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COURNOT EQUILIBRIUM IN CASE OF (-1)-CONCAVE PRICE FUNCTION

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Abstract. We consider a class of homogeneous Cournot oligopolies with (-1)-concave price function. We show some useful properties of the revenue function in case of (-1)-concave price function and prove the existence of an equilibrium in the continuous and non-differentiable case. A simple proof of an equilibrium uniqueness result in the smooth case with $(-1)/N$-concave (N-number of the firms in the market) price function is provided.

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1. INTRODUCTION

Various results in the literature guarantee the existence of a unique Cournot equilibrium in homogeneous Cournot oligopolies. One of the first results is in Murphy et. al (1982) [8], and employs concave revenue functions and convex cost functions. Amir (1996) [1], shows that log-concavity of the inverse demand guarantees the existence of an equilibrium and in case of convex cost functions the uniqueness of the equilibrium. Deneckere and Kovenock (1999) [4] derive an existence and uniqueness theorem using assumptions based on the direct demand function. This theorem is restated, using the concept of ρ-concavity, in [2]. Biconcavity is a condition on the inverse demand that corresponds to concavity after a simultaneous parametrization of price and quantity, the price function is (α, β)-biconcave if and only if the demand function is (β, α)-biconcave. Ewerhart (2014) [5] uses this concept to analyze the Cournot model. The existence theorem in his paper admits values of $0 \leq \alpha \leq 1$ in the non-smooth case and $\alpha < 0$ in the case of twice differentiable price function. von Mouche and Quartieri (2013) [9] define the concept of concave integrated price flexibility to provide new results about semi-uniqueness and uniqueness of Cournot equilibria.

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(-1)-concavity of the price function corresponds to $(-1,1)$-biconcavity, i.e. $\alpha < 0$. We prove some properties of the revenue function in case of (-1)-concavity of the price function without assuming differentiability and we prove the existence theorem in the non-smooth case with convex costs functions. Employing the same approach as in [9] and [10] we derive a simple proof of the uniqueness theorem in the smooth case with $(-1)/N$-concave price function and convex costs functions (N-number of the firms in the market), without assuming twice differentiability. The uniqueness theorem for log-concave (0-concave) price function is a consequence of our result. We also show that the property of concave price flexibility, assumed in [9] does not entail (-1)-concavity. To the best of our knowledge the conditions we impose are not entailed by any other conditions used in the proofs of the existence and uniqueness theorems of a Cournot equilibrium so far.

The paper is organized as follows: Section 2 introduces the concept of a Cournot game and lists some results from the literature that will be used in our proofs. Section 3 introduces the concept of generalized concavity and provides some properties of the revenue function in the case of $(-1)/N$-concavity of the price function. In Section 4 existence and uniqueness theorems are derived. In Section 5 concluding remarks are given.

2. BACKGROUND AND SETTINGS

Cournot oligopoly is a game with perfect information, N players (firms) that produce a homogenous product and compete on quantities. Each firm has a cost $c_i(x_i)$ for producing $x_i \geq 0$, $i = 1, ..., N$. The market price is a function of the firms' total output $x_1 + ... + x_N = X$. Let $\widetilde{N} = \{1, ..., N\}$; the profit of the $i-$th firm is given by:

$$f_i(X) = x_i P\Big(x_i + \sum_{j \in \widetilde{N}, j \neq i} x_j\Big) - c_i(x_i).$$

A Nash equilibrium of an oligopoly is called a Cournot equilibrium.

We will provide the following results from [9] in the framework for our needs.

LEMMA 2.1. *Suppose each cost function is convex and:*

- *p is left and right differentiable at each point of R_+;*
- $\mathrm{d}^+p(x) \leq \mathrm{d}^-p(x) < 0$ $(x \in R_+)$.

If there exists more than one equilibrium, then 0 is not an equilibrium.

THEOREM 2.2. *Suppose each cost function is convex and:*

- *p is left and right differentiable at each point of R_+;*
- $\mathrm{d}^-p(x) \leq \mathrm{d}^+p(x) < 0$ $(x \in R_+)$.

Then the Nash-sums are injective.

