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WIRTINGER TYPE INEQUALITIES FOR CONFORMABLE FRACTIONAL INTEGRALS

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Abstract. The aim of this paper is to establish a generalization and a refinement of Wirtinger's inequality for conformable fractional integrals.

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1. INTRODUCTION

One of the most important issues in inequality theory is given by the integral inequalities involving a function and its derivative. Wirtinger's inequality has been attracting attention due to the close coupling of linear differential equations and differential geometry. Wirtinger's inequality compares the integral of a square of a function with that of a square of its first derivative. Over the last twenty years, a large number of papers considered simpler proofs, various generalizations and discrete analogues of Wirtinger's inequality and its generalizations, see [2, 4, 5] and [10-14].

First of all, we recall the following inequality ascribed to Wirtinger.

THEOREM 1.1. *Let f be a real valued function with period 2π such that $\int_0^{2\pi} f(x)\mathrm{d}x = 0$ and $f' \in L_2[0, 2\pi]$. Then the following inequality holds*

$$\int_0^{2\pi} f^2(x)\mathrm{d}x \le \int_0^{2\pi} \left(f'(x)\right)^2 \mathrm{d}x, \tag{1}$$

with equality if and only if $f(x) = A\cos x + B\sin x,\ A, B \in \mathbb{R}$.

Beesack obtained in [10] the following generalization of Wirtinger's inequality: if $p > 1$, $f' \in C\left[0, \frac{\pi}{2}\right]$, $f(0) = 0$, then

$$\int_0^{\frac{\pi}{2}} |f(x)|^p \,\mathrm{d}x \le \frac{1}{p-1}\left(\frac{p}{2\sin(\frac{\pi}{p})}\right)\int_0^{\frac{\pi}{2}} |f'(x)|^p \,\mathrm{d}x.$$

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The purpose of this paper is to establish a generalization and a refinement of Wirtinger's inequality for conformable integrals. The structure of this paper is as follows. In Section 2, we present the definitions for conformable derivatives and conformable integral and we introduce several useful notations for our main result. In Section 3, the main result is given.

2. DEFINITIONS AND PROPERTIES OF CONFORMABLE FRACTIONAL DERIVATIVE AND INTEGRAL

The following definitions and theorems regarding conformable fractional derivatives and integrals have been considered in [1, 3], [6]-[9].

Definition 2.1 (Conformable fractional derivative). Given a function $f : [0,\infty) \to \mathbb{R}$. Then the *conformable fractional derivative* of f of order α is defined by

$$D_\alpha (f)(t) = \lim_{\varepsilon\to 0} \frac{f\left(t+\varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}, \tag{2}$$

for all $t > 0$, $\alpha \in (0,1)$. If f is α-differentiable in some $(0,a)$, $\alpha > 0$, and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then define

$$f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t). \tag{3}$$

Let $f^{(\alpha)}(t)$ for $D_\alpha(f)(t)$ denote the conformable fractional derivatives of f of order α. In addition, if the conformable fractional derivative of f of order α exists, then we simply say that f is α-differentiable.

Theorem 2.2. *Let $\alpha \in (0,1]$ and f, g be α-differentiable at a point $t > 0$. Then:*

i) $D_\alpha(af+bg) = aD_\alpha(f) + bD_\alpha(g)$, *for all $a, b \in \mathbb{R}$;*
ii) $D_\alpha(\lambda) = 0$, *for all constant functions $f(t) = \lambda$;*
iii) $D_\alpha(fg) = fD_\alpha(g) + gD_\alpha(f)$;
iv) $D_\alpha\left(\frac{f}{g}\right) = \frac{fD_\alpha(g) - gD_\alpha(f)}{g^2}$.

If f is differentiable, then

$$D_\alpha(f)(t) = t^{1-\alpha}\frac{\mathrm{d}f}{\mathrm{d}t}(t). \tag{4}$$

Definition 2.3 (Conformable fractional integral). Let $\alpha \in (0,1]$ and $0 \le a < b$. A function $f : [a,b] \to \mathbb{R}$ is *α-fractional integrable* on $[a,b]$, if the integral

$$\int_a^b f(x)\,\mathrm{d}_\alpha x := \int_a^b f(x)\,x^{\alpha-1}\mathrm{d}x \tag{5}$$

exists and is finite. The set of all α-fractional integrable functions on $[a,b]$ is denoted by $L^1_\alpha([a,b])$.

