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LOCALIZATION OF THE EIGENVALUES OF A MATRIX THROUGH ITS SPREAD

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Abstract. The spread of a given matrix A is the largest distance between its eigenvalues. We can localize the eigenvalues of the matrix A using its spread. In the present work we propose a refinement of Samuelson's inequality. Also, we give some lower and upper bounds for the multiplication of the spread of two different matrices A and B. In the particular case when $A = B$, we reobtain some known results.

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1. INTRODUCTION

We assume throughout this paper that $n \geq 3$. Let $A = (a_{ij})$ be an $n \times n$ complex matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. The *spread* of the matrix A is defined as

$$\mathrm{sp}(A) = \max_{i,j} |\lambda_i - \lambda_j|.$$

It was introduced for the first time by L. Mirsky – see [8]. We write $\mathrm{sp}_{\mathrm{Re}}(A) = \max_{i,j} |\mathrm{Re}(\lambda_i) - \mathrm{Re}(\lambda_j)|$ and $\mathrm{sp}_{\mathrm{Im}}(A) = \max_{i,j} |\mathrm{Im}(\lambda_i) - \mathrm{Im}(\lambda_j)|$. Let $m = \mathrm{tr}A/n$, where trA is the *trace* of A.

In [8], L. Mirsky gave an upper bound for the spread of an arbitrary $n \times n$ matrix A:

$$\mathrm{sp}(A) \leq \left\{ 2\|A\|_F^2 - \frac{2}{n}|\mathrm{tr}A|^2 \right\}^{1/2}, \tag{1}$$

where $\|A\|_F$ denotes the Frobenius norm. Also, he deduced from (1) that

$$\mathrm{sp}(A) \leq \sqrt{2}\|A\|_F. \tag{2}$$

E. Deutsch [5] and E. Jiang , X. Zhan [7] presented new proofs of inequality (1). Different bounds for the spread of a matrix are given – for more details, the reader should consult [7, 9]. We state some of these bounds. Let A be an

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$n\times n$ Hermitian matrix. E.R. Barnes and A.J. Hoffman [2] gave the following lower bound for the spread of A:

$$\sqrt{\frac{2}{n}}\left(2\|A\|_F^2-\frac{2}{n}(\operatorname{tr}(A))^2\right)^{\frac{1}{2}}\le \operatorname{sp}(A). \tag{3}$$

A lower bound, better than (3), was given by A. Brauer and A.C. Mewborn – see [3].

$$\operatorname{sp}(A)\ge\begin{cases}\sqrt{\frac{2}{n}}\left\{2\sum_{n=1}^{n}(\lambda_i)^2-\frac{2}{n}(\operatorname{tr}A)^2\right\}^{1/2}, & n \text{ even},\\ \sqrt{\frac{2n}{n^2-1}}\left\{2\sum_{n=1}^{n}(\lambda_i)^2-\frac{2}{n}(\operatorname{tr}A)^2\right\}^{1/2}, & n \text{ odd},\end{cases} \tag{4}$$

where $\lambda_i\in\mathbb{R}$ are the eigenvalues of A.

Samuelson [1] asserts that, for any real numbers $x_1,x_2,\dots,x_n$,

$$\max_i|x_i-\overline{x}|\le\sqrt{n-1}\sqrt{\frac{1}{n}\sum_{j=1}^{n}(x_j-\overline{x})^2},$$

i.e.

$$(x_i-\overline{x})^2\le\frac{n-1}{n}\sum_{j=1}^{n}(x_j-\overline{x})^2, \tag{5}$$

where $\overline{x}=\frac{\sum_{i=1}^{n}x_i}{n}$.

2. PRELIMINARY LEMMAS

LEMMA 2.1. *If $z_1,z_2,\dots,z_n$ are complex numbers such that $\sum_{i=1}^{n}z_i=0$, then*

$$|z_i|^2\le\frac{n-1}{n}\sum_{j=1}^{n}|z_j|^2, \tag{6}$$

for $i=1,2,\dots,n$.

