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OPERATORS IN MINIMAL SPACES WITH HEREDITARY CLASSES

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Abstract. Quite recently, a new minimal structure $m_H^\star$ has been introduced in [12] by using a minimal structure m and a hereditary class $\mathcal{H}$. In this paper, we introduce and investigate an operator $\Gamma^\star_{mH}$, $(\star)$-strongly m-codense hereditary class $\mathcal{H}$ and a minimal structure m which is said to be m-compatible with a hereditary class $\mathcal{H}$ in a hereditary m-space $(X, m, \mathcal{H})$.

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1. INTRODUCTION

The notion of ideals in topological spaces was introduced by Kuratowski [10]. Janković and Hamlett [8] defined the local function on an ideal topological space $(X, \tau, \mathcal{I})$. By using it they obtained a new topology $\tau^\star$ for X and investigated relations between τ and $\tau^\star$. A subfamily μ of the power set $\mathcal{P}(X)$ on a nonempty set X is called a generalized topology (briefly GT) [6] if $\emptyset \in \mu$ and any union of elements of μ belongs to μ. Császár [7] defined a hereditary class $\mathcal{H}$ which is weaker than an ideal and constructed a new GT $\mu^\star$ from a GT μ and a hereditary class $\mathcal{H}$. Moreover, he showed that many properties related to τ and $\tau^\star$ remain valid (possibly with small modifications) for μ and $\mu^\star$.

In [12], Noiri and Popa introduced the minimal local function on a minimal space (X, m) with a hereditary class $\mathcal{H}$ and constructed a minimal structure $m_H^\star$ which contains m. They showed that many properties related to τ and $\tau^\star$ (or μ and $\mu^\star$) remain similarly valid on m and $m_H^\star$.

In this paper, we investigate relationships between a minimal stracture m and a hereditary class $\mathcal{H}$. In Section 3, we define and study an operator, called $\Gamma^\star_{mH}$, on a herediatary minimal space $(X, m, \mathcal{H})$. In Section 4, we investigate a minimal structure m which is said to be m-compatible with a hereditary class $\mathcal{H}$. In the last section, we define and investigate a heraditary class $\mathcal{H}$ which

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is said to be ($\star$)-strongly m-codense. Several characterizations of minimal stracture were provided in [1, 2, 3, 4, 5].

2. MINIMAL STRUCTURES

DEFINITION 2.1. A subfamily m of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) [13] on X if $\emptyset \in m$ and $X \in m$.

By (X, m) we denote a nonempty set X with a minimal structure m on X and call it an *m-space*. Each member of m is said to be *m-open* and the complement of an m-open set is said to be *m-closed*. For a point $x \in X$, the family $\{U : x \in U$ and $U \in m\}$ is denoted by $m(x)$.

DEFINITION 2.2. Let (X, m) be an m-space and A a subset of X. The *m-closure* $\mathrm{mCl}(A)$ of A [11] is defined by $\mathrm{mCl}(A) = \cap\{F \subset X : A \subset F, X \setminus F \in m\}$.

LEMMA 2.3 (Maki et al. [11]). *Let X be a nonempty set and m a minimal structure on X. For subsets A and B of X, the following properties hold:*
(1) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mCl}(A) = A$ *if* A *is m-closed,*
(2) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$,
(3) *If* $A \subset B$, *then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$,
(4) $\mathrm{mCl}(A) \cup \mathrm{mCl}(B) \subset \mathrm{mCl}(A \cup B)$,
(5) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$.

DEFINITION 2.4. A minimal structure m of a set X is said to have
(1) *property* $\mathcal{B}$ [11] if the union of any collection of elements of m is an element of m,
(2) *property* $[F]$ if m is closed under finite intersections.

LEMMA 2.5 (Popa and Noiri [13]). *Let* (X, m) *be an m-space and A a subset of X.*
(1) $x \in \mathrm{mCl}(A)$ *if and only if* $U \cap A \neq \emptyset$ *for every* $U \in m(x)$.
(2) *Let m have property* $\mathcal{B}$. *Then the following properties hold:*
(*i*) *A is m-closed if and only if* $\mathrm{mCl}(A) = A$,
(*ii*) $\mathrm{mCl}(A)$ *is m-closed.*

DEFINITION 2.6. A nonempty subfamily $\mathcal{H}$ of $\mathcal{P}(X)$ is called a *hereditary class* on X [7] if it satisfies the following properties: $A \in \mathcal{H}$ and $B \subset A$ implies $B \in \mathcal{H}$. A hereditary class $\mathcal{H}$ is called an *ideal* if it satisfies the additional condition: $A \in \mathcal{H}$ and $B \in \mathcal{H}$ implies $A \cup B \in \mathcal{H}$.

A minimal space (X, m) with a hereditary class $\mathcal{H}$ on X is called a *hereditary minimal space* (briefly *hereditary m-space*) and is denoted by $(X, m, \mathcal{H})$.

