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UJ-ENDOMORPHISM RINGS

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Abstract. In this paper, we introduce and study UJ-modules, that is modules M for which their endomorphism rings E_M are right UJ. We show, in particular, that: (1) if M is a left UJ-module over a ring R, then M is Dedekind finite; (2) M is a UJ-module iff all clean elements of E_M are J-clean; (3) M is a clean UJ-module iff $E_M/J(E_M)$ is a Boolean ring and the idempotents lift modulo $J(E_M)$ (equivalently, M is a J-clean module); and (4) M is a clean UJ-module such that $J(E_M)$ is nil iff M is a conjugate nil clean UJ-module. We also give characterizations of the trivial extension and the (trivial) Morita context, $R[x]/(x^2)$ and the tail rings which are right UJ.

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1. INTRODUCTION

Throughout the paper all rings considered are associative and unital. For a ring R, the Jacobson radical, the group of units and the set of all nilpotent elements of R are denoted by $J(R)$, $U(R)$ and $N(R)$, respectively. For a module M, $\mathrm{Rad}(M)$ and 1_M represent the radical of a module and identity morphism of M, respectively. Throughout this article the homomorphisms of the modules are written on the left of their arguments.

One always has $1 + J(R) \subseteq U(R)$. Recently, Koşan, Leroy and Matczuk [6] showed that the problem of lifting the UJ property from a ring R to the polynomial ring $R[x]$ is equivalent to the Köthe problem for F_2-algebras.

We recall some notations used in [11] and [12]. Let $E_M := \mathrm{End}_R(M)$. Then, by [11], we have

$$\begin{aligned} J(E_M) &= \{\alpha \in E_M : 1_M - \alpha\beta \in U(E_M), \forall \beta \in E_M\} \\ &= \{\alpha \in E_M : 1_M - \beta\alpha \in U(E_M), \forall \beta \in E_M\} \\ &= \{\alpha \in E_M : \beta\alpha \in J(E_M), \forall \beta \in E_M\} \\ &= \{\alpha \in E_M : \alpha\beta \in J(E_M), \forall \beta \in E_M\}. \end{aligned}$$

Clearly, $J(E_R) = J(End(R)) = J(R)$. From the definition of $J(E_M)$, one always has $1_M + J(E_M) \subseteq U(E_M)$. Then it makes sense to study the equality $1_M + J(E_M) = U(E_M)$, for a left R-module M. A module M with this

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property will be called a UJ module. The aim of the paper is: to obtain some (basic) properties of UJ-modules and to investigate the behavior of the UJ property under various ring extensions.

In section 2, we give basic properties and construct some examples of UJ-modules. For a left R-module M, we show that M_{E_M} has no maximal submodule and that $E_M/J(E_M)$ is reduced (i.e. it has no nonzero nilpotent elements) and hence abelian (i.e. every idempotent is central).

We begin section 3 by showing that, for an abelian ring R and $e^2 = e \in R$, R is a UJ-ring iff eR and $(1-e)R$ are UJ-rings. Here, we recall that R is a UJ-ring iff eRe and $(1-e)R(1-e)$ are UJ-rings and $eR(1-e), (1-e)Re \subseteq J(R)$ (see [6, Proposition 2.7]). In Theorems 3.3, 3.4 and Corollary 3.5, we show that the behavior of the UJ property is very nice with respect to the trivial extension and ring $R[x]/(x^2)$. Corollary 3.6 states, in particular, that the trivial Morita context $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is a UJ-module if and only if A, M, N, B are UJ-modules. In Theorem 3.7, the UJ-property of the Dorroh extension is investigated. The section ends with the tail ring extension $\mathcal{R}[D, C]$. We prove, in Theorem 3.8, that, for a subring C of a ring D, $\mathcal{R}[D, C]$ is a UJ-ring if and only if D and C are UJ-rings.

For the last section, we establish some results between UJ-modules, J-clean and (conjugate) nil clean modules. We prove in Theorem 4.4 that M is a clean UJ-module iff $E_M/J(E_M)$ is a Boolean ring and idempotents lift modulo $J(E_M)$ iff M is a J-clean UJ-module iff M is a J-clean module. It is also shown that a module M is a clean UJ-module with $J(E_M)$ nil iff $E_M/J(E_M)$ is a Boolean ring and M is a UU-module (i.e. E_M is a UU-ring) iff M is a nil clean UJ-module iff M is a conjugate nil clean UJ-module (Theorem 4.6).

2. *UJ*-MODULES

Let M be a right R-module and $\mathcal{C}(E_M) = \{\alpha \in E_M : 1_M - \alpha \in U(E_M)\}$. It is easy to see that $(\mathcal{C}(E_M), \circ)$ is a group which is isomorphic to $U(E_M)$, by $\alpha \in \mathcal{C}(E_M) \mapsto 1 - \alpha \in U(E_M)$. Notice that M is a UJ-module if and only if $\mathcal{C}(E_M)$ is an ideal of E_M.