Proof. For $i=1,\dots,n$, we have $-z_i=\sum_{j=1 j\neq i}^{n}z_j$. Applying the Cauchy-Schwarz inequality, it follows that

$$|z_i|^2\le(n-1)\sum_{j=1 j\neq i}^{n}|z_j|^2=(n-1)\sum_{j=1}^{n}|z_j|^2-(n-1)|z_i|^2.$$

Hence the result follows immediately. □

THEOREM 2.2. *Let A be an $n\times n$ complex matrix. Then*

$$\operatorname{sp}(A)\le 2\left\{\left(\frac{n-1}{n}\right)\left(\|A\|_F^2-\frac{|\operatorname{tr}A|^2}{n}\right)\right\}^{1/2}. \tag{7}$$

Proof. Taking $z_i = (\lambda_i - m)$ in (6) it follows that

$$|\lambda_i - m| \leq \sqrt{\frac{n-1}{n}\sum_{j=1}^{n}|\lambda_j - m|^2}.$$

Furthermore,

$$\begin{aligned}\sum_{i=1}^{n}|\lambda_i - m|^2 &= \sum_{i=1}^{n}\left(|\lambda_i|^2 - m\overline{\lambda_i} - \overline{m}\lambda_i + |m|^2\right)\\ &= \sum_{i=1}^{n}|\lambda_i|^2 - \frac{|trA|^2}{n} \leq \|A\|_F^2 - \frac{|trA|^2}{n}.\end{aligned}$$

Hence $|\lambda_i - m| \leq \sqrt{\frac{n-1}{n}\left(\|A\|_F^2 - \frac{|\mathrm{tr}A|^2}{n}\right)}$. We have $\mathrm{sp}(A) = \max_{i,j}|\lambda_i - \lambda_j| \leq |\lambda_i - m| + |\lambda_j - m| \leq 2|\lambda_i - m|$, and thus $\mathrm{sp}(A) \leq 2\sqrt{\frac{n-1}{n}\left(\|A\|_F^2 - \frac{|\mathrm{tr}A|^2}{n}\right)}$. □

Lemma 2.3 (Lagrange's identity). *Let $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ and $b = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$. Then*

$$\left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right) - \left(\sum_{i=1}^{n}a_ib_i\right)^2 = \frac{1}{2}\sum_{i,j=1}^{n}(a_ib_j - a_jb_i)^2.$$

In the following theorem we extend the inequality of Samuelson. Let $\Omega_i = \{1, 2, \ldots, n\} \setminus \{i\}$, where $i = 1, \ldots, n$.

Theorem 2.4. *Let $x_1, x_2, \ldots, x_n$ be real numbers and let $\overline{x} = \frac{\sum_{i=1}^{n}x_i}{n}$. Then*

$$(x_i - \overline{x})^2 + \frac{1}{2n}\sum_{j,k\in\Omega_i}(x_j - x_k)^2 = \frac{n-1}{n}\sum_{j=1}^{n}(x_j - \overline{x})^2, \tag{8}$$

for $i = 1, \ldots, n$.

Proof. For $i = 1, \ldots, n$, we have

$$(x_i - \overline{x})^2 + \frac{1}{2}\sum_{j,k\in\Omega_i}(x_j - x_k)^2 = \left(-\sum_{j\in\Omega_i}(x_j - \overline{x})\right)^2 + \frac{1}{2}\sum_{j,k\in\Omega_i}(x_j - x_k)^2.$$

On the other hand, apply the identity of Lagrange, we get

$$\frac{1}{2}\sum_{j,k\in\Omega_i}(x_j - x_k)^2 = (n-1)\sum_{j\in\Omega_i}(x_j - \overline{x})^2 - \left(\sum_{j\in\Omega_i}(x_j - \overline{x})\right)^2.$$