DEFINITION 2.7 ([12]). Let $(X, m, \mathcal{H})$ be a hereditary m-space. For a subset A of X, the *minimal local function* $A^{\star}_{mH}(\mathcal{H}, m)$ of A is defined as follows:

$$A^{\star}_{mH}(\mathcal{H}, m) = \{x \in X : U \cap A \notin \mathcal{H} \text{ for every } U \in m(x)\}.$$

Hereafter, $A^{\star}_{mH}(\mathcal{H}, m)$ is simply denoted by $A^{\star}_{mH}$.

Remark 2.8 ([12]). Let $(X, m, \mathcal{H})$ be a hereditary m-space and A a subset of X. If $\mathcal{H} = \{\emptyset\}$ (resp. $\mathcal{P}(X)$), then $A^{\star}_{mH} = \mathrm{mCl}(A)$ (resp. $A^{\star}_{mH} = \emptyset$).

Lemma 2.9 ([12]). *Let $(X, m, \mathcal{H})$ be a hereditary m-space. For subsets A and B of X, the following properties hold:*

(1) *If $A \subset B$, then $A^{\star}_{mH} \subset B^{\star}_{mH}$,*
(2) $A^{\star}_{mH} = \mathrm{mCl}(A^{\star}_{mH}) \subset \mathrm{mCl}(A)$,
(3) $A^{\star}_{mH} \cup B^{\star}_{mH} \subset (A \cup B)^{\star}_{mH}$,
(4) $(A^{\star}_{mH})^{\star}_{mH} \subset (A \cup A^{\star}_{mH})^{\star}_{mH} = A^{\star}_{mH}$,
(5) *If $A \in \mathcal{H}$, then $A^{\star}_{mH} = \emptyset$.*

Lemma 2.10. *Let $(X, m, \mathcal{H})$ be a hereditary m-space and A a subset of X. If $U \in m$ and $U \cap A \in \mathcal{H}$, then $U \cap A^{\star}_{mH} = \emptyset$.*

3. THE OPERATOR $\Gamma^{\star}_{MH}$

Definition 3.1. Let $(X, m, \mathcal{H})$ be a hereditary m-space. An operator $\Gamma^{*}_{mH} : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined as follows: for every $A \in X$, $\Gamma^{*}_{mH}(A) = \{x \in X :$ there exists $M \in m(x)$ such that $M - A \in \mathcal{H}\}$.

Theorem 3.2. *Let $(X, m, \mathcal{H})$ be a hereditary m-space. Then, for every subset A of X, $\Gamma^{*}_{mH}(A) = X - (X - A)^{*}_{mH}$.*

Proof. Suppose $x \in X - (X - A)^{*}_{mH}$. Then $x \notin (X - A)^{*}_{mH}$, and so there exists $M \in m(x)$ such that $M \cap (X - A) \in \mathcal{H}$, which implies that $M - A \in \mathcal{H}$. Therefore, $X - (X - A)^{*}_{mH} \subseteq \{x \in X :$ there exists $M \in m(x)$ such that $M - A \in \mathcal{H}\} = \Gamma^{*}_{mH}(A)$. Conversely, assume that $y \in \Gamma^{*}_{mH}(A)$. Then there exists $M \in m(y)$ such that $M - A \in \mathcal{H}$. Since $M - A \in \mathcal{H}$, $M \cap (X - A) \in \mathcal{H}$ which implies that $y \notin (X - A)^{*}_{mH}$. Therefore, $y \in X - (X - A)^{*}_{mH}$. Thus, $\Gamma^{*}_{mH}(A) = X - (X - A)^{*}_{mH}$. □

Definition 3.3 ([12]). Let $(X, m, \mathcal{H})$ be a hereditary m-space and A a subset of X. The *minimal $\star$-closure* $\mathrm{mCl}^{\star}_{H}(A)$ of A is defined as $\mathrm{mCl}^{\star}_{H}(A) = A \cup A^{\star}_{mH}$. A new m-structure, $m^{\star}_{H}$, is defined as follows: $m^{\star}_{H} = \{U \subset X : \mathrm{mCl}^{\star}_{H}(X \setminus U) = X \setminus U\}$. Each member of $m^{\star}_{H}$ is said to be $m^{\star}_{H}$-*open* and the complement of an $m^{\star}_{H}$-open set is said to be $m^{\star}_{H}$-*closed*.