We will use the following classical existence theorem (Debreu, Glicksberg, Fan (1952) [3, 6, 7]):

THEOREM 2.3. *Consider a game with $N \geq 2$ players with strategies $s_i \in S_i$, $i = 1, ..., N$. If:*

- *$S_i, i = 1, ..., N$ are nonempty, compact, convex subsets of finite dimensional Euclidean spaces,*
- *all pay-off functions are continuous on $S_1 \times ... \times S_N$,*
- *all payoff functions are quasi-concave of s_i over S_i, if all the other strategy sets are held fixed,*

then there exists at least one equilibrium.

3. GENERALIZED CONCAVITY OF FUNCTIONS

We consider a function $p : \Omega \to R_+$ defined on a convex set $\Omega \subsetneqq R_+$.

DEFINITION 3.1. A positive function defined on a convex set $\Omega \subsetneqq R_+$ is said to be $\alpha -$ *concave*, where $\alpha \in [-\infty, +\infty]$, if for all $x, y \in \Omega$ and all $\lambda \in [0, 1]$ the following inequality holds true:

$$p(\lambda x + (1 - \lambda)y) \geq m_\alpha(p(x), p(y), \lambda),$$

where $m_\alpha : R_+ \times R_+ \times [0, 1] \to R$ is defined as follows:

$$m_\alpha(a, b, \lambda) = \begin{cases} a^\lambda b^{1-\lambda}, & \text{if } \alpha = 0 \\ \max\{a, b\}, & \text{if } \alpha = \infty \\ \min\{a, b\}, & \text{if } \alpha = -\infty \\ (\lambda a^\alpha + (1 - \lambda)b^\alpha)^{1/\alpha}, & \text{otherwise.} \end{cases}$$

In the case $\alpha = 0$ the function is log-concave, in the case $\alpha = 1$ the function is simply concave.

The following lemma is very important, because it implies that α-concavity entails β-concavity, for all $\beta \leq \alpha$.

LEMMA 3.2. *The mapping $\alpha \to m_\alpha(a, b, \lambda)$ is non-decreasing and continuous.*

We are interested especially in $(-1)/n$-concave functions, $n \geq 1$:

DEFINITION 3.3. A positive function defined on a convex set $\Omega \subsetneqq R_+$ is said to be $(-1)/n -$ *concave*, $n \geq 1$, if for all $x, y \in \Omega$ and all $\lambda \in [0, 1]$ the following inequality holds true:

$$p(\lambda x + (1 - \lambda)y) \geq (\lambda(p(x))^{-1/n} + (1 - \lambda)(p(y))^{-1/n})^{-n}.$$

From Lemma 3.2 it follows that all concave ($\alpha = 1$) and log-concave ($\alpha = 0$) functions are $(-1)/n$-concave. It is clear that a positive function $p(\cdot)$ is $(-1)/n$-concave if and only if $1/p(\cdot)^{1/n}$ is convex.

THEOREM 3.4. *Let $p : \Omega \subsetneqq R_+ \cup \{0\} \to R_+$ be a decreasing, continuous and $(-1)/n$-concave function and let $n \geq 1$. Then $xp(x + k)^{1/n}$ is strictly concave on the interval where it is strictly increasing for all $k \geq 0$.*

Proof. Let $x_1, x_2 \in \Omega$ with $x_1 < x_2$ and

$$x_1 p(x_1+k)^{1/n} < x_2 p(x_2+k)^{1/n}. \tag{1}$$

Note that convexity (concavity) is equivalent to midpoint convexity (concavity) in the case of a continuous function. The function $p(\cdot)$ is a continuous and $(-1)/n$-concave function, therefore $1/p(x+k)^{1/n}$, $k \geq 0$, is convex and midpoint convex. We get

$$\frac{1}{p\Big(\frac{x_1+x_2}{2}+k\Big)^{1/n}} \leq \left(\frac{1}{p(x_1+k)^{1/n}} + \frac{1}{p(x_2+k)^{1/n}}\right)/2.$$

Hence,

$$p\Big(\frac{x_1+x_2}{2}+k\Big)^{1/n} \geq \frac{2p(x_1+k)^{1/n} p(x_2+k)^{1/n}}{p(x_1+k)^{1/n}+p(x_2+k)^{1/n}}. \tag{2}$$

Suppose $xp(x+k)^{1/n}$ is not strictly midpoint concave on the interval where it is strictly increasing:

$$\frac{x_1+x_2}{2} p\Big(\frac{x_1+x_2}{2}+k\Big)^{1/n} \leq \frac{x_1 p(x_1+k)^{1/n} + x_2 p(x_2+k)^{1/n}}{2}.$$