We begin with another characterization of the UJ-modules.

Proposition 2.1. *The following conditions are equivalent, for a left R-module M:*

(1) $U(E_M) = 1_M + J(E_M)$, *i.e. M is a UJ-module;*
(2) $U(E_M/J(E_M)) = \{1_M\}$*;*
(3) $\mathcal{C}(E_M)$ *is an ideal of* E_M *(then* $\mathcal{C}(E_M) = J(E_M)$*);*
(4) $\alpha\beta - \gamma\alpha \in J(E_M)$*, for any* $\alpha \in J(E_M)$ *and* $\beta, \gamma \in \mathcal{C}(E_M)$*;*
(5) $\alpha u - v\alpha \in J(E_M)$*, for any* $u, v \in U(E_M)$ *and* $\alpha \in E_M$*;*
(6) $U(E_M) + U(E_M) \subseteq J(E_M)$ *(and hence* $U(E_M) + U(E_M) = J(E_M)$*).*

Proof. (1) $\Rightarrow$ (2) By [6, Proposition 1.3.(5)], $E_M/J(E_M)$ is a UJ-ring. Then, by [6, Lemma 1.1 (2)], we get $U(E_M/J(E_M)) = 1_M$, as desired.

(1) $\Rightarrow$ (3) Let $\alpha \in \mathcal{C}(E_M)$. Then $1_M - \alpha \in U(E_M)$ and so there exists $u \in U(E_M)$ such that $1_M - \alpha = u$, which gives $\alpha = 1_M - u \in 1_M - U(E_M)$. Therefore

$$\mathcal{C}(E_M) \subseteq 1_M - U(E_M) = 1_M - (1_M + J(E_M)) \subseteq J(E_M).$$

By the definition, $J(E_M) \subseteq \mathcal{C}(E_M)$. Hence $J(E_M) = \mathcal{C}(E_M)$.

(2) $\Rightarrow$ (1) Clearly, $1_M + J(E_M) \subseteq U(E_M)$. For the converse, first we prove the following claim:

Claim: $U(E_M)/J(E_M) = U(E_M/J(E_M))$: Let $\alpha + J(E_M) \in U(E_M/J(E_M))$. By the hypothesis, $U(E_M/J(E_M)) = \{1_M\}$ and so $\alpha + J(E_M) = 1_M$, which gives $1_M - \alpha \in J(E_M)$. By the definition of $J(E_M)$, one obtains $\alpha \in U(E_M)$ so $\alpha + J(E_M) \in U(E_M)/J(E_M)$. The reverse is clear, since α is an element of $U(E_M)$.

Now we are ready to prove the $U(E_M) \subseteq 1_M + J(E_M)$. Let $\alpha \in U(E_M)$. Then

$$\alpha + J(E_M) \in U(E_M)/J(E_M) = U(E_M/J(E_M)) = \{1_M\}.$$

Therefore $\alpha + j = 1_M$, for all $j \in J(E_M)$, which implies $\alpha = 1_M - j \in 1_M + J(E_M)$.

(3) $\Rightarrow$ (4) Since $\mathcal{C}(E_M)$ is an ideal of E_M, we get $\alpha\beta - \gamma\alpha \in \mathcal{C}(E_M)$, for $\beta, \gamma \in \mathcal{C}(E_M)$ and $\alpha \in J(E_M)$. By (3), $\mathcal{C}(E_M) = J(E_M)$, so $\alpha\beta - \gamma\alpha \in J(E_M)$.

(4) $\Rightarrow$ (5) If we set $\beta := 1_M + u$ and $\gamma := 1_M + v$, for $u, v \in U(E_M)$, then (5) is an immediate consequence of (4).

(5) $\Rightarrow$ (6) If we take $\alpha = 1_M$ in (5), then we get $u - v \in J(E_M)$, for any $u, v \in U(E_M)$, which gives $U(E_M) + U(E_M) \subseteq J(E_M)$. Now, every $\alpha \in J(E_M)$ can be written as a sum of two invertible morphisms as $\alpha = 1_M + (\alpha - 1_M) \in U(E_M) + U(E_M)$, so we are done.

(6) $\Rightarrow$ (1) Clearly, $1_M + J(E_M) \subseteq U(E_M)$. By (6), $U(E_M) - 1_M \subseteq J(E_M)$, i.e. $U(E_M) \subseteq 1_M + J(E_M)$, which completes the proof. □

As an immediate application of Proposition 2.1, we obtain the following corollary.