Lemma 3.4. *Let $(X, m, \mathcal{H})$ be a hereditary m-space. A subset F of X is $m^{\star}_{H}$-closed if and only if $F^{*}_{mH} \subseteq F$.*

Proof. F is $m^{\star}_{H}$-closed if and only if $F = mCl^{\star}_{H}(F) = F \cup F^{*}_{mH}$ if and only if $F^{*}_{mH} \subseteq F$. □

Lemma 3.5. *Let $(X, m, \mathcal{H})$ be a hereditary m-space, then $m^{\star}_{H} = \{A \subseteq X : A \subseteq \Gamma^{*}_{mH}(A)\}$.*

Proof. Let $A \subseteq X$ and $A \subseteq \Gamma^*_{mH}(A)$. By Theorem 3.2, $A \subseteq X-(X-A)^*_{mH}$ and $X - A \supseteq (X-A)^*_{mH}$. Therefore, $X - A = mCl^\star_H(X-A)$ and hence $A \in m^\star_H$. Conversely, let $A \in m^\star_H$. Then $X - A$ is $m^\star_H$-closed. Therefore, $(X-A)^*_{mH} \subseteq X - A$, which implies that $X-(X-A) \subseteq X-(X-A)^*_{mH}$ and hence $A \subseteq \Gamma^*_{mH}(A)$. □

Corollary 3.6. *Let $(X, m, \mathcal{H})$ be a hereditary m-space. Then $U \subseteq \Gamma^*_{mH}(U)$ for every m-open set $U \subseteq X$.*

Proof. We know that $\Gamma^*_{mH}(U) = X - (X-U)^*_{mH}$. Now, $(X-U)^*_{mH} \subseteq mCl(X-U) = X-U$, since $X-U$ is m-closed. Therefore, $U = X-(X-U) \subseteq X-(X-U)^*_{mH} = \Gamma^*_{mH}(U)$. □

Several basic properties concerning the behavior of the operator Γ^*_{mH} are included in the following theorem.

Theorem 3.7. *Let $(X, m, \mathcal{H})$ be a hereditary m-space. Then, for a subset A of X, the following properties hold:*

(1) *If m has property $\mathcal{B}$, then $\Gamma^*_{mH}(A)$ is m-open.*
(2) *If $A \subseteq B \subseteq X$, then $\Gamma^*_{mH}(A) \subseteq \Gamma^*_{mH}(B)$.*
(3) $\Gamma^*_{mH}(A \cap B) \subseteq \Gamma^*_{mH}(A) \cap \Gamma^*_{mH}(A)$.
(4) $\Gamma^*_{mH}(A) = \Gamma^*_{mH}[\Gamma^*_{mH}(A)]$ *if and only if* $(X-A)^\star_{mH} = [(X-A)^\star_{mH}]^\star_{mH}$.
(5) $\Gamma^*_{mH}(A) \subset \Gamma^*_{mH}(\Gamma^*_{mH}(A))$.

Proof. (1) This follows from Lemma 2.9 (2) and Theorem 3.2.
(2) This follows from Lemma 2.9 (1).
(3) This is obvious by (2).
(4) This follows from the facts:
(i) $\Gamma^*_{mH}(A) = X - (X-A)^\star_{mH}$.
(ii) $\Gamma^*_{mH}[\Gamma^*_{mH}(A)] = X - [X - (X - (X-A)^\star_{mH})]^\star_{mH}$
$= X - [(X-A)^\star_{mH}]^\star_{mH}$.
(5) By Lemma 2.9 and the above fact, $((X-A)^\star_{mH})^\star_{mH} \subset (X-A)^\star_{mH}$ and $\Gamma^*_{mH}(A) = X-(X-A)^\star_{mH} \subset X - ((X-A)^\star_{mH})^\star_{mH} = \Gamma^*_{mH}(\Gamma^*_{mH}(A))$. □

The following example due to Renukadevi and Vimaladevi [14] shows that the inequality in Theorem 3.7(5) will not be an equality.

Example 3.8. Let $X = \{a,b,c,d\}$, $m = \{\emptyset, X, \{a\}, \{a,b\}, \{b,c\}, \{a,b,c\}\}$ and $\mathcal{H} = \{\emptyset, \{b\}, \{c\}\}$. Then $(X, m, \mathcal{H})$ is a hereditary m-space. Let $A = \{a,d\}$, then $\Gamma^\star_{mH}(A) = \{a,b\}$ and $\Gamma^\star_{mH}(\Gamma^\star_{mH}(A)) = \Gamma^\star_{mH}(\{a,b\}) = \{a,b,c\}$. Therefore, $\Gamma^*_{mH}(A) \neq \Gamma^*_{mH}(\Gamma^*_{mH}(A))$.

