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PERIOD OF BALANCING NUMBERS MODULO PRODUCT OF CONSECUTIVE LUCAS-BALANCING NUMBERS

BIJAN KUMAR PATEL, SHANTA KUMARI SUNANDA, and PRASANTA KUMAR RAY

Abstract. The period of the balancing numbers modulo m, denoted by $\pi(m)$, is the least positive integer l such that $\{B_l, B_{l+1}\} \equiv \{0, 1\} \pmod{m}$, where B_l denotes the l-th balancing number. In the present study, we examine the periods of the balancing numbers modulo a product of consecutive Lucas-balancing numbers.

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Key words. Balancing number, Lucas-balancing number, periodicity.

1. INTRODUCTION

As usual, the balancing sequence $\{B_n\}_{n\geq 0}$ satisfies the linear recurrence $B_{n+1} = 6B_n - B_{n-1}$ with $B_0 = 0$, $B_1 = 1$ - see [1]. The existence of the Lucas-balancing sequence $\{C_n\}_{n\geq 0}$ is given by the sequence $\{B_n\}_{n\geq 0}$, where $C_n = \sqrt{8B_n^2 + 1}$ - see [5]. Lucas-balancing numbers satisfy the same recurrence pattern as that of the balancing numbers with different initials, that is $C_{n+1} = 6C_n - C_{n-1}$ with $C_0 = 1$, $C_1 = 3$. The identities resembling trigonometric identities $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$ are $B_{m\pm n} = B_m C_n \pm C_m B_n$, while those for the de Moivre identities $(\cos\theta \pm i \sin\theta)^n = \cos n\theta \pm i \sin n\theta$ are $(C_m \pm \sqrt{8}B_m)^n = C_{mn} \pm \sqrt{8}B_{mn}$ - see [5].

In [3, 4], Marques established some identities concerning the order of appearance of the Fibonacci numbers modulo a product of consecutive Fibonacci and Lucas numbers. In a subsequent paper, Khaochim et al. [2] have extended Marques's ideas and examined the period of the Fibonacci sequence modulo a product of consecutive Fibonacci numbers. In [6], Panda et al. studied the period of balancing sequence modulo certain primes and also examined its periodicity modulo the terms of some sequences. According to them, the period modulo m, denoted by $\pi(m)$, is defined as the smallest positive integer l for which $\{B_l, B_{l+1}\} \equiv \{0, 1\} \pmod{m}$. The rank of balancing sequence $\alpha(n)$ of a positive integer n is the smallest positive integer k such that n divides B_k, whereas its order $o(m)$ is the order of the least residue of $B_{\alpha(m)+1}$ - see [7]. Some relations between the period, rank and order of the balancing sequence modulo m are also established in [7]. Among others, an important relation is that the period of the balancing sequence equals the product of its

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rank and order. Subsequently, the moduli for which all the residues appear with equal frequency with a single period in the balancing sequence have been investigated by Ray et al. [9].

In the present study, we examine the period of the balancing sequence modulo a product of consecutive Lucas-balancing numbers. For instance,

$$\pi(C_nC_{n+1}C_{n+2}C_{n+3}) =$$
$$\begin{cases} 6n(n+1)(n+2)(n+3), & \text{if } n \equiv \{1,2,7,8,13,14\} \pmod{18} \\ 2n(n+1)(n+2)(n+3), & \text{if } n \equiv \{3,4,5,6,9,10,11,12,16,17\} \pmod{18} \\ \frac{2}{3}n(n+1)(n+2)(n+3), & \text{if } n \equiv \{0,15\} \pmod{18}. \end{cases}$$

2. PRELIMINARIES

Certain divisibility properties of balancing numbers were extensively studied in [5, 10]. In this section, we present some identities that are used subsequently. Throughout, for any two positive integers a, and b, (a, b) and $[a, b]$ denote their greatest common divisor and least common multiple, respectively.

The following results relating $\pi(m)$ are found in [6].

Lemma 2.1. *If m divides n, then $\pi(m)$ divides $\pi(n)$.*

Lemma 2.2. *For any natural number n, $\pi(B_n) = 2n$.*

The following result is found in [7].

Lemma 2.3. *For any positive integers m, and n, $\pi([m, n]) = [\pi(m), \pi(n)]$.*

The following results are found in [10].

Lemma 2.4. *If m, n are any integers with $m \geq 1$, then C_m divides B_n if and only if m divides n and $\frac{n}{m}$ is an even integer.*

Lemma 2.5. *If m, n are any integers with $m \geq 1$, then C_m divides C_n if and only if m divides n and $\frac{n}{m}$ is an odd integer.*

Lemma 2.6. *For any $n \in \mathbb{N}$ and $m, k \in \mathbb{Z}$, $B_{2mn+k} \equiv (-1)^n B_k \pmod{C_m}$.*

3. MAIN RESULTS

In this section, we will examine the period of balancing sequence modulo a product of consecutive Lucas-balancing numbers. Before proving the main results, we will establish some identities that are used subsequently.