Corollary 2.2 ([6, Lemma 1.1]). *For a ring R, the following conditions are equivalent:*

(1) $U(R) = 1 + J(R)$, *i.e. R is a UJ-ring;*
(2) $U(R/J(R)) = \{1\}$;
(3) $\mathcal{C}(R)$ *is an ideal of R (then $\mathcal{C}(R) = J(R)$);*
(4) $rb - cr \in J(R)$, *for any $r \in R$ and $b, c \in \mathcal{C}(R)$;*
(5) $ru - vr \in J(R)$, *for any $u, v \in U(R)$ and $r \in R$,*
(6) $U(R) + U(R) \subseteq J(R)$ *(and hence $U(R) + U(R) = J(R)$).*

The next two observations contain several properties of the UJ-modules and rings.

Proposition 2.3. *Let M be a left UJ-module over R. Then:*

(1) M_{E_M} *has no a maximal submodule;*

(2) *if E_M is a division ring, then $E_M = \mathbb{F}_2$;*
(3) *$E_M/J(E_M)$ is reduced and hence abelian;*
(4) *if $\alpha, \beta \in E_M$ and if $\alpha\beta \in J(E_M)$, then $\beta\alpha \in J(E_M)$, $\alpha E_M \beta, \beta E_M \alpha \subseteq J(E_M)$;*
(5) *if $I \subseteq J(E_M)$ is an ideal of E_M, then M is a UJ-module if and only if E_M/I is a UJ-ring,*
(6) *M is Dedekind finite (i.e. E_M is a Dedekind finite ring; if $a, b \in E_M$, $ab = 1 \Rightarrow ba = 1$).*
(7) *the module $\prod_{i \in I} M_i$ is UJ if and only if each M_i is a UJ-module, for all $i \in I$.*

Proof. (1) By Proposition 2.1 (6), we have $U(E_M) + U(E_M) = J(E_M)$. Since M is a right E_M module and $M_{E_M} J(E_M) \subseteq Rad(M_{E_M})$, we get

$$M_{E_M} U(E_M) + M_{E_M} U(E_M) = M_{E_M} J(E_M) \subseteq Rad(M_{E_M}).$$

One gets the following, for $1_M \in U(E_M)$:

$$M_{E_M} \subseteq M_{E_M} + M_{E_M} \subseteq Rad(M_{E_M}) \subseteq M_{E_M}.$$

This gives $Rad(M_{E_M}) = M_{E_M}$, that is M_{E_M} has no maximal submodule.

(2) If E_M is a division ring, then every nonzero morphism of E_M has an inverse. By Proposition 2.1(1), $U(E_M) = 1_M + J(E_M)$. Hence $1_M + J(E_M) \in E_M/J(E_M)$ has only an element which has an inverse. By Proposition 2.1(2), $U(E_M/J(E_M)) = \{1_M\}$, as desired.

(3) Let $\alpha + J(E_M)$ be a nilpotent element in $E_M/J(E_M)$. We show that $\alpha \in J(E_M)$. Since $\alpha + J(E_M)$ is nilpotent, there exits $n \in \mathbb{N}$ such that $\alpha^n + J(E_M) = J(E_M)$. Then

$$\begin{aligned}
1_M + J(E_M) &= [(\alpha^n) + J(E_M)] + (1_M + J(E_M)) \\
&= (\alpha^n + 1_M) + J(E_M) \\
&= (\alpha + 1_M)((-1)^{n-1}\alpha^{n-1} + \cdots + (-1)^0 1_M) + J(E_M) \\
&= [(\alpha + 1_M) + J(E_M)][(\alpha^{n-1} + \cdots + 1_M) + J(E_M)].
\end{aligned}$$

So $(\alpha + 1_M) + J(E_M) \in U(E_M/J(E_M))$, which is 1_M, by Proposition 2.1(2). Then there exists $j \in J(E_M)$ such that $(\alpha + 1_M) + j = 1_M$, that is $\alpha = -j \in J(E_M)$.

By (1), $M/Rad(M) = M/M = 0$ so $M/Rad(M)$ has no nonzero nilpotent elements, hence it is reduced and so it is abelian.

(4) Let $\alpha\beta \in J(E_M)$. Then $\alpha\beta + J(E_M) = J(E_M)$. Multiplying this equation by $\beta + J(E_M)$ on the left and by $\alpha + J(E_M)$ on the right, we get

$$\beta\alpha\beta\alpha + J(E_M) = (\beta\alpha)^2 + J(E_M) = J(E_M).$$

By (3), $E_M/J(E_M)$ is reduced and thus $\beta\alpha + J(E_M) = J(E_M)$. Hence $\beta\alpha \in J(E_M)$. Now, the rest follows from (3).

(5) Let $I \subseteq J(E_M)$. We show $J(E_M)/I = J(E_M/I)$. Clearly, $J(E_M)/I \subseteq J(E_M/I)$. For the converse, let $\alpha + I \in J(E_M/I)$. Then $(1_M - \alpha) + I$ is an element of $U(E_M)$, so $[(1_M - \alpha) + I](\beta + I) = [(1_M - \alpha)\beta] + I = 1_M + I$.