Lemma 3.9. *Let $(X, m, \mathcal{I})$ be an ideal m-space and A, B any subsets of X. If m has property $[F]$, then $A^*_{mH} \cup B^*_{mH} = (A \cup B)^*_{mH}$.*

Proof. It follows from Lemma 2.9 that $(A\cup B)^*_{mH} \supseteq A^*_{mH} \cup B^*_{mH}$. To prove the reverse inclusion, let $x \notin A^*_{mH} \cup B^*_{mH}$. Then x belongs neither to A^*_{mH} nor to B^*_{mH}. Therefore, there exist U_x, $V_x \in m(x)$ such that $U_x \cap A \in \mathcal{I}$ and

$V_x \cap B \in \mathcal{I}$. Since $\mathcal{I}$ is additive, $U_x \cap A) \cup (V_x \cap B) \in \mathcal{I}$. Moreover, since $\mathcal{I}$ is hereditary and

$$\begin{aligned}(U_x \cap V_x) \cap (A \cup B) =&((U_x \cap V_x) \cap A) \cup ((U_x \cap V_x) \cap B)\\ \subseteq&(U_x \cap A) \cup (V_x \cap B),\end{aligned}$$

$(U_x \cap V_x) \cap (A \cup B) \in \mathcal{I}$. Since $U_x \cap V_x \in m(x)$, $x \notin (A \cup B)^*_{mH}$. This shows that $(A \cup B)^*_{mH} \subseteq A^*_{mH} \cup B^*_{mH}$. Hence, we obtain $A^*_{mH} \cup B^*_{mH} = (A \cup B)^*_{mH}$. □

LEMMA 3.10. *Let $(X, m, \mathcal{I})$ be an ideal m-space. If m has property [F] and A, B are subsets of X, then $A^*_{mH} - B^*_{mH} = (A - B)^*_{mH} - B^*_{mH}$.*

Proof. We have, by Lemma 3.9, $A^*_{mH} = [(A - B) \cup (A \cap B)]^*_{mH} = (A - B)^*_{mH} \cup (A \cap B)^*_{mH} \subseteq (A - B)^*_{mH} \cup B^*_{mH}$. Thus, $A^*_{mH} - B^*_{mH} \subseteq (A - B)^*_{mH} - B^*_{mH}$. By Lemma 2.9, $(A - B)^*_{mH} \subseteq A^*_{mH}$ and hence $(A - B)^*_{mH} - B^*_{mH} \subseteq A^*_{mH} - B^*_{mH}$. Hence, $A^*_{mH} - B^*_{mH} = (A - B)^*_{mH} - B^*_{mH}$. □

COROLLARY 3.11. *Let $(X, m, \mathcal{I})$ be an ideal minimal space. If m has property [F] and A, B are subsets of X with $B \in \mathcal{I}$, then $(A \cup B)^*_{mH} = A^*_{mH} = (A - B)^*_{mH}$.*

Proof. Since $B \in \mathcal{I}$, by Lemma 2.9, $B^*_{mH} = \emptyset$. By Lemma 3.10, $A^*_{mH} = (A - B)^*_{mH}$ and, by Lemma 3.9, $(A \cup B)^*_{mH} = A^*_{mH} \cup B^*_{mH} = A^*_{mH}$ □

THEOREM 3.12. *Let $(X, m, \mathcal{H})$ be a hereditary m-space and $A \subseteq X$. Then the following properties hold:*

(1) $\Gamma^*_{mH}(A) = \cup\{U \in m : U - A \in \mathcal{H}\}$.
(2) $\Gamma^*_{mH}(A) \supseteq \cup\{U \in m : (U - A) \cup (A - U) \in \mathcal{H}\}$.

Proof. (1) This follows immediately from the definition of Γ^*_{mH}-operator.

(2) Since $\mathcal{H}$ is heredity, it is obvious that $\cup\{U \in m_X : (U - A) \cup (A - U) \in \mathcal{H}\} \subseteq \cup\{U \in m : U - A \in \mathcal{H}\} = \Gamma^*_{mH}(A)$ for every $A \subseteq X$. □

THEOREM 3.13. *Let $(X, m, \mathcal{H})$ be a hereditary m-space and $\sigma = \{A \subseteq X : A \subseteq \Gamma^*_{mH}(A)\}$. Then the following properties hold:*

(1) σ is a minimal structure with property $\mathcal{B}$,
(2) If $\mathcal{H}$ is an ideal and m has property [F], then σ is a topology for X.

Proof. (1) By Lemma 3.5, $\sigma = m^\star_H$. It is known from [12, Theorem 2.1] that $m^\star_H$ is a minimal structure with property $\mathcal{B}$.

(2) Let $A, B \in \sigma$, then $A \cap B \subseteq \Gamma^*_{mH}(A) \cap \Gamma^*_{mH}(B)$. By Lemma 3.9, we have

$$\begin{aligned}\Gamma^*_{mH}(A \cap B) =&X - (X - (A \cap B))^*_{mH} = X - [(X - A) \cup (X - B)]^*_{mH}\\ =&X - [(X - A)^*_{mH} \cup (X - B)^*_{mH}]\\ =&[X - (X - A)^*_{mH}] \cap [X - (X - B)^*_{mH}]\\ =&\Gamma^*_{mH}(A) \cap \Gamma^*_{mH}(B).\end{aligned}$$

$\Gamma^*_{mH}(A) \cap \Gamma^*_{mH}(B) = \Gamma^*_{mH}(A \cap B)$. Therefore, $A \cap B \subseteq \Gamma^*_{mH}(A \cap B)$ and hence $A \cap B \in \sigma$. This shows that σ is a topology. □

4. COMPATIBLITY OF M WITH $\mathcal{H}$

DEFINITION 4.1. Let $(X, m, \mathcal{H})$ be a hereditary m-space. We say that m is m-compatible with a hereditary class $\mathcal{H}$, denoted $m \widetilde{\sim} \mathcal{H}$, if the following holds: for every $A \subseteq X$, $A \in \mathcal{H}$ whenever for each $x \in A$ there exists $U \in m(x)$ such that $U \cap A \in \mathcal{H}$.