REMARK 2.4.
$$I_\alpha^a(f)(t) = I_1^a\left(t^{\alpha-1}f\right) = \int_a^t \frac{f(x)}{x^{1-\alpha}}\mathrm{d}x,$$
where the integral is the usual Riemann improper integral, and $\alpha \in (0,1]$.

THEOREM 2.5. *Let $f:(a,b)\to\mathbb{R}$ be differentiable and $0<\alpha\le 1$. Then, for all $t>a$, we have*
$$I_\alpha^a D_\alpha f(t) = f(t) - f(a). \tag{6}$$

THEOREM 2.6 (Integration by parts). *Let $f,g:[a,b]\to\mathbb{R}$ be two functions such that fg is differentiable. Then*
$$\int_a^b f(x) D_\alpha(g)(x)\,\mathrm{d}_\alpha x = fg|_a^b - \int_a^b g(x) D_\alpha(f)(x)\,\mathrm{d}_\alpha x. \tag{7}$$

THEOREM 2.7. *Assume that $f:[a,\infty)\to\mathbb{R}$ such that $f^{(\alpha)}(t)$ is continuous and $\alpha\in(n,n+1]$. Then, for all $t>a$, we have*
$$D_\alpha f(t)\, I_\alpha^a = f(t).$$

We can give the Hölder's inequality with conformable integrals as follows.

LEMMA 2.8. *If $f,g\in C[a,b]$, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then*
$$\int_a^b |f(x)g(x)|\,\mathrm{d}_\alpha x \le \left(\int_a^b |f(x)|^p\,\mathrm{d}_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q\,\mathrm{d}_\alpha x\right)^{\frac{1}{q}}.$$

REMARK 2.9. If we take $p=q=2$ in Lemma 2.8 , the we have the Cauchy-Schwartz inequality with conformable integrals.

3. WIRTINGER TYPE INEQUALITIES FOR CONFORMABLE FRACTIONAL INTEGRAL

Now, we present the main results.

THEOREM 3.1. *Let $\alpha\in(0,1]$, $u:[0,h]\to\mathbb{R}$ be an α- fractional differentiable function, $p\ge 1$, $u(0)=0$. Then we have the following inequality*
$$\int_0^h |u(t)|^p\,\mathrm{d}_\alpha t \le \frac{1}{p}\left(\frac{h^\alpha}{\alpha}\right)^p \int_0^h |D_\alpha u(t)|^p\,\mathrm{d}_\alpha t.$$

Proof. From Hölder's inequality with indices p and $\frac{p}{p-1}$, and using the partial integration method, we get
$$\int_0^h |u(x)|^p\,\mathrm{d}_\alpha x \quad\le\quad \int_0^h \left(\int_0^x |D_\alpha u(t)|\,d_\alpha t\right)^p \mathrm{d}_\alpha x$$

$$
\begin{aligned}
&\leq \int_0^h \left(\frac{x^\alpha}{\alpha}\right)^{p-1} \left(\int_0^x |D_\alpha u(t)|^p \, d_\alpha t\right) \mathrm{d}_\alpha x \\
&= \frac{h^{\alpha p}}{p\alpha^p} \int_0^h |D_\alpha u(t)|^p \, \mathrm{d}_\alpha t - \frac{1}{p\alpha^p} \int_0^h x^{\alpha p} |D_\alpha u(x)|^p \, \mathrm{d}_\alpha x \\
&\leq \frac{h^{\alpha p}}{p\alpha^p} \int_0^h |D_\alpha u(t)|^p \, \mathrm{d}_\alpha t,
\end{aligned}
$$

which is the desired inequality. □

Corollary 3.2. *Under assumption of Theorem 3.1 with* $p = 2$, *we get*

$$\int_0^h |u(t)|^2 \, \mathrm{d}_\alpha t \leq \frac{h^{2\alpha}}{2\alpha^2} \int_0^h |D_\alpha u(t)|^2 \, \mathrm{d}_\alpha t.$$