Lemma 3.1. *For any natural number n, $\pi(C_n) = 4n$.*

Proof. Since $B_{2n} = 2B_nC_n$, C_n divides B_{2n}. It follows from Lemma 2.1 that $\pi(C_n)$ divides $\pi(B_{2n})$ and hence $\pi(C_n)$ divides $4n$, in view of Lemma 2.2. On the other hand, $B_{2n} \equiv 0 \pmod{C_n}$ and $B_{2n+1} = B_{2n}C_1 + B_1C_{2n} \equiv C_{2n} = C_n^2 + 8B_n^2 \equiv 8B_n^2 = C_n^2 - 1 \not\equiv 1 \pmod{C_n}$. Consequently, $\pi(C_n) > 2n$ and we have $2n < \pi(C_n) \leq 4n$. The result follows, since $4n$ has no positive divisor between $2n$ and $4n$. □

Lemma 3.2. *For any natural number n,*

$$\pi(C_1C_n) = \begin{cases} 4n, & \textit{if } n \equiv 0 \pmod 2 \\ 12n, & \textit{if } n \equiv 1 \pmod 2. \end{cases}$$

Proof. Let $n \equiv 0 \pmod 2$. Then $(C_1, C_n) = 1$. By Lemmas 2.3 and 3.1, we obtain $\pi(C_1C_n) = \pi([3, C_n]) = [\pi(3), \pi(C_n)] = [4, 4n] = 4n$. On the other hand, let $n \equiv 1 \pmod 2$. As C_1C_n divides B_{6n}, $B_{12n} \equiv 0 \pmod{C_1C_n}$. Furthermore, $B_{12n+1} \equiv C_{12n} = C_{6n}^2 + 8B_{6n}^2 = 1 + 16B_{6n}^2 \equiv 1 \pmod{C_1C_n}$. Therefore, $\pi(C_1C_n)$ divides $12n$. In order to complete the proof, we shall show that $\pi(C_1C_n)$ does not divide $6n$, as well as $4n$. Since $B_{6n} \equiv 0 \pmod{C_1C_n}$ and by Lemma 2.6, $B_{6n+1} \not\equiv 1 \pmod{C_1C_n}$, which implies that $\pi(C_1C_n)$ does not divide $6n$. Further, as $n \equiv 1 \pmod 2$, $(3, B_n) = (3, C_{2n}) = 1$. It follows that $\frac{B_{4n}}{3C_n} = \frac{2B_{2n}C_{2n}}{3C_n} = \frac{4B_nC_{2n}}{3} \notin \mathbb{Z}$. Consequently, $B_{4n} \not\equiv 0 \pmod{C_1C_n}$ and hence $\pi(C_1C_n)$ does not divide $4n$. This completes the proof. □

Lemma 3.3. *For any natural number n,*

$$\pi(C_nC_{n+2}) = \begin{cases} 2n(n+2), & \textit{if } n \equiv 0 \pmod 2 \\ 12n(n+2), & \textit{if } n \equiv 1 \pmod 2. \end{cases}$$

Proof. Let $n \equiv 0 \pmod 2$. Then $(n, n+2) = 2$. Consequently, $(C_n, C_{n+2}) = 1$. Using Lemmas 2.3 and 3.1 again, we get

$$\begin{aligned} \pi(C_nC_{n+2}) &= \pi([C_n, C_{n+2}]) = [\pi(C_n), \pi(C_{n+2})] \\ &= [4n, 4(n+2)] = \frac{4n \times (n+2)}{(n, n+2)} = 2n(n+2). \end{aligned}$$

Further, when $n \equiv 1 \pmod 2$, n and $n+2$ are relatively prime. Since $\pi(C_nC_{n+2}) = \pi([C_n, C_{n+2}](C_n, C_{n+2}))$ and $(C_n, C_{n+2}) = 3$, we have

$$\pi([C_n, C_{n+2}](C_n, C_{n+2})) = \pi(3[C_n, C_{n+2}]).$$

Using Lemma 2.3, the right-hand side is equal to $\pi([3C_n, 3C_{n+2}])$. By virtue of Lemma 3.2,

$$[\pi(3C_n), \pi(3C_{n+2})] = [12n, 12(n+2)] = \frac{12n(n+2)}{(n, n+2)}.$$

Since $(n, n+2) = 1$, the result follows. □

Now we are in the position to derive our main results. The first result shows that the period of balancing numbers modulo a product of two consecutive Lucas-balancing numbers equals four times the product of those consecutive natural numbers.

Theorem 3.4. *For any natural number n, $\pi(C_nC_{n+1}) = 4n(n+1)$.*

Proof. By Lemma 2.4, C_n divides $B_{2n(n+1)}$ and C_{n+1} divides $B_{2n(n+1)}$. Since $(C_n, C_{n+1}) = 1$, C_nC_{n+1} divides $B_{2n(n+1)}$ and hence $\pi(C_nC_{n+1})$ divides $\pi(B_{2n(n+1)}) = 4n(n+1)$. On the other hand, for $\alpha \in \{0, 1\}$, $C_{n+\alpha}$ divides

C_nC_{n+1}. It follows from Lemma 3.1 that $4(n+\alpha)$ divides $\pi(C_nC_{n+1})$. But, for n even or odd, we have either $(4n, n+1) = 1$ or $(n, 4(n+1)) = 1$. It follows that $4n(n+1)$ divides $\pi(C_nC_{n+1})$, which ends the proof. □

In the next two results, we study the period of balancing numbers modulo a product of three and four consecutive Lucas-balancing numbers.