From (2), we have

$$\frac{x_1+x_2}{2} \frac{2p(x_1+k)^{1/n} p(x_2+k)^{1/n}}{p(x_1+k)^{1/n}+p(x_2+k)^{1/n}} \leq \frac{x_1 p(x_1+k)^{1/n} + x_2 p(x_2+k)^{1/n}}{2},$$

which yields

$$(x_1+x_2) p(x_1+k)^{1/n} p(x_2+k)^{1/n} \leq x_1 p(x_1+k)^{2/n} + x_2 p(x_2+k)^{2/n},$$

or

$$\begin{aligned} &x_1((p(x_1+k))^{2/n} - p(x_1+k)^{1/n} p(x_2+k)^{1/n}) \\ &\geq x_2(p(x_1+k)^{1/n}(p(x_2+k)^{1/n} - p(x_2+k)^{2/n}). \end{aligned}$$

Dividing by $p(x_1+k)^{1/n} - p(x_2+k)^{1/n} > 0$, we obtain

$$x_1 p(x_1+k)^{1/n} \geq x_2 p(x_2+k)^{1/n}.$$

The last inequality is in contradiction with (1). □

THEOREM 3.5. *Let $p : \Omega \subseteqq R_+ \cup \{0\} \to R_+$ be a $(-1)/n$-concave function, $n \geq 1$, $k \geq 0$. Then, if $xp(x+k)^{1/n}$ has an extremum greater than 0, it is a global maximum.*

Proof. First we will prove $xp(x+k)^{1/n}$ does not possess a positive local minimum x^* such that: $x^* p(x^*+k)^{1/n} \leq xp(x+k)^{1/n}$, where x is in a small neighbourhood of x^* and the inequality is strict at least for $x > x^*$ or $x < x^*$. Suppose $x^* = (x_1 + x_2)/2$ is a local minimum, $x_1, x_2 > 0$ and

$$\frac{x_1+x_2}{2}p\Big(\frac{x_1+x_2}{2}+k\Big)^{1/n} < p(x_1+k)^{1/n}x_1 \tag{3}$$

$$\frac{x_1+x_2}{2}p\Big(\frac{x_1+x_2}{2}+k\Big)^{1/n} \le p(x_2+k)^{1/n}x_2 \tag{4}$$

From (3) we obtain:

$$0 < p\Big(\frac{x_1+x_2}{2}+k\Big)^{1/n} < \frac{x_1}{x_2}\Big(2p(x_1+k)^{1/n} - p\Big(\frac{x_1+x_2}{2}+k\Big)^{1/n}\Big)$$

and

$$\frac{x_1}{x_2} > \frac{p\Big(\frac{x_1+x_2}{2}+k\Big)^{1/n}}{2p(x_1+k)^{1/n} - p(\frac{x_1+x_2}{2}+k)^{1/n}}. \tag{5}$$

Using (4), we have:

$$\frac{x_1}{x_2}p\Big(\frac{x_1+x_2}{2}+k\Big)^{1/n} \le 2p(x_2+k)^{1/n} - p\Big(\frac{x_1+x_2}{2}+k\Big)^{1/n},$$

that is

$$\frac{x_1}{x_2} \le \frac{2p(x_2+k)^{1/n} - p(\frac{x_1+x_2}{2}+k)^{1/n}}{p\Big(\frac{x_1+x_2}{2}+k\Big)^{1/n}}. \tag{6}$$

From (5) and (6), we get

$$\frac{p\Big(\frac{x_1+x_2}{2}+k\Big)^{1/n}}{2p(x_1+k)^{1/n} - p(\frac{x_1+x_2}{2}+k)^{1/n}} < \frac{2p(x_2+k)^{1/n} - p\Big(\frac{x_1+x_2}{2}+k\Big)^{1/n}}{p\Big(\frac{x_1+x_2}{2}+k\Big)^{1/n}}$$

or

$$p\Big(\frac{x_1+x_2}{2}+k\Big)^{1/n} < \frac{2p(x_1+k)^{1/n}p(x_2+k)^{1/n}}{p(x_1+k)^{1/n} + p(x_2+k)^{1/n}}.$$

The last is in contradiction with (2).