Theorem 3.3. *Let* $\alpha \in (0, 1]$, $u : [0, h] \to \mathbb{R}$ *be an* α*-fractional differentiable function with* $u(0) = u(h) = 0$, $p \geq 1$. *Further, let* $g(x)$ *be a non-negative and continuous function on* $[0, h]$. *Then the following inequality holds:*

$$
\begin{aligned}
&\int_0^h g(x) \, |u(x)|^p \, \mathrm{d}_\alpha x \leq \frac{1}{2} \left[\left(\int_0^h \left(\frac{x^\alpha (h^\alpha - x^\alpha)}{\alpha} \right)^{\frac{p-1}{2}} g(x) \mathrm{d}_\alpha x \right) \right] \\
&\times \left(\int_0^h |D_\alpha u(x)|^p \, \mathrm{d}_\alpha x \right).
\end{aligned} \tag{8}
$$

Proof. We have

$$u(x) = \int_0^x D_\alpha u(t) \mathrm{d}_\alpha t, \; u(x) = -\int_x^h D_\alpha u(t) \mathrm{d}_\alpha t \tag{9}$$

and hence, from Hölder's inequality with indices p and $\frac{p}{p-1}$, it follows that

$$|u(x)|^{\frac{p}{2}} \leq \left[\left(\int_0^x |D_\alpha u(t)| \, \mathrm{d}_\alpha t \right)^p \right]^{\frac{1}{2}} \leq \left(\frac{x^\alpha}{\alpha} \right)^{\frac{p-1}{2}} \left(\int_0^x |D_\alpha u(t)|^p \, \mathrm{d}_\alpha t \right)^{\frac{1}{2}} \tag{10}$$

and similarly

$$|u(x)|^{\frac{p}{2}} \leq \left(\frac{h^\alpha - x^\alpha}{\alpha} \right)^{\frac{p-1}{2}} \left(\int_x^h |D_\alpha u(t)|^p \, \mathrm{d}_\alpha t \right)^{\frac{1}{2}}. \tag{11}$$

Now, multiplying (10) and (11) and using the elementary inequality $\sqrt{mn} \leq \frac{1}{2}(m+n)$, $m, n \geq 0$, we get

$$\begin{aligned} |u(x)|^p &\leq \frac{1}{2}\left(\frac{x^\alpha(h^\alpha - x^\alpha)}{\alpha}\right)^{\frac{p-1}{2}} \left[\int_0^x |D_\alpha u(t)|^p \,\mathrm{d}_\alpha t + \int_x^h |D_\alpha u(t)|^p \,\mathrm{d}_\alpha t\right] \\ &= \frac{1}{2}\left(\frac{x^\alpha(h^\alpha - x^\alpha)}{\alpha}\right)^{\frac{p-1}{2}} \int_0^h |D_\alpha u(t)|^p \,\mathrm{d}_\alpha t, \end{aligned}$$

which is the same as (8). This completes the proof. □

Corollary 3.4. *Under the conditions of Theorem 3.3, we have the following inequality*

$$\int_0^h g(x)\,|u(x)|^p \,\mathrm{d}_\alpha x \leq \frac{1}{2}\left(\frac{h^\alpha}{2\alpha}\right)^{p-1} \left(\int_0^h g(x) d_\alpha x\right)\left(\int_0^h |D_\alpha u(x)|^p \,\mathrm{d}_\alpha x\right). \tag{12}$$

Proof. From (9), it is clear that

$$|u(x)| \leq \frac{1}{2}\int_0^h |D_\alpha u(t)| \,\mathrm{d}_\alpha t$$

and hence, from Hölder's inequality with indices p and $\frac{p}{p-1}$, we have

$$\begin{aligned} |u(x)|^p &\leq \frac{1}{2^p}\left(\int_0^h |D_\alpha u(t)| \,\mathrm{d}_\alpha t\right)^p \leq \frac{1}{2^p}\left(\frac{h^\alpha}{\alpha}\right)^{p-1}\left(\int_0^h |D_\alpha u(t)|^p \,\mathrm{d}_\alpha t\right) \\ &= \frac{1}{2}\left(\frac{h^\alpha}{2\alpha}\right)^{p-1}\left(\int_0^h |D_\alpha u(t)|^p \,\mathrm{d}_\alpha t\right). \end{aligned}$$

Now multiplying both sides of the above inequality by $g(x)$ and integrating the resulted inequality from 0 to h, we obtain (12). □

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