Then

$$(x_i - \overline{x})^2 + \frac{1}{2} \sum_{j,k \in \Omega_i} (x_j - x_k)^2 = (n-1) \sum_{j \in \Omega_i} (x_j - \overline{x})^2$$

$$= (n-1) \sum_{j=1}^{n} (x_j - \overline{x})^2 - (n-1)(x_i - \overline{x})^2.$$

Hence the desired result is obtained. □

LEMMA 2.5. *Let $z_1, z_2, \dots, z_n$ be complex numbers. Then*

$$\sum_{i=1}^{n} |z_i - m|^2 = \frac{1}{n} \sum_{1 \le i < k \le n}^{n} |z_i - z_k|^2. \tag{9}$$

Proof. We have $\sum_{1 \le i < k \le n}^{n} |z_i - z_k|^2 = n \sum_{i=1}^{n} |z_i|^2 - |trA|^2$. On the other hand, $\mathrm{tr}(A - mI) = 0$ and thus the result follows immediately. □

COROLLARY 2.6. *Let $x_1, x_2, \dots, x_n$ be real numbers and let $\overline{x} = \dfrac{\sum_{i=1}^{n} x_i}{n}$. Then*

$$(x_i - \overline{x})^2 \le \frac{n-1}{n^2} \sum_{1 \le j < k \le n}^{n} (x_j - x_k)^2, \tag{10}$$

for $i = 1, \dots, n$.

Proof. The result follows by (5) and (9). □

In the following two theorems we will see that we can localize the eigenvalues of a matrix A, using its spread.

3. MAIN RESULT

THEOREM 3.1. *Let A be an $n \times n$ matrix with real eigenvalues $\lambda_1 < \lambda_2 < \dots < \lambda_n$. Then all the eigenvalues of A lie in the interior or on the boundary of the circle with center m and radius R, where*

$$R = \frac{1}{2} \sqrt{(n-1)} \mathrm{sp}(A), \quad \text{if } n \text{ is even and}$$

$$R = \frac{n-1}{2n} \sqrt{n+1} \mathrm{sp}(A), \quad \text{if } n \text{ is odd.}$$

Proof. From (10) we have $\frac{n^2}{n-1}(\lambda_i - m)^2 \le \sum_{1 \le j < \ell \le n}^{n} (\lambda_j - \lambda_\ell)^2 = \Theta$. Let φ be an integer such that $2 \le \varphi \le n-1$. Then

$$\frac{d^2\Theta}{d\lambda_\varphi^2} = \frac{d}{d\lambda_\varphi} \left\{ 2 \sum_{j \ne \varphi} (\lambda_\varphi - \lambda_j) \right\} = 2(n-1) > 0.$$

So Θ, as function of λ_φ, attains its maximum on the boundary of the interval $[\lambda_1, \lambda_n]$. Assume $\lambda_1 = \lambda_2 = \cdots = \lambda_k$ and $\lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_n$, $1 < k < n$. Thus

$$\Theta = \sum_{1\le j<\ell\le n}^{n} (\lambda_j - \lambda_\ell)^2 = k(n-k)\mathrm{sp}^2(A).$$

Θ attains the maximum value for $k = \left[\frac{n}{2}\right]$ and thus

$$\sum_{1\le j<\ell\le n}^{n} (\lambda_j - \lambda_\ell)^2 \le \begin{cases} \frac{1}{4} n^2 \mathrm{sp}^2(A), & n \text{ even}, \\ \frac{1}{4}(n^2-1)\mathrm{sp}^2(A), & n \text{ odd}. \end{cases}$$

Hence

$$\frac{n^2}{n-1}(\lambda_i - m)^2 \le \begin{cases} \frac{1}{4} n^2 \mathrm{sp}^2(A), & n \text{ even}, \\ \frac{1}{4}(n^2-1)\mathrm{sp}^2(A), & n \text{ odd}. \end{cases}$$