If $xp(x+k)^{1/n}$ is non-increasing around 0, it will be non-increasing for all $x > 0$, since there is no local minimum and the function will not change its behavior. If $xp(x+k)^{1/n}$ is increasing around 0 and there is a local maximum x^*, $xp(x+k)^{1/n}$ will be non-increasing for all $x > x^*$ and x^* is a global maximum. □

Corollary 3.6. *Let $p : \Omega \subseteqq R_+ \cup \{0\} \to R_+$ be a decreasing, continuous and $(-1)/n$-concave function, $n \ge 1$. If $0 < x_1 \le x_2$, $k \ge 0$, $1 \le s \le n$, then from $x_2 \mathrm{d}^- p(x_2+k) + sp(x_2+k) > 0$, it follows that $x_1 \mathrm{d}^+ p(x_1+k) + sp(x_1+k) > x_2 \mathrm{d}^- p(x_2+k) + sp(x_2+k)$.*

Proof. If p is $(-1)/n$-concave, it is $(-1)/s$-concave, $1 \le s \le n$ (Lemma 3.2). From Theorem 3.4 and Theorem 3.5, it follows that if there is an interval where the function $xp(x+k)^{1/s}$ is increasing, it is unique and its left boundary point is 0; moreover, $xp(x+k)^{1/s}$ is strictly concave in this interval, therefore, if $0 < x_1 \le x_2$, $k \ge 0$ and

$$\mathrm{d}^-(x_2 p(x_2+k)^{1/s}) \ge 0 \Leftrightarrow \frac{x_2 \mathrm{d}^- p(x_2+k) + sp(x_2+k)}{s} p(x_2)^{1/s-1} \ge 0,$$

which is

$$x_2 \mathrm{d}^- p(x_2+k) + sp(x_2+k) \ge 0,$$

then it follows

$$\mathrm{d}^+(xp(x+k)^{1/s})(x_1) \ge \mathrm{d}^-(xp(x+k)^{1/s})(x_2)$$

or

$$p(x_1)^{1/s-1} \frac{x_1 \mathrm{d}^+ p(x_1+k) + sp(x_1+k)}{s} \ge p(x_2)^{1/s-1} \frac{x_2 \mathrm{d}^- p(x_2+k) + sp(x_2+k)}{s}$$

and, since $p(x)$ is decreasing and, for $s \ge 1$, $p(x_1)^{1/s-1} \le p(x_2)^{1/s-1}$,

$$x_1 \mathrm{d}^+ p(x_1+k) + sp(x_1+k) \ge x_2 \mathrm{d}^- p(x_2+k) + sp(x_2+k).$$

□

4. EXISTENCE AND UNIQUENESS OF AN EQUILIBRIUM

THEOREM 4.1. *Suppose each cost function is convex and strictly increasing, p_i is a positive, strictly decreasing, continuous and (-1)-concave function on $R_+ \cup \{0\}$ for each $i = 1, ..., N$. Then there exists at least one equilibrium.*

Proof. From Theorem 3.4 and Theorem 3.5, it follows that $xp_i(x+k) - c_i(x)$ is a quasi-concave function. As $p_i(\cdot)$ is positive, (-1)-concave and decreasing, $1/p_i(\cdot)$ is positive, convex and increasing, therefore for each $x_0 \ge 0$ and each i there exists a supporting line $a_i(x_0)x + b_i(x_0)$ such that: $1/p_i(x) \ge a_i(x_0)x + b_i(x_0)$ (with an equality at $x = x_0$). From the fact that p_i is positive and strictly increasing it follows that $a_i(x_0)x + b_i(x_0) > 0$ for each $x \ge 0$ and $a_i(x_0) \ge 0$. Moreover, there exists $\bar{x}_0$ such that $a_i(\bar{x}_0) > 0$. Then

$$\limsup_{x\to\infty} xp_i(x+k) - c_i(x) \le \lim_{x\to\infty} x \frac{1}{a_i(\bar{x}_0)x + b_i(\bar{x}_0)} - c_i(x) = -\infty$$

Consequently, there exists X_i such that: $xp_i(x) - c_i(x) \le -c_i(0)$ for all $x \ge X_i$ and the effective strategy of each firm is contained in $[0, X_i]$.

