z}X$, that

$$-A_{nX}Z+\nabla^{\perp}_{Z}nX=Z(\ln f)nX+t'h(Z,X)+n'h(Z,X). \tag{46}$$

Moreover, by comparing normal components from the above expression, followed by the inner product with nX, we get

$$g(\nabla^{\perp}_{Z}nX,nX)=Z(\ln f)g(nX,nX)+g(n'h(Z,X),nX). \tag{47}$$

By using the facts that $\eta(X)=0$, $M_{\perp}$ is anti-invariant and (2), (3) in (47), we deduce that

$$g(\nabla^{\perp}_{Z}nX,nX)=Z(\ln f)g(nX,nX). \tag{48}$$

From equations (45) and (47), we achieve that

$$g(\nabla^{\perp}_{X}nZ,nX)-g(\nabla^{\perp}_{Z}nX,nX)=-\overline{g}(h(X,tZ),nX). \tag{49}$$

Employing (6), (8) and the fact that $M_{\perp}$ is anti-invariant in (49), we arrive at $g(X,\xi)=0$,

$$\overline{g}(\overline{\nabla}_{X}\phi Z,\phi X)-\overline{g}(\overline{\nabla}_{X}tZ,\phi X)-\overline{g}(\overline{\nabla}_{Z}\phi X,\phi X)=-\overline{g}(h(X,tZ),nX). \tag{50}$$

Now, by using Proposition (2.3), $g(X,\xi)=0$ and (2), (3) in (50), we find that

$$\overline{g}(\overline{\nabla}_{X}tZ,\phi X)=\overline{g}(h(X,tZ),nX). \tag{51}$$

The left hand side of the above expression, in view of the equations (3), (7), (8), and the definition of a slant submanifold, reduces to

$$-\lambda\overline{g}(\overline{\nabla}_{X}Z,X)+g(A_{ntZ}X,X)=g(A_{nX}tZ,X). \tag{52}$$

By applying Lemma 17, Propositions (2.3), (3.5) and (3), (5), (8) in (52), we obtain that

$$(1-\lambda)Z(\ln f)g(X,X)=0, \tag{53}$$

for any non-degenerate vector fields $X\in\Gamma(TM_{\perp})$ and $Z\in\Gamma(TM_{\lambda})$. The last equation implies that either $1-\lambda=0$ or f is a constant function on M_{λ}. Since M_{λ} is a proper slant submanifold, $1-\lambda=0$ is impossible. Hence, f must be constant on M_{λ}. This is contradiction to our assumption and thus completes the proof of the theorem. □

As an immediate conclusion of the above theorem we can easily write the following remark:

Remark 4.4. If in Theorem 4.3 $\lambda = 1$, then the manifold reduces to the form $M_T \times_f M_\perp$, when ξ is tangent to the first factor. This means, from (53), that the warped product with improper slant coefficient of type-(b) may exist.

5. EXAMPLE

In support to Remark 4.4 of Sect. 4, here we present a non-trivial example of $\mathcal{PR}$-pseudo-slant warped product submanifold $M_\lambda \times_f M_\perp$ with slant coefficient $\lambda = 1$, i.e. non-trivial $\mathcal{PR}$-semi-invariant warped product submanifold of the form $M_T \times_f M_\perp$ such that $\xi \in TM_T$ in a paracosymplectic manifold.

Let $\overline{M} = \mathbb{R}^4 \times \mathbb{R}_+ \subset \mathbb{R}^5$ be a 5-dimensional manifold with the standard Cartesian coordinates (x_1, x_2, y_1, y_2, z). Define a paracosymplectic pseudo-Riemannian metric structure $(\phi, \xi, \eta, \overline{g})$ on $\overline{M}$ by

$$\phi e_1 = e_3,\ \phi e_2 = e_4,\ \phi e_3 = e_1,\ \phi e_4 = e_2,\ \phi e_5 = 0, \tag{54}$$

$$\xi = e_5,\ \eta = dz,\ \overline{g} = \sum_{i=1}^{2}(dx_i)^2 - \sum_{j=1}^{2}(dy_j)^2 + \eta \otimes \eta. \tag{55}$$