Therefore,

$$(\lambda_i - m)^2 \le \begin{cases} \frac{1}{4}(n-1)\mathrm{sp}^2(A), & n \text{ even}, \\ \frac{1}{4}\frac{(n^2-1)(n-1)}{n^2}\mathrm{sp}^2(A), & n \text{ odd}. \end{cases}$$

This establishes the theorem. □

THEOREM 3.2. *Let A be an $n\times n$ normal matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then all the eigenvalues of A lie in the interior or on the boundary of the rectangle*

$$\left[\frac{\mathrm{Re}(\mathrm{tr}(A))}{n} - a, \frac{\mathrm{Re}(\mathrm{tr}(A))}{n} + a\right] \times \left[\frac{\mathrm{Im}(\mathrm{tr}(A))}{n} - b, \frac{\mathrm{Im}(\mathrm{tr}(A))}{n} + b\right], \tag{11}$$

where

$$\begin{cases} a = \frac{1}{4}(n-1)\mathrm{sp}^2_{\mathrm{Re}}(A), & n \text{ even}, \\ a = \frac{1}{4}\frac{(n^2-1)(n-1)}{n^2}\mathrm{sp}^2_{\mathrm{Re}}(A), & n \text{ odd}, \end{cases}$$

and

$$\begin{cases} b = \frac{1}{4}(n-1)\mathrm{sp}^2_{\mathrm{Im}}(A), & n \text{ even}, \\ b = \frac{1}{4}\frac{(n^2-1)(n-1)}{n^2}\mathrm{sp}^2_{\mathrm{Im}}(A), & n \text{ odd}. \end{cases}$$

Proof. Since A is normal, the eigenvalues of the Hermitian matrices $\frac{1}{2}(A + A^*)$ and $\frac{1}{2i}(A - A^*)$ are $\mathrm{Re}(\lambda_1), \ldots, \mathrm{Re}(\lambda_n)$ and $\mathrm{Im}(\lambda_1), \ldots, \mathrm{Im}(\lambda_n)$, respectively. Further, we have $\mathrm{tr}(\frac{A+A^*}{2}) = \mathrm{Re}(\mathrm{tr}(A))$ and $\mathrm{tr}(\frac{A-A^*}{2i}) = \mathrm{Im}(\mathrm{tr}(A))$. Also, we know that, if A is a normal matrix, then $\mathrm{sp}_{\mathrm{Re}}(A) = \mathrm{sp}(\frac{A+A^*}{2})$ and $\mathrm{sp}_{\mathrm{Im}}(A) = \mathrm{sp}(\frac{A-A^*}{2i})$. The real parts and imaginary parts of the complex numbers $(\lambda_i - m)$, $i = 1, 2, \ldots, n$ satisfy condition (10). Hence,

$$\left(\mathrm{Re}(\lambda_i) - \frac{\mathrm{Re}(\mathrm{tr}(A))}{n}\right)^2 \le \begin{cases} \frac{1}{4}(n-1)\mathrm{sp}^2_{\mathrm{Re}}(A), & n \text{ even}, \\ \frac{1}{4}\frac{(n^2-1)(n-1)}{n^2}\mathrm{sp}^2_{\mathrm{Re}}(A), & n \text{ odd}, \end{cases}$$

and

$$\left(\operatorname{Im}(\lambda_i) - \frac{\operatorname{Im}(\operatorname{tr}(A))}{n}\right)^2 \leq \begin{cases} \frac{1}{4}(n-1)\operatorname{sp}^2_{\operatorname{Im}}(A), & n \text{ even}, \\ \frac{1}{4}\frac{(n^2-1)(n-1)}{n^2}\operatorname{sp}^2_{\operatorname{Im}}(A), & n \text{ odd}. \end{cases}$$

We now may write the previous inequalities as

$$\left|\operatorname{Re}(\lambda_i) - \frac{\operatorname{Re}(\operatorname{tr}(A))}{n}\right| \leq a \quad \text{and} \quad \left|\operatorname{Im}(\lambda_i) - \frac{\operatorname{Im}(\operatorname{tr}(A))}{n}\right| \leq b,$$

for $i = 1, \ldots, n$. Hence the rectangle (11) contains all the eigenvalues of the matrix A. □