Then $1_M - [(1_M - \alpha)\beta] \subseteq I \subseteq J(E_M)$, which implies that $(1_M - \alpha)\beta$ is an element of $U(E_M)$, that is $\alpha \in J(E_M)$. Hence $\alpha + I \in J(E_M) + I$. By the proof of Proposition 2.1(2),

$$\frac{E_M/I}{J(E_M/I)} = \frac{E_M/I}{J(E_M)/I} = E_M/J(E_M),$$

which implies

$$U(\frac{E_M/I}{J(E_M/I)}) = U(E_M/J(E_M)).$$

(6) We note that $E_M/J(E_M)$ is Dedekind finite, since it is reduced. Let $\alpha\beta = 1_M$, for $\alpha, \beta \in E_M$. Then $\alpha\beta + J(E_M) = 1_M + J(E_M)$. Since $E_M/J(E_M)$ is Dedekind finite, we obtain $\beta\alpha + J(E_M) = 1_M + J(E_M)$, that is $\beta\alpha$ is invertible. Clearly, $\beta\alpha$ is an idempotent, so $\beta\alpha = 1_M$.

(7) Recall that

$$U(\prod_{i\in I} E_{M_i}) = \Big\{ \prod_{i\in I} \alpha_i : \prod_{i\in I} M_i \to \prod_{i\in I} M_i | \prod_{i\in I} \alpha_i \quad \text{is an element of } U(E_{M_i}) \Big\},$$

and

$$\prod_{i\in I}(U(E_{M_i})) = \Big\{ \prod_{i\in I} \alpha_i : \prod_{i\in I} M_i \to \prod_{i\in I} M_i | \\ \forall i \in I,\ \alpha_i \quad \text{is an element of } U(E_{M_i}) \Big\}.$$

Now, it is easy to see that $U(\prod_{i\in I} E_{M_i}) = \prod_{i\in I}(U(E_{M_i}))$. Similarly, we have $J(\prod_{i\in I} E_{M_i}) = \prod_{i\in I} J(E_{M_i})$. □

COROLLARY 2.4 ([6, Proposition 1.3]). *Let R be a UJ-ring. Then:*

(1) $2 \in J(R)$;
(2) *if R is a division ring, then $R = \mathbb{F}_2$;*
(3) *$R/J(R)$ is reduced and hence abelian;*
(4) *if $x, y \in R$ are such that $xy \in J(R)$, then $yx \in J(R)$ and $xRy, yRx \subseteq J(R)$;*
(5) *if $I \subseteq J(R)$ is an ideal of R, then R is a UJ-ring if and only if R/I is a UJ-ring;*
(6) *R is Dedekind finite;*
(7) *the ring $\prod_{i\in I} R_i$ is UJ if and only each R_i is a UJ-ring, $i \in I$.*

Recall that the ring R is said to be semilocal, if $R/J(R)$ is semisimple artinian.

PROPOSITION 2.5. *A semilocal ring E_M is UJ if and only if $E_M/J(E_M) \cong \mathbb{F}_2 \times \ldots \times \mathbb{F}_2$.*

Proof. Since $E_M/J(E_M)$ is semisimple, by the definition, and reduced, by Proposition 2.3(3), we obtain that $E_M/J(E_M)$ is a finite direct product of a division ring. Proposition 2.3(2) completes the proof. □

Corollary 2.6 ([6, Proposition 1.4]). *A semilocal ring R is UJ if and only if* $R/J(R) \cong \mathbb{F}_2 \times \ldots \times \mathbb{F}_2$.

For a left module M, let $M[x]$ be the set of all formal polynomials in indeterminate x with coefficients from M. Then $M[x]$ becomes a left $R[x]$-module under the usual addition and multiplication of polynomials, where $R[x]$ denotes the polynomial ring in the set x of commuting indeterminates.

Let

$$N(E_M) = \{\alpha \in E_M : \alpha^n = 0,\ for\ some\ n \in \mathbb{N}\}.$$

Lemma 2.7. *If 1_M is the only element of $U(E_M)$, then $U(E_M)[x] = \{1_M\}$.*

Proof. Since a unit in $M[x]$ depends only on finitely many indeterminates, we may assume that x is a finite set.

By the assumption, $U(E_M)[x] = \{1_M\}$, so E_M does not contain nontrivial nilpotent elements, because $1_M + N(E_M) \subseteq U(E_M)$, i.e. it is a reduced ring. But $U(E_M)[x] = U(E_M)$, so we are done. □

Corollary 2.8 ([6, Lemma 2.3]). *Let R be a ring with trivial units. Then* $U(R[x]) = \{1\}$.

3. SOME RING EXTENSIONS

The left Peirce decompositions. We consider the sets eR and $(1-e)R$, where $e^2 = e \in R$.