THEOREM 4.2. *Let $(X, m, \mathcal{H})$ be a hereditary m-space. Then the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) hold. If m has property $[F]$ and $\mathcal{H}$ is an ideal, then the following properties are equivalent:*

1. *For every $A \subseteq X$, if A contains no nonempty subset B with $B \subseteq B^*_{mH}$, then $A \in \mathcal{H}$;*
2. *$m \widetilde{\sim} \mathcal{H}$;*
3. *If a subset A of X has a cover of m-open sets whose intersection with A is in $\mathcal{H}$, then $A \in \mathcal{H}$;*
4. *For every $A \subseteq X$, $A \cap A^*_{mH} = \emptyset$ implies that $A \in \mathcal{H}$;*
5. *For every $A \subseteq X$, $A - A^*_{mH} \in \mathcal{H}$.*

Proof. (1) $\Rightarrow$ (2): Let $A \subseteq X$ and assume that for every $x \in A$, there exists $U \in m(x)$ such that $U \cap A \in \mathcal{H}$. Then $A \cap A^*_{mH} = \emptyset$. Suppose that A contains B such that $B \subseteq B^*_{mH}$. Then $B = B \cap B^*_{mH} \subseteq A \cap A^*_{mH} = \emptyset$. Therefore, A contains no nonempty subset B with $B \subseteq B^*_{mH}$. Hence, $A \in \mathcal{H}$.

(2) $\Rightarrow$ (3): The proof is obvious.

(3) $\Rightarrow$ (4): Let $A \subseteq X$ and $x \in A$. Then $x \notin A^*_{mH}$ and there exists $V_x \in m(x)$ such that $V_x \cap A \in \mathcal{H}$. Therefore, we have $A \subseteq \cup\{V_x : x \in A\}$ and $V_x \in m(x)$ and by (3) $A \in \mathcal{H}$.

(4) $\Rightarrow$ (5): For any $A \subseteq X$, $A - A^*_{mH} \subseteq A$ and $(A - A^*_{mH}) \cap (A - A^*_{mH})^*_{mH} \subseteq (A - A^*_{mH}) \cap A^*_{mH} = \emptyset$. By (4), $A - A^*_{mH} \in \mathcal{H}$.

(5) $\Rightarrow$ (1): By (5), for every $A \subseteq X$, $A - A^*_{mH} \in \mathcal{H}$. Let $A - A^*_{mH} = J \in \mathcal{H}$, then $A = J \cup (A \cap A^*_{mH})$ and, by lemma 3.9 and Lemma 2.9, $A^*_{mH} = J^*_{mH} \cup (A \cap A^*_{mH})^*_{mH} = (A \cap A^*_{mH})^*_{mH}$. Therefore, we have $A \cap A^*_{mH} = A \cap (A \cap A^*_{mH})^*_{mH} \subseteq (A \cap A^*_{mH})^*_{mH}$ and $A \cap A^*_{mH} \subseteq A$. By the assumption, $A \cap A^*_{mH} = \emptyset$ and hence $A = A - A^*_{mH} \in \mathcal{H}$. □

COROLLARY 4.3. *Let $(X, m, \mathcal{H})$ be a hereditary m-space and $m \widetilde{\sim} \mathcal{H}$. If $A \cap A^*_{mH} = \emptyset$ for $A \subseteq X$, then $A^*_{mH} = \emptyset$.*

THEOREM 4.4. *Let $(X, m, \mathcal{H})$ be a hereditary m-space. Then $m \widetilde{\sim} \mathcal{H}$ if and only if $\Gamma^*_{mH}(A) - A \in \mathcal{H}$ for every $A \subseteq X$.*

Proof. Necessity. Assume $m \widetilde{\sim} \mathcal{H}$ and let $A \subseteq X$. Observe that $x \in \Gamma^*_{mH}(A) - A$ if and only if $x \notin A$ and $x \notin (X - A)^*_{mH}$ if and only if $x \notin A$ and there exists $U_x \in m(x)$ such that $U_x - A \in \mathcal{H}$ if and only if there exists $U_x \in m(x)$ such that $x \in U_x - A \in \mathcal{H}$. Now, for each $x \in \Gamma^*_{mH}(A) - A$ and $U_x \in m(x)$, $U_x \cap (\Gamma^*_{mH}(A) - A) \in \mathcal{H}$, by heredity, and hence $\Gamma^*_{mH}(A) - A \in \mathcal{H}$, by the assumption that $m \widetilde{\sim} \mathcal{H}$.