Theorem 3.3. *Let A and B be $n \times n$ Hermitian matrices with eigenvalues $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ and $\beta_1 < \beta_2 < \cdots < \beta_n$, respectively. Then*

$$\left(\frac{1}{n}\sum_{i=1}^{n} \alpha_i\beta_i - \frac{\operatorname{tr}(A)\operatorname{tr}(B)}{n^2}\right) \leq \frac{1}{4}\operatorname{sp}(A)\operatorname{sp}(B). \tag{12}$$

Proof. We will use the well-known identity

$$n\sum_{i=1}^{n} \alpha_i\beta_i - \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \beta_i = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_i - \alpha_j)(\beta_i - \beta_j).$$

We apply the Cauchy-Schwarz inequality to get

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_i - \alpha_j)(\beta_i - \beta_j) \leq \left[\sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_i - \alpha_j)^2\right]^{\frac{1}{2}} \left[\sum_{i=1}^{n}\sum_{j=1}^{n}(\beta_i - \beta_j)^2\right]^{\frac{1}{2}}.$$

On the other hand, we have

$$n\sum_{i=1}^{n}\alpha_i^2 - \left(\sum_{i=1}^{n}\alpha_i\right)^2 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_i - \alpha_j)^2$$

and similarly for β_i. A simple calculation shows that

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\alpha_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n}\alpha_i\right)^2 = {} & \left(\alpha_n - \frac{1}{n}\sum_{i=1}^{n}\alpha_i\right)\left(\frac{1}{n}\sum_{i=1}^{n}\alpha_i - \alpha_1\right) \\ & - \frac{1}{n}\sum_{i=1}^{n}(\alpha_i - \alpha_1)(\alpha_n - \alpha_i).\end{aligned}$$

Since $\sum_{i=1}^{n}(\alpha_i - \alpha_1)(\alpha_n - \alpha_i) \geq 0$, it follows that

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\alpha_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n}\alpha_i\right)^2 & \leq \left(\alpha_n - \frac{1}{n}\sum_{i=1}^{n}\alpha_i\right)\left(\frac{1}{n}\sum_{i=1}^{n}\alpha_i - \alpha_1\right) \\ & \leq \frac{1}{4}\left[\left(\alpha_n - \frac{1}{n}\sum_{i=1}^{n}\alpha_i\right) + \left(\frac{1}{n}\sum_{i=1}^{n}\alpha_i - \alpha_1\right)\right]^2\end{aligned}$$

$$= \frac{1}{4}\mathrm{sp}^2(A),$$

where we have used the arithmetic-geometric mean. Hence

$$\left[\sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_i-\alpha_j)^2\right]^{\frac{1}{2}} \leq \frac{n}{\sqrt{2}}\mathrm{sp}(A).$$

A similar reasoning for β_i yields

$$\left[\sum_{i=1}^{n}\sum_{j=1}^{n}(\beta_i-\beta_j)^2\right]^{\frac{1}{2}} \leq \frac{n}{\sqrt{2}}\mathrm{sp}(B).$$

Therefore,

$$n\sum_{i=1}^{n}\alpha_i\beta_i - \sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\beta_i \leq \frac{n^2}{4}\mathrm{sp}(A)\mathrm{sp}(B).$$

□

REMARK 3.4. If we take $\mathrm{sp}(A) = \mathrm{sp}(B)$ in (12) and we use the fact that if A is a Hermitian matrix, then $\sum_{i=1}^{n}\alpha_i^2 = \|A\|_F^2$, we obtain (3).