Proposition 3.1. *Let M be an abelian module and $e^2 = e \in E_M$. Then the following are equivalent.*

(1) *M is a UJ-module.*
(2) *eM and $(1-e)M$ are UJ-modules.*

Proof. Since $e^2 = e \in E_M$, we have $M = eM \oplus (1-e)M$. So, we have,

$$\begin{aligned} E_M &= \mathrm{Hom}_R(eM \oplus (1-e)M, eM \oplus (1-e)M) \\ &= \mathrm{Hom}_R(eM, eM) \oplus \mathrm{Hom}_R((1-e)M, (1-e)M) \\ &\oplus \mathrm{Hom}_R(eM, (1-e)M) \oplus \mathrm{Hom}_R((1-e)M, eM) \\ &= E_{eM} \oplus E_{(1-e)M}. \end{aligned}$$

Hence we obtain

$$U(E_M) = U(E_{eM}) \oplus U(E_{(1-e)M})$$

and

$$J(E_M) = J(E_{eM}) \oplus J(E_{(1-e)M}).$$

□

Corollary 3.2. *Let R be an abelian ring and $e^2 = e \in R$. Then the following are equivalent.*

(1) *R is a UJ-ring.*
(2) *eR and $(1-e)R$ are UJ-rings.*

The trivial extension and the (trivial) Morita context. Let R be a ring and M a bimodule over R. The trivial extension of R and M is

$$T(R,M) = \{(r,m) : r \in R \text{ and } m \in M\}$$

with the addition defined componentwise and the multiplication defined by

$$(r,m)(s,n) = (rs, rn + ms).$$

The trivial extension $T(R,M)$ is isomorphic to the subring $\left\{ \begin{pmatrix} r & m \\ 0 & r \end{pmatrix} : r \in R \text{ and } m \in M \right\}$ of the formal 2×2 matrix ring $\begin{pmatrix} R & M \\ 0 & R \end{pmatrix}$ and also $T(R,R) \cong R[x]/(x^2)$. We also note that the set of units of the trivial extension $T(R,M)$ is

$$U(T(R,M)) = T(U(R), M),$$

by [1, Proposition 4.9 (2)], and

$$J(T(R,M)) = T(J(R), M),$$

by [1, Corollary 4.8 (2)].

A Morita context is a 4-tuple $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$, where A and B are rings, ${}_AM_B$ and ${}_BN_A$ are bimodules and there exist context products $M \times N \to A$ and $N \times M \to B$, written multiplicatively as $(w,z) = wz$ and $(z,w) = zw$, such that $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is an associative ring with the obvious matrix operations.

A Morita context $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is called trivial, if the context products are trivial, i.e. $MN = 0$ and $NM = 0$ (see [7, p. 1993]). We have

$$\begin{pmatrix} A & M \\ N & B \end{pmatrix} \cong T(A \times B, M \oplus N),$$

where $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is a trivial Morita context, by [5].

THEOREM 3.3. *If the trivial extension $T := T(R,M)$ is a UJ-ring, then R is a UJ-ring and M is a UJ-module.*

Proof. Let $r \in R$ and $u, v \in U(R)$. Then

$$\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} - \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \in J(T),$$

where $\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \in T$ and $\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} \in U(T)$. Hence

$$\begin{pmatrix} ru & 0 \\ 0 & ru \end{pmatrix} - \begin{pmatrix} vr & 0 \\ 0 & vr \end{pmatrix} = \begin{pmatrix} ru - vr & 0 \\ 0 & ru - vr \end{pmatrix} \in J(T)$$

which implies $ru - vr \in J(R)$, for $r \in R$ and $u, v \in U(R)$.

Let $\alpha \in U(E_M)$. Clearly, $\varphi : T \to T$ is an element of $U(E_M)$, which is defined by $\varphi(\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}) = \begin{pmatrix} r & \alpha(m) \\ 0 & r \end{pmatrix}$. Since T is an UJ-module, we get $U(E_T) \subseteq 1_T + J(E_T)$. Then $\varphi = 1_T + \psi$, for $\psi \in J(E_T)$. One obtains $\psi(\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}) = \begin{pmatrix} 0 & (\alpha - 1_M)(m) \\ 0 & 0 \end{pmatrix}$, by a direct calculation. If we prove $\alpha - 1_M \in J(E_M)$, then we are done. First, note that, for an endomorphism $\gamma : T(R, M) \to T(R, M)$, there exist $f \in E_R$ and $g \in E_M$ such that $\gamma(\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}) = \begin{pmatrix} f(r) & g(m) \\ 0 & f(r) \end{pmatrix}$. We also have that γ is an element of $U(E_M)$ if and only if f and g are elements of $U(E_M)$. By the hypothesis, we have $\varphi - 1_T = \psi \in J(E_T)$, that is $1_T - \psi\gamma \in U(E_T)$, for every $\gamma \in E_T$, where $\gamma(\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}) = \begin{pmatrix} f(r) & g(m) \\ 0 & f(r) \end{pmatrix}$. By a direct calculation,

$$1_T - \psi\gamma = \begin{pmatrix} r & 1_M(m) - (\alpha - 1_M)g(m) \\ 0 & r \end{pmatrix} \in U(E_T),$$

so $1_M - (\alpha - 1_M)g \in U(E_M)$, for every $g \in E_M$, which completes the proof. $\square$