THEOREM 3.5. *For any natural number n,*

$$\pi(C_nC_{n+1}C_{n+2}) = \begin{cases} 2n(n+1)(n+2), & \text{if } n \equiv 0 \pmod 2 \\ 12n(n+1)(n+2), & \text{if } n \equiv \{1,3\} \pmod 6 \\ 4n(n+1)(n+2), & \text{if } n \equiv 5 \pmod 6. \end{cases}$$

Proof. Let $n \equiv 0 \pmod 2$. Then $(4(n+1), 2n(n+2)) = 4$. By Lemmas 3.1, 3.3 and $(C_{n+1}, C_nC_{n+2}) = 1$, we have

$$\begin{aligned} \pi(C_nC_{n+1}C_{n+2}) &= \pi([C_{n+1}, C_nC_{n+2}]) = [\pi(C_{n+1}), \pi(C_nC_{n+2})] \\ &= [4(n+1), 2n(n+2)] = \frac{4(n+1) \times 2n(n+2)}{(4(n+1), 2n(n+2))} \\ &= 2n(n+1)(n+2). \end{aligned}$$

Further, if $n \equiv \{1,3\} \pmod 6$, then we have $(4(n+1), 12n(n+2)) = 4$. Since $\pi(C_nC_{n+2}) = 12n(n+2)$, when $n \equiv \{1,3\} \pmod 6$, proceeding as above, we get the desired result.

Finally, for $n \equiv 5 \pmod 6$, we observe that $(4(n+1), 12n(n+2)) = 12$. Proceeding similarly as above, we get $\pi(C_nC_{n+1}C_{n+2}) = 4n(n+1)(n+2)$, which ends the proof. □

THEOREM 3.6. *For any natural number n,*

$$\pi(C_nC_{n+1}C_{n+2}C_{n+3}) = \begin{cases} 6n(n+1)(n+2)(n+3), & \text{if } n \equiv \{1,2,7,8,13,14\} \pmod{18} \\ 2n(n+1)(n+2)(n+3), & \text{if } n \equiv \{3,4,5,6,9,10,11,12,16,17\} \pmod{18} \\ \frac{2n(n+1)(n+2)(n+3)}{3}, & \text{if } n \equiv \{0,15\} \pmod{18}. \end{cases}$$

Proof. For any natural number n, $(C_nC_{n+2}, C_{n+1}C_{n+3}) = 1$. Let $n \equiv \{1,7,13\} \pmod{18}$. Then $(12n(n+2), 2(n+1)(n+3)) = 4$. By Lemmas 2.3 and 3.3,

$$\begin{aligned} \pi(C_nC_{n+1}C_{n+2}C_{n+3}) &= \pi([C_nC_{n+2}, C_{n+1}C_{n+3}]) \\ &= [\pi(C_nC_{n+2}), \pi(C_{n+1}C_{n+3})] \\ &= [12n(n+2), 2(n+1)(n+3)] \\ &= \frac{12n(n+2) \times 2(n+1)(n+3)}{(12n(n+2), 2(n+1)(n+3))} \\ &= 6n(n+1)(n+2)(n+3). \end{aligned}$$

Let $n \equiv \{2, 8, 14\} \pmod{18}$. Then $(2n(n+2), 12(n+1)(n+3)) = 4$. Proceeding as above, we get the desired result.

Let $n \equiv \{4, 6, 10, 12, 16\} \pmod{18}$ and $n \equiv \{3, 5, 9, 11, 17\} \pmod{18}$. In these two cases, $(2n(n+2), 12(n+1)(n+3))$ and $(12n(n+2), 2(n+1)(n+3))$ are equal to 12. A similar simplification as above, yields $\pi(C_n C_{n+1} C_{n+2} C_{n+3}) = 2n(n+1)(n+2)(n+3)$.

On the other hand, let $n \equiv 0 \pmod{18}$ and $n \equiv 15 \pmod{18}$. For both these cases, $(2n(n+2), 12(n+1)(n+3))$ and $(12n(n+2), 2(n+1)(n+3))$ are 36. Further simplifications show that $\pi(C_n C_{n+1} C_{n+2} C_{n+3}) = \frac{2n(n+1)(n+2)(n+3)}{3}$. This completes the proof. □

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International Institute
of Information Technology
Bhubaneswar-751003, India
E-mail: iiit.bijan@gmail.com
E-mail: shanta@iiit-bh.ac.in

Sambalpur University
Burla-768019, India
E-mail: rayprasanta2008@gmail.com