COROLLARY 3.5. *Let A be an $n\times n$ normal matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then*

$$\frac{1}{n}\sum_{i=1}^{n}\mathrm{Re}(\lambda_i)\mathrm{Im}(\lambda_i) - \frac{\mathrm{Re}(\mathrm{tr}(A))\mathrm{Im}(\mathrm{tr}(A))}{n^2} \leq \frac{1}{4}sp\left(\frac{A+A^*}{2}\right)sp\left(\frac{A-A^*}{2i}\right).$$

Proof. The eigenvalues of the Hermitian matrices $\frac{1}{2}(A+A^*)$ and $\frac{1}{2i}(A-A^*)$ are $\mathrm{Re}(\lambda_1),\ldots,\mathrm{Re}(\lambda_n)$ and $\mathrm{Im}(\lambda_1),\ldots,\mathrm{Im}(\lambda_n)$, respectively. Further, $\mathrm{tr}(\frac{A+A^*}{2}) = \mathrm{Re}(\mathrm{tr}(A))$ and $\mathrm{tr}(\frac{A-A^*}{2i}) = \mathrm{Im}(\mathrm{tr}(A))$. Hence, the desired result is obtained. □

COROLLARY 3.6. *Let A be an $n\times n$ normal matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then*

$$\frac{1}{n}\sum_{i=1}^{n}\mathrm{Re}(\lambda_i)\mathrm{Im}(\lambda_i) - \frac{1}{n}\sum_{i=1}^{n}\mathrm{Re}(\lambda_i)\frac{1}{n}\sum_{i=1}^{n}\mathrm{Im}(\lambda_i) \leq \frac{1}{4}\mathrm{sp}_{\mathrm{Re}}(A)\mathrm{sp}_{\mathrm{Im}}(A).$$

Proof. Since A is a normal matrix, $\mathrm{sp}_{\mathrm{Re}}(A) = \mathrm{sp}(\frac{A+A^*}{2})$ and $\mathrm{sp}_{\mathrm{Im}}(A) = \mathrm{sp}(\frac{A-A^*}{2i})$. Hence the result follows immediately. □

In the following theorem we provide a refinement of inequality (12).

THEOREM 3.7. *Let A and B be $n\times n$ Hermitian matrices with eigenvalues $\alpha_1<\alpha_2<\cdots<\alpha_n$ and $\beta_1<\beta_2<\cdots<\beta_n$, respectively. Then*

$$\frac{1}{n}\sum_{i=1}^{n}\alpha_i\beta_i - \frac{\mathrm{tr}(A)\mathrm{tr}(B)}{n^2} \leq \begin{cases}\frac{1}{4}\mathrm{sp}(A)\mathrm{sp}(B), & n \ \textit{even},\\ \frac{1}{4}\left(1-\frac{1}{n^2}\right)\mathrm{sp}(A)\mathrm{sp}(B), & n \ \textit{odd}.\end{cases} \tag{13}$$

Proof. We shall use the well-known identity

$$n\sum_{i=1}^{n}\alpha_i\beta_i-\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\beta_i=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_i-\alpha_j)(\beta_i-\beta_j).$$

We apply the Cauchy-Schwarz inequality to get

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_i-\alpha_j)(\beta_i-\beta_j)\le\left[\sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_i-\alpha_j)^2\right]^{\frac{1}{2}}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}(\beta_i-\beta_j)^2\right]^{\frac{1}{2}}.$$

Let $\sum_{1\le i<j\le n}(\alpha_i-\alpha_j)^2=\Theta_1$ and $\sum_{1\le i<j\le n}(\beta_i-\beta_j)^2=\Theta_2$. We have

$$\Theta_1\le\begin{cases}\frac{1}{4}n^2\mathrm{sp}^2(A), & n \text{ even},\\ \frac{1}{4}(n^2-1)\mathrm{sp}^2(A), & n \text{ odd},\end{cases}$$

and

$$\Theta_2\le\begin{cases}\frac{1}{4}n^2\mathrm{sp}^2(B), & n \text{ even},\\ \frac{1}{4}(n^2-1)\mathrm{sp}^2(B), & n \text{ odd}.\end{cases}$$

Hence the assertion now follows immediately. □

REMARK 3.8. If we take $A=B$ in (13) we obtain inequality (4), given by A. Brauer and A.C. Mewborn.