Theorem 3.4. *If R is a UJ-ring, then so is the trivial extension $T(R, M)$.*

Proof. Assume that R is a UJ-ring. Then $U(T(R, M)) = T(U(R), M)$ and $U(R) = 1_R + J(R)$. So one can write

$$\begin{aligned} U(T(R,M)) &= \begin{pmatrix} U(R) & M \\ 0 & U(R) \end{pmatrix} \\ &= \begin{pmatrix} 1_R + J(R) & M \\ 0 & 1_R + J(R) \end{pmatrix} \\ &= \begin{pmatrix} 1_R & 0 \\ 0 & 1_R \end{pmatrix} + \begin{pmatrix} J(R) & M \\ 0 & J(R) \end{pmatrix} \\ &= 1_T + J(T(R, M)), \end{aligned}$$

as desired. $\square$

Corollary 3.5. *R is a UJ-ring if and only if $R[x]/(x^2)$ is a UJ-ring.*

Corollary 3.6. *The trivial Morita context $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is a UJ-module if and only if A, M, N, B are UJ-modules.*

Proof. It is easy to see that

$$\begin{pmatrix} A & M \\ N & B \end{pmatrix} \cong T(A \times B, M \oplus N) \cong \begin{pmatrix} A \times B & M \oplus N \\ 0 & A \times B \end{pmatrix}.$$

Then the rest follows from Theorems 3.3, 3.4 and Proposition 2.3 (7). $\square$

The Dorroh extension. Given a ring R and a ring without identity I, we will say that I is an R ring without identity if it is an (R,R)-bimodule, for which the actions of R are compatible with the multiplication in I (i.e. $r(ij) = (ri)j$, $i(rj) = (ir)j$ and $(ij)r = i(jr)$, for every $r \in R$ and $i, j \in I$). If R is a ring with identity and I is a ring without identity, then one can turn the abelian group $R \oplus I$ into a ring, by defining the multiplication by

$$(r,i).(p,j) = (rp, ip + rj + ij),$$

for $r, p \in R$ and $i, j \in I$. Such a ring is called an ideal extension (it is also called the Dorroh extension), and denoted by $E(R,I)$ - see [9] .

Theorem 3.7. *Let I be a ring without identity, finitely generated as a (left) R-module by elements that commute with all elements of R. Then R is a UJ-ring, if the Dorroh extension $E(R,I)$ is a UJ-ring.*

Proof. Let $u, v \in U(R)$ and $r \in R$. Then $(u,0),(v,0) \in U(E(R,I))$ and $(r,0) \in E(R,I)$. We get $(r,0)(u,0) - (v,0)(r,0) = (ru - vr, 0) \in J(E(R,I))$, by the hypothesis. But $J(E(R,I)) = J(R) \oplus J(I)$, which implies $ru - vr \in J(R)$. □

The tail ring extension $\mathcal{R}[D,C]$. For a subring C of a ring D, the set

$$\mathcal{R}[D,C] := \{(d_1, \cdots, d_n, c, c, \cdots) : d_i \in D, c \in C, n \geq 1\},$$

with the addition and the multiplication defined componentwise, is a ring.

Theorem 3.8. *$\mathcal{R}[D,C]$ is a UJ-ring if and only if D and C are UJ-rings.*

Proof. ($\Rightarrow$) First, we show that D is a UJ-ring. Let $u, v \in U(D)$ and $d \in D$. Then

$$\alpha = (d, 0, 0, \cdots) \in \mathcal{R}[D,C],$$
$$\beta = (u, 1, 1, \cdots), \gamma = (v, 1, 1, \cdots) \in U(\mathcal{R}[D,C]).$$

Now, $\alpha\beta - \gamma\alpha \in J(\mathcal{R}[D,C])$, by the hypothesis, that is $m := (du - vd, 0, 0, \cdots) \in J(\mathcal{R}[D,C])$. Hence, for any $t := (y_1, \cdots, y_n, x, \cdots) \in \mathcal{R}[D,C]$,

$$(1,1,1,\cdots) - mt = (1 - (du - vd)y_1, 1, 1, \cdots) \in U(\mathcal{R}[D,C]) = \mathcal{R}[U(D), U(C)],$$

which implies that D is a UJ ring.