The conditions of Theorem 2.3 are fulfilled and there exists at least one equilibrium. □

THEOREM 4.2. *Suppose each cost function is convex and increasing, p_i is a positive, strictly decreasing, continuous and $(-1)/N$-concave function on R_+, for each $i = 1, ..., N$ $N \geq 2$. Then the Nash-sums σ are constant.*

Proof. As in [9], Theorem 2, suppose a, b are equilibria with:

$$x_a = \sum_{l \in N} a_l < \sum_{l \in N} b_l = x_b.$$

Let $J = \{l \in N | a_l < b_l\}$, $\widetilde{x_a} = \sum_{l \in J} a_l$, $\widetilde{x_b} = \sum_{l \in J} b_l$, s – the number of elements of J. Note that $x_a \neq 0$ (Lemma 2.1), $x_b > 0$, $s \geq 1$, $\widetilde{x_a} \leq x_a$, $\widetilde{x_b} \leq x_b$, $\widetilde{x_a} < \widetilde{x_b}$, $\widetilde{x_b} - \widetilde{x_a} \geq x_b - x_a$.

As a and b are equilibria, $\mathrm{d}^+{}_i\pi_i(a) \leq 0 \leq \mathrm{d}^-{}_i\pi_i(b)$ $(i \in J)$, i.e.

$$\mathrm{d}^+ p(x_a)a_i + p(x_a) - \mathrm{d}^+ c_i(a_i) \leq 0 \leq \mathrm{d}^- p(x_b)b_i + p(x_b) - \mathrm{d}^- c_i(b_i),$$

or

$$\mathrm{d}^+ p(x_a)\widetilde{x_a} + sp(x_a) - \sum_{i \in J} \mathrm{d}^+ c_i(a_i) \leq 0 \leq \mathrm{d}^- p(x_b)\widetilde{x_b} + sp(x_b) - \sum_{i \in J} \mathrm{d}^- c_i(b_i).$$

As each c_i is convex this inequality implies:

$$\mathrm{d}^+ p(x_a)\widetilde{x_a} + sp(x_a) \leq \mathrm{d}^- p(x_b)\widetilde{x_b} + sp(x_b). \tag{7}$$

Consider the two points $x_1 = x_a - (x_b - \widetilde{x_b}) > 0$, $x_2 = \widetilde{x_b} > 0$, $x_1 < x_2$, $k = x_b - \widetilde{x_b} > 0$. From Corollary 3.6:

$$(x_a - (x_b - \widetilde{x_b}))\mathrm{d}^+ p(x_a) + sp(x_a) > \widetilde{x_b}\mathrm{d}^- p(x_b) + sp(x_b) \geq 0.$$

From $\widetilde{x_b} - \widetilde{x_a} \geq x_b - x_a$ it follows $x_a - (x_b - \widetilde{x_b}) \geq \widetilde{x_a}$; furthermore, p_i is strictly decreasing and consequently:

$$\widetilde{x_a}\mathrm{d}^+ p(x_a) > (x_a - (x_b - \widetilde{x_b}))\mathrm{d}^+ p(x_a),$$

therefore

$$\widetilde{x_a}\mathrm{d}^+ p(x_a) + sp(x_a) > \widetilde{x_b}\mathrm{d}^- p(x_b) + sp(x_b),$$

in contradiction with (7). □

COROLLARY 4.3. *Suppose each cost function is convex and increasing, p_i is a positive, strictly decreasing, continuous, differentiable and $(-1)/N$-concave function on $R_+ \cup \{0\}$, for each $i = 1, ..., N$, $N \geq 2$. Then there exists a unique equilibrium.*

Proof. This is a direct consequence of Theorems 2.2, 4.2 and 4.1. □

5. CONCLUDING REMARKS

As noticed in the Introduction, our results are not equivalent to the results presented in [9]. An example of a function that is (-1)-concave, but without concave integrated price flexibility, is $p(x) = \frac{1}{(x+1)\ln(x+2)}$. In this case the revenue function is quasi-concave (but not concave). There are examples of functions that are with concave integrated price flexibility, but not (-1)-concave ($(x+1)^\alpha, 0 < \alpha < 1$), therefore, our results neither imply, nor are implied by the results in [9]. The existence theorem [5, Theorem 3.3] supposes the payoff and costs functions are twice differentiable. The existence theorem 4.1 is in the non-smooth case. So, to the best of our knowledge, the results presented here are new.

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