Sufficiency. Let $A \subseteq X$ and assume that for each $x \in A$ there exists $U_x \in m(x)$ such that $U_x \cap A \in \mathcal{H}$. Observe that $\Gamma^*_{mH}(X - A) - (X - A) = A - A^*_{mH} = \{x : \text{there exists } U_x \in m(x) \text{ such that } x \in U_x \cap A \in \mathcal{H}\}$. Thus we have $A \subseteq \Gamma^*_{mH}(X - A) - (X - A) \in \mathcal{H}$ and hence $A \in \mathcal{H}$ by heredity of $\mathcal{H}$. □

PROPOSITION 4.5. *Let $(X, m, \mathcal{H})$ be a hereditary m-space with $m \approx \mathcal{H}$, $A \subseteq X$. If N is a nonempty m-open subset of $A^*_{mH} \cap \Gamma^*_{mH}(A)$, then $N - A \in \mathcal{H}$ and $N \cap A \notin \mathcal{H}$.*

Proof. If $N \subseteq A^*_{mH} \cap \Gamma^*_{mH}(A)$, then $N - A \subseteq \Gamma^*_{mH}(A) - A \in \mathcal{H}$, by Theorem 4.4, and hence $N - A \in \mathcal{H}$, by heredity. Since $N \in m - \{\emptyset\}$ and $N \subseteq A^*_{mH}$, we have $N \cap A \notin \mathcal{H}$, by the definition of A^*_{mH}. □

5. STRONGLY *M*-CODENSE HEREDITARY CLASSES

DEFINITION 5.1. Let $(X, m, \mathcal{H})$ be a hereditary m-space. The hereditary class $\mathcal{H}$ is said to be

(1) *m-codense* if $m \cap \mathcal{H} = \{\emptyset\}$,

(2) *strongly m-codense* if $U, V \in m$ and $U \cap V \in \mathcal{H}$ implies $U \cap V = \emptyset$,

(3) $(\star)$*-strongly m-codense* if for $U, V \in m$, $(U \cap V) \cap A \in \mathcal{H}$ and $(U \cap V) - A \in \mathcal{H}$ implies $U \cap V = \emptyset$ for every subset A of X.

LEMMA 5.2. *Let $(X, m, \mathcal{H})$ be a hereditary m-space. Then, for the hereditary class $\mathcal{H}$, the following properties hold:*

(1) *If $\mathcal{H}$ is $(\star)$-strongly m-codense, then it is strongly m-codense,*

(2) *If $\mathcal{H}$ is strongly m-codense, then it is m-codense.*

Proof. (1) If $U, V \in m$ and $U \cap V \in \mathcal{H}$, then $(U \cap V) \cap \emptyset \in \mathcal{H}$ and $(U \cap V) - \emptyset \in \mathcal{H}$ and, by hypothesis, $U \cap V = \emptyset$.

(2) Let $\mathcal{H}$ be strongly m-codense. Suppose that $m \cap \mathcal{H} \neq \{\emptyset\}$. There exists $U \in m \cap \mathcal{H}$ such that $U \neq \emptyset$. Since $x \in U \in m$ and $U \in \mathcal{H}$, for any $V \in m(x), V \cap U \subset U \in \mathcal{H}$ and $V \cap U \in \mathcal{H}$. Since $x \in U \cap V$, this is contrary to the hypothesis. □

REMARK 5.3. The following example due to Kim and Min [9] shows that the converse of (1) in Lemma 5.2 is not always true. And also [12, Example 2.1] shows that the converse of (2) in Lemma 5.2 is not always true.

EXAMPLE 5.4. (1) Let $X = \{a, b, c\}, m = \{\emptyset, \{a, c\}, X\}$ and $\mathcal{H} = \{\emptyset, \{a\}, \{c\}\}$. Then $\mathcal{H}$ is strongly m-codense. Let $U = \{a, c\}$ and $V = X$. Then for $A = \{b, c\}$, $(U \cap V) \cap A = \{c\} \in \mathcal{H}$ and $(U \cap V) - A = \{a\} \in \mathcal{H}$ but $U \cap V = U \neq \emptyset$. Hence, $\mathcal{H}$ is not $(\star)$-strongly m-codense.

(2) Let $X = \{a, b, c\}, m = \{\emptyset, \{a, b\}, \{a, c\}, X\}$ and $\mathcal{H} = \{\emptyset, \{a\}\}$. Then $\mathcal{H}$ is m-codense. Let $U = \{a, b\}$ and $V = \{a, c\}$, then $U \cap V = \{a\} \in \mathcal{H}$ and $\mathcal{H}$ is not strongly m-codense.