We show that C is UJ. Let $u^*, v^* \in U(C)$ and $c \in C$. We prove that $cu^* - v^*c \in J(C)$. Then

$$\alpha^* = (0, \cdots, 0, c, c \cdots) \in \mathcal{R}[D,C],$$
$$\beta^* = (1, \cdots, 1, u^*, u^* \cdots), \gamma^* = (1, \cdots, 1, v^*, v^* \cdots) \in U(\mathcal{R}[D,C]).$$

Now, $\alpha^*\beta^* - \gamma^*\alpha^* \in J(\mathcal{R}[D,C]$, that is $n := (0, \cdots, 0, cu^* - v^*c, cu^* - v^*c, \cdots) \in J(\mathcal{R}[D,C])$. Hence, for any $t = (y_1, \cdots, y_n, x, \cdots) \in \mathcal{R}[D,C]$,

$$(1,1,1,\cdots) - nt = (1, \cdots, 1, 1 - (cu^* - v^*c)x, \cdots) \in U(\mathcal{R}[D,C]) = \mathcal{R}[U(D), U(C)],$$

which implies that C is a UJ-ring.

($\Leftarrow$) Assume D and C are UJ-rings. Let

$$\beta = (u_1, \cdots, u_n, u, u, \cdots), \gamma = (v_1, \cdots, v_n, v, v, \cdots) \in U(\mathcal{R}[D, C]),$$

where $u_i, v_i, u, v \in U(R)$, for $1 \leq i \leq n$ and $\alpha = (d_1, \cdots, d_n, c, c, \cdots) \in \mathcal{R}[D, C]$. Set $x := \alpha\beta - \gamma\alpha$. Since D and C are UJ-rings and $J(\mathcal{R}[D, C]) = \mathcal{R}[J(D), J(C)]$, we obtain

$$\begin{aligned} x &= (d_1, \cdots, d_n, c, c, \cdots)(u_1, \cdots, u_n, u, u, \cdots) \\ &- (v_1, \cdots, v_n, v, v, \cdots)(d_1, \cdots, d_n, c, c, \cdots) \\ &= (d_1u_1, \cdots, d_nu_n, cu, cu, \cdots) - (v_1d_1, \cdots, v_nd_n, vd, vd, \cdots) \\ &= (d_1u_1 - v_1d_1, \cdots, d_nu_n - v_nd_n, cu - vd, cu - vd, \cdots). \end{aligned}$$

Now, $d_iu_i - v_id_i \in J(D)$ and $du - vd \in J(C)$ imply $m := \alpha\beta - \gamma\alpha \in \mathcal{R}[J(D), J(C)]$. □

Corollary 3.9. *$\mathcal{R}[D, D]$ is a UJ-ring if and only if D is a UJ-ring.*

4. CLEAN MODULES

Recall that an element $r \in R$ is clean (J-clean) provided there exist an idempotent $e \in R$ and an element $t \in U(R)$ ($t \in J(R)$) such that $r = e + t$. A ring R is clean (J-clean), if every element of R has such a clean (J-clean) decomposition [10] ([3]). Clearly, every J-clean ring is clean.

We say that a module M is J-clean, if the endomorphism ring of M is a J-clean ring.

UU-rings are defined by Călugăreanu [2] as $U(R) = 1 + N(R)$ (i.e. rings with unipotent units). It is clear that, if R is a UJ-ring with nil Jacobson radical, then R is a UU-ring.

Recall that

$$N(E_M) = \{\alpha \in E_M : \alpha^n = 0, \; for \; some \; n \in \mathbb{N}\}.$$

We call M a UU-module, if E_M is UU-ring, that is $U(E_M) = 1_M + N(E_M)$.

Corollary 4.1. *A module M is UJ with $J(E_M)$ nil iff M is a UU-module and $N(E_M)$ is an ideal of E_M.*

Let $Id(R)$ be the set of all idempotent elements of R.

Proposition 4.2. *The following are equivalent for a module M.*

(1) *M is a UJ-module.*
(2) *All clean elements of E_M are J-clean.*

Proof. (1) $\Rightarrow$ (2) Assume $\alpha \in E_M$ is clean. Then $\alpha = e + u$, for $e^2 = e \in E_M$ and $u \in U(E_M)$. By the hypothesis, $u = 1_M + j$, where $j \in J(E_M)$. Hence

$$\alpha = e + 1_M + j = (1_M - e) + e + e + j,$$

but $e+e \in J(E_M)$, by [6, Proposition 1.3 (1)]. Hence $e+e+j \in J(E_M)$ and $(1_M - e) \in Id(E_M)$, as desired.