THEOREM 3.9. *Let A and B be $n\times n$ matrices with eigenvalues $\alpha_1,\alpha_2,\ldots,$ α_n and $\beta_1,\beta_2,\ldots,\beta_n$, respectively. Then*

$$\mathrm{sp}(A)\mathrm{sp}(B)\le\|A\|_F\|B\|_F+\left|\sum_{i=1}^{n}\alpha_i\overline{\beta_i}\right|. \tag{14}$$

Proof. We shall use M.L. Buzano's inequality [4, 6], which states that, for any vectors a,b,e in $\mathbb{C}^n$, where $\|e\|=1$, we have

$$|\langle a,e\rangle\langle e,b\rangle|\le\frac{1}{2}(\|a\|\|b\|+|\langle a,b\rangle|), \tag{15}$$

where $\|.\|$ is the spectral norm. Assume without loss of generality that $\mathrm{sp}(A)=|\alpha_n-\alpha_1|$ and $\mathrm{sp}(B)=|\beta_n-\beta_1|$. Next, we choose in (15) $a=(\alpha_1,\ldots,\alpha_n)^t$, $b=(\beta_1,\ldots,\beta_n)^t$, and $e=\frac{1}{\sqrt{2}}(-1,0,\ldots,0,1)^t$, to get

$$\begin{aligned}\mathrm{sp}(A)\mathrm{sp}(B) &\le \sqrt{\sum_{i=1}^{n}|\alpha_i|^2}\sqrt{\sum_{i=1}^{n}|\beta_i|^2}+\left|\sum_{i=1}^{n}\alpha_i\overline{\beta_i}\right|\\ &\le \|A\|_F\|B\|_F+\left|\sum_{i=1}^{n}\alpha_i\overline{\beta_i}\right|.\end{aligned}$$

□

REMARK 3.10. If we take $A=B$ in (14), we obtain inequality (2).

THEOREM 3.11. *Let A and B be $n\times n$ matrices with eigenvalues $\alpha_1, \alpha_2, \dots,$ α_n and $\beta_1, \beta_2, \dots, \beta_n$, respectively. Then*

$$\operatorname{sp}(A)\operatorname{sp}(B) \leq \sqrt{M(A)M(B)} + \left| \sum_{i=1}^{n} \alpha_i \overline{\beta_i} - \frac{\operatorname{tr}(A)\overline{\operatorname{tr}(B)}}{n} \right| ., \tag{16}$$

where $M(A) = \left(\|A\|_F^2 - \frac{|\operatorname{tr}(A)|^2}{n} \right)$.

Proof. First we note that $\operatorname{sp}(A) = \operatorname{sp}(A - mI)$, since the spectrum of $(A - mI)$ is obtained by a translation of the spectrum of A. On the other hand,

$$\begin{aligned} \|A - mI\|_F^2 &= \sum_{i=1}^{n} \left(|a_{ii} - m|^2 + \sum_{i \neq k} |a_{ik}|^2 \right) \\ &= \sum_{i=1}^{n} \sum_{k=1}^{n} |a_{ik}|^2 - \frac{|\operatorname{tr}(A)|^2}{n} = \|A\|_F^2 - \frac{|\operatorname{tr}(A)|^2}{n}. \end{aligned}$$

Further,

$$\sum_{i=1}^{n} \left(\alpha_i - \frac{\operatorname{tr}(A)}{n} \right) \left(\overline{\beta_i} - \frac{\overline{\operatorname{tr}(B)}}{n} \right) = \sum_{i=1}^{n} \alpha_i \overline{\beta_i} - \frac{\operatorname{tr}(A)\overline{\operatorname{tr}(B)}}{n}.$$

This completes the proof. □

REMARK 3.12. If we take $A = B$ in (16), we obtain inequality (1).

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