THEOREM 5.5. *Let $(X, m, \mathcal{H})$ be a hereditary m-space. Then, the following properties hold:*

(1) *If $\mathcal{H}$ is an ideal and strongly m-codense, then it is $(\star)$-strongly m-codense,*

(2) *If m has property $[F]$ and $\mathcal{H}$ is m-codense, then $\mathcal{H}$ is strongly m-codense.*

Proof. (1) Let $U, V \in m$ and $(U \cap V) \cap A \in \mathcal{H}$ and $(U \cap V) - A \in \mathcal{H}$. Then, since $\mathcal{H}$ is an ideal, $U \cap V = ((U \cap V) \cap A) \cup ((U \cap V) - A) \in \mathcal{H}$. Since $\mathcal{H}$ is strongly m-codense, $U \cap V = \emptyset$ and hence $\mathcal{H}$ is $(\star)$-strongly m-codense.

(2) Let $U, V \in m$ and $(U \cap V) \in \mathcal{H}$. Since m has property $[F]$, $U \cap V \in m$ and $(U \cap V) \in m \cap \mathcal{H}$. Hence $U \cap V = \emptyset$. □

Corollary 5.6. *Let $(X, m, \mathcal{H})$ be an ideal m-space and m have property $[F]$. Then the following properties are equivalent:* (1) *m-codense,* (2) *strongly m-codense and* (3) *$(\star)$-strongly m-codense.*

Lemma 5.7. *Let $(X, m, \mathcal{H})$ be a hereditary m-space. Then the following properties are equivalent:*

(1) *$\mathcal{H}$ is m-codense;*

(2) *$X = X^*_{mH}$;*

(3) *$\Gamma^*_{mH}(\emptyset) = \emptyset$.*

Proof. (1) ⇒ (2): Suppose that $\mathcal{H}$ is m-codense. For any point $x \in X$ and any $U \in m(x)$, $U \cap X = U \notin \mathcal{H}$. Hence, $x \in X^\star_{mH}$ and $X \subset X^\star_{mH}$. Therefore, $X = X^*_{mH}$.

(2) ⇒ (3): Since $\Gamma^*_{mH}(\emptyset) = X - X^*_{mH}$, $\Gamma^*_{mH}(\emptyset) = \emptyset$.

(3) ⇒ (1): Suppose that $m \cap \mathcal{H} \neq \{\emptyset\}$. Then there exists $U \in m \cap \mathcal{H}$ such that $U \neq \emptyset$. There exists $x \in U \in m$ and $U \cap X = U \in \mathcal{H}$ and hence $x \notin X^\star_{mH}$. Hence, $x \in X - X^\star_{mH} = \Gamma^*_{mH}(\{\emptyset\})$. This is contrary to $\Gamma^*_{mH}(\emptyset) = \emptyset$. □

Lemma 5.8. *Let $(X, m, \mathcal{H})$ be a hereditary m-space and m have property $\mathcal{B}$. Then $\mathcal{H}$ is m-codense if and only if $Int(H) = \emptyset$ for every $H \in \mathcal{H}$.*

Proof. Let $\mathcal{H}$ be m-codense. Since m has property $\mathcal{B}$, for every $H \in \mathcal{H}$, $Int(H) \in m$. Since $Int(H) \subset H \in \mathcal{H}$, $Int(H) \in \mathcal{H}$ and hence $Int(H) \in m \cap \mathcal{H} = \{\emptyset\}$. Therefore, $Int(H) = \emptyset$.

Conversely, suppose that $\mathcal{H}$ is not m-codense. Then there exists $U \in m \cap \mathcal{H}$ such that $U \neq \emptyset$. Since $U \in m$, $Int(U) = U \neq \emptyset$ for $U \in \mathcal{H}$. □

Theorem 5.9. *Let $(X, m, \mathcal{H})$ be a hereditary m-space and m have property $[F]$. The following properties are equivalent:*

(1) *$\mathcal{H}$ is m-codense;*

(2) *If A is m-closed, then $\Gamma^\star_{mH}(A) - A = \emptyset$;*

(3) *If U is m-open, then $U \subset U^\star_{mH}$;*

(4) *$\mathcal{H}$ is strongly m-codense.*

Proof. (1) ⇒ (2): Suppose that A is m-closed and $x \in \Gamma^\star_{mH}(A) - A$. Then $x \in X - (X - A)^\star_{mH}$ and hence $x \notin (X - A)^\star_{mH}$. There exists $U \in m(x)$ such that $U \cap (X - A) \in \mathcal{H}$. Since A is m-closed, $U \cap (X - A) \in m$ and hence $U \cap (X - A) \in m \cap \mathcal{H} = \{\emptyset\}$. This is contrary to $x \in U \cap (X - A)$. Hence, $\Gamma^\star_{mH}(A) - A = \emptyset$.