(2) $\Rightarrow$ (1) Clearly $1_M + J(E_M) \subseteq U(E_M)$. Let $u \in U(E_M)$. Then u is a clean element, so $u = e+j$, for $e^2 =$ and $j \in J(E_M)$. Since $1_M = u^{-1}e + u^{-1}j$, we obtain $u^{-1}e = 1_M - u^{-1}j$. Hence $u^{-1}e$ is an element of $U(E_M)$, which implies $e = 1$. □

Corollary 4.3 ([6, Proposition 3.1]). *The following conditions are equivalent for a ring R.*

(1) *R is a UJ-ring,*
(2) *All clean elements of R are J-clean.*

Theorem 4.4. *The following conditions are equivalent for a module M.*

(1) *M is a clean UJ-module.*
(2) *$E_M/J(E_M)$ is a Boolean ring and idempotents lift modulo $J(E_M)$.*
(3) *M is a J-clean UJ-module.*
(4) *M is a J-clean module.*

Proof. (1) $\Rightarrow$ (2) Since $E_M/J(E_M)$ is clean, every element $\alpha + J(E_M) \in E_M/J(E_M)$ is of the form

$$\alpha + J(E_M) = (e + J(E_M)) + (1_M + J(E_M)) = (e+1) + J(E_M).$$

Hence

$$\begin{aligned}\alpha^2 + J(E_M) &= [(e + J(E_M)) + (1_M + J(E_M))][(e + J(E_M)) + (1_M + J(E_M))]\\ &= [(e+1)(e+1)] + J(E_M)\\ &= (e+e+e+1) + J(E_M)\\ &= (e+1) + J(E_M) + (e+e) + J(E_M)\\ &= (e+1) + J(E_M),\end{aligned}$$

so $\alpha + J(E_M)$ is an idempotent element, that is $E_M/J(E_M)$ is a Boolean ring. The rest follows from the definition of clean rings.

(2) $\Rightarrow$ (3) Let $\alpha \in E_M$. Then $\alpha + J(E_M) \in E_M/J(E_M)$ is an idempotent. By hypothesis, there exists an idempotent $e \in E_M$ such that $\alpha - e \in J(E_M)$. Then $\alpha = e + j$, for $j \in J(E_M)$, i.e. α is a J-clean element. This shows that M is a J-clean module. If $u \in U(E_M)$, then $u + J(E_M) \in U(E_M/J(E_M))$ in a Boolean ring $E_M/J(E_M)$. Then $u - 1 \in J(E_M)$, that is $u \in 1_M + J(E_M)$.

(3) $\Rightarrow$ (4) This is clear.

(4) $\Rightarrow$ (1) This follows from Proposition 4.2. □

Corollary 4.5 ([6, Proposition 3.2]). *The following conditions are equivalent for a ring R.*

(1) *R is a clean UJ-ring.*
(2) *$R/J(R)$ is a Boolean ring and idempotents lift modulo $J(R)$.*
(3) *R is a J-clean UJ-ring.*
(4) *R is a J-clean ring.*

Recall that idempotents e and f are said to be conjugate in R, if there exists $u \in U(R)$ such that $e = ufu^{-1}$. Conjugate (nil) clean rings are defined as (nil) clean rings such that idempotents that appear in the decompositions are unique up to conjugation, i.e., if $a = e + s = f + t$ are such decompositions, then the idempotents e, f are conjugate in R (see [8]). We call M a conjugate (nil) clean module, if E_M is a conjugate (nil) clean ring.

THEOREM 4.6. *The following conditions are equivalent for a module M.*

(1) *M is a clean UJ-module with $J(E_M)$ nil.*
(2) *$E_M/J(E_M)$ is a Boolean ring and $J(E_M)$ is nil.*
(3) *M is a nil clean UJ-module.*
(4) *M is a conjugate nil clean UJ-module.*
(5) *M is a conjugate nil clean module and $N(E_M)$ is an ideal of E_M.*
(6) *$E_M/J(E_M)$ is a Boolean ring and M is a UU-module.*

Proof. In view of [4, Corollary 3.17], the ring E_M is nil clean if and only if $E_M/J(E_M)$ is nil clean and $J(E_M)$ is nil. Also, if M is a UJ-module, then M is nil clean if and only if M is J-clean and $J(E_M)$ is a nil ideal of E_M. Now, (1) $\Leftrightarrow$ (3) holds, by Theorem 4.4 and the fact that idempotents lift modulo nil ideals.

(4) $\Rightarrow$ (3) Trivial.

(2) $\Rightarrow$ (4) By (2), M is a UJ-module. Since Boolean rings are conjugate nil clean, the rest follows from [8, Corollary 2.16].

(4) $\Leftrightarrow$ (5) If M is nil clean, then $J(E_M)$ is nil and hence the statement follows by Remark 4.1.

(2) $\Leftrightarrow$ (6) This is clear. □

COROLLARY 4.7 ([6, Theorem 3.3]). *The following conditions are equivalent for a ring R.*

(1) *R is a clean UJ-ring with nil Jacobson radical $J(R)$.*
(2) *$R/J(R)$ is a Boolean ring and $J(R)$ is nil.*
(3) *R is a nil clean UJ-ring.*
(4) *R is a conjugate nil clean UJ-ring.*
(5) *R is a conjugate nil clean ring and $N(R)$ is an ideal of R.*
(6) *$R/J(R)$ is a Boolean ring and R is a UU-ring.*

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