Here, $\{e_1, e_2, e_3, e_4, e_5\}$ is a local orthonormal frame for $\Gamma(T\overline{M})$ given by $e_1 = \dfrac{\partial}{\partial x_1}$, $e_2 = \dfrac{\partial}{\partial x_2}$, $e_3 = \dfrac{\partial}{\partial y_1}$, $e_4 = \dfrac{\partial}{\partial y_2}$ and $e_5 = \dfrac{\partial}{\partial z}$. Let M be an isometrically immersed pseudo-Riemannian submanifold in a paracosymplectic manifold $\overline{M}$ defined by $\Omega(u, v, \alpha, z) = (u\cos(\alpha), u\sin(\alpha), v\cos(\alpha), v\sin(\alpha), z)$, where $\alpha \in (0, \pi/2)$ and $v < u \in \mathbb{R}_+$ or $v > u \in \mathbb{R}_-$. Then the tangent bundle $\Gamma(TM)$ of M is spanned by the vectors

$$\begin{aligned} &X_u = \cos(\alpha)e_1 + \sin(\alpha)e_2,\ X_v = \cos(\alpha)e_3 + \sin(\alpha)e_4,\\ &X_\alpha = -u\sin(\alpha)e_1 + u\cos(\alpha)e_2 - v\sin(\alpha)e_3 + v\cos(\alpha)e_4,\ X_z = e_5. \end{aligned} \tag{56}$$

The space $\phi(TM)$ with respect to the paracosymplectic pseudo-Riemannian metric structure $(\phi, \xi, \eta, \overline{g})$ of $\overline{M}$ becomes

$$\begin{aligned} &\phi(X_u) = \cos(\alpha)e_3 + \sin(\alpha)e_4,\ \phi(X_v) = \cos(\alpha)e_1 + \sin(\alpha)e_2,\\ &\phi(X_\alpha) = -v\sin(\alpha)e_1 + v\cos(\alpha)e_2 - u\sin(\alpha)e_3 + u\cos(\alpha)e_4,\\ &\phi(X_z) = 0. \end{aligned} \tag{57}$$

From (56) and (57), we obtain that $\phi(X_\alpha)$ is orthogonal to $\Gamma(TM)$ and $\phi(X_u)$, $\phi(X_v)$, $\phi(X_z)$ are tangent to M. So, $\mathfrak{D}_\lambda$ and $\mathfrak{D}^\perp$ can be taken as a subspace span$\{X_u, X_v, X_z\}$ and a subspace span$\{X_\alpha\}$, respectively, where $\xi = X_z$ for $\phi(X_z) = 0$ and $\eta(X_z) = 1$. Therefore, M becomes a $\mathcal{PR}$-pseudo-slant submanifold. Furthermore, we can say, from Theorem 3.8 and Theorem 3.9, that $\mathfrak{D}^\perp$ and $\mathfrak{D}_\lambda$ are integrable. Therefore, by taking the integral manifolds of $\mathfrak{D}_\lambda$

and $\mathfrak{D}^{\perp}$ by M_λ and $M_\perp$, respectively, the induced pseudo-Riemannian metric tensor g of M is given by

$$[g(e_i, e_j)] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & u^2 - v^2 \end{bmatrix},$$

that is

$$g = g_{M_\lambda} \oplus (u^2 - v^2) g_{M_\perp}.$$

Hence, M is a 4-dimensional $\mathcal{PR}$-pseudo-slant warped product submanifold of $\overline{M}$ with warping function $f = \sqrt{(u^2 - v^2)}$ and slant coefficient λ. By direct computation from the above, we achieve that the slant coefficient for $\mathfrak{D}_\lambda$ is $\lambda = 1$. Thus, M is a $\mathcal{PR}$-pseudo-slant warped product submanifold of the form $M_\lambda \times_f M_\perp$ with slant coefficient $\lambda = 1$, in particular, a non-trivial $\mathcal{PR}$-semi-invariant warped product submanifold of $\overline{M}$.

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