(2) $\Rightarrow$ (3): Let $U \in m$. Then $X - U$ is m-closed and by (2) $\emptyset = \Gamma^{\star}_{mH}(X - U) - (X - U) = (X - U^{\star}_{mH}) \cap U = U - U^{\star}_{mH}$. Hence $U \subset U^{\star}_{mH}$.

(3) $\Rightarrow$ (4): Suppose that $U, V \in m$ and $(U \cap V) \in \mathcal{H}$. If $x \in U \cap V$, then by (3) $x \in U \subset U^{\star}_{mH}$ and hence $V \cap U \notin \mathcal{H}$. This is a contradiction.

(4) $\Rightarrow$ (1): This follows from Lemma 5.2 (2). □

THEOREM 5.10. *Let $(X, m, \mathcal{H})$ be a hereditary m-space. If $\mathcal{H}$ is $(\star)$-strongly m-codense, then for a subset A of X the following properties hold:*

(1) $\Gamma^{*}_{mH}(A) \subseteq A^{*}_{mH} \subseteq mCl(A)$,

(2) $\Gamma^{*}_{mH}(A) \cap \Gamma^{*}_{mH}(X - A) = \emptyset$.

Proof. (1) Suppose there exists an element $x \in \Gamma^{*}_{mH}(A)$ such that $x \notin A^{*}_{mH}$. For $x \in \Gamma^{*}_{mH}(A)$, since $x \notin (X - A)^{*}_{mH}$, there exists $U \in m(x)$ such that $U \cap (X - A) \in \mathcal{H}$. For $x \notin A^{*}_{mH}$, there exists $V \in m(x)$ such that $V \cap A \in \mathcal{H}$. Since $\mathcal{H}$ is $(\star)$-strongly m-codense, for $U, V \in m$, $(U \cap V) \cap A \in \mathcal{H}$ and $(U \cap V) - A \in \mathcal{H}$ implies $U \cap V = \emptyset$. But this contradicts the fact that both U and V are containing x. Hence, we have $\Gamma^{*}_{mH}(A) \subseteq A^{*}_{mH}$. It follows from Lemma 2.9(2) that $A^{*}_{mH} \subseteq mCl(A)$.

(2) Assume that $z \in \Gamma^{*}_{mH}(A) \cap \Gamma^{*}_{mH}(X - A)$ for some $z \in X$. Then there exist $U, V \in m(z)$ such that $U \cap A \in \mathcal{H}$ and $V \cap (X - A) \in \mathcal{H}$. Hence, $(U \cap V) - A \in \mathcal{H}$ and $(U \cap V) \cap A \in \mathcal{H}$. Since $\mathcal{H}$ is $(\star)$-strongly m-codense, for $U, V \in m$, $(U \cap V) \cap A \in \mathcal{H}$ and $(U \cap V) - A \in \mathcal{H}$ implies $U \cap V = \emptyset$ and we have $U \cap V = \emptyset$. This is contrary to $z \in U \cap V$. Hence, $\Gamma^{*}_{mH}(A) \cap \Gamma^{*}_{mH}(X - A) = \emptyset$. □

COROLLARY 5.11. *Let $(X, m, \mathcal{H})$ be a hereditary m-space. If $\mathcal{H}$ is $(\star)$-strongly m-codense, then for a subset A of X the following properties hold:*

(1) $\Gamma^{*}_{mH}(A) \subseteq A^{*}_{mH} \subseteq A$ *if A is m-closed in X,*

(2) $A^{*}_{mH} \cup (X - A)^{*}_{mH} = X$,

(3) *If $A \in \mathcal{H}$, then* $\Gamma^{*}_{mH}(A) = \emptyset$,

(4) *If $X - A \in \mathcal{H}$, then* $A^{*}_{mH} = X$.

Proof. (1) This is obvious by Theorem 5.10(1).

(2) By Theorem 5.10 (2), $\emptyset = \Gamma^{*}_{mH}(A) \cap \Gamma^{*}_{mH}(X - A) = [X - (X - A)^{\star}_{mH}] \cap (X - A^{\star}_{mH}) = X - [(X - A)^{\star}_{mH} \cup A^{\star}_{mH}]$. Hence $A^{*}_{mH} \cup (X - A)^{*}_{mH} = X$.

(3) By Theorem 5.10 (1), $\Gamma^{*}_{mH}(A) \subseteq A^{*}_{mH}$. Since $A \in \mathcal{H}$, by Lemma 2.9 (5), $A^{\star}_{mH} = \emptyset$ and hence $\Gamma^{*}_{mH}(A) = \emptyset$.

(4) If $X - A \in \mathcal{H}$, by (3), $\emptyset = \Gamma^{*}_{mH}(X - A) = X - A^{\star}_{mH}$. Hence, $A^{*}_{mH} = X$. □

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