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NEW RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL RESULTS FOR ACZÉL TYPE INEQUALIES

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Abstract. In this paper, using the Riemann-Liouville integral operator, we establish several fractional refinements of the Aczél inequality. Some classical results on this famous inequality can be deduced as some special cases of our results.

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Key words. Integral inequality, Aczél inequality, Riemann-Liouville integral.

1. INTRODUCTION

The integral inequality theory is an important field of research. This theory plays a crucial role in differential equations, probability theory and applied science. For more details, we refer to [7, 11, 12, 13, 14, 15, 19] and the references therein. Moreover, the fractional integral inequalities are also of great importance. For some applications, one can consult the papers [2, 3, 4, 5, 6, 9, 10, 16]. The idea that we develop in the present paper is motivated by the work of J. Tian and S. Wu [17], where the authors established the following theorems related to the well known Aczél inequality [1].

THEOREM 1.1. *Let $a_{rj} > 0$ $(r = 1, 2, \dots, n,\ j = 1, 2, \dots, m)$ and $\lambda_1 \leq \lambda_2 \leq \dots \lambda_m$ be such that $a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} > 0$ $(j = 1, 2, \dots, m)$. Also, let $m \geq 2$, $n \geq 2$ and $\tau = max\left\{\sum_{r=2}^{n} \frac{1}{\lambda_j}, 1\right\}$. Then,*

$$\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} \geq n^{1-\tau}\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj} - \frac{a_{11}a_{12}\dots a_{1m}}{2\lambda_1}\sum_{j=1}^{m-1}\left[\sum_{r=2}^{n}\left(\frac{a_{rj}^{\lambda_j}}{a_{1j}^{\lambda_j}} - \frac{a_{r(j+1)}^{\lambda_{j+1}}}{a_{1(j+1)}^{\lambda_{j+1}}}\right)\right]^2. \tag{1}$$

The inequality (1) is also valid for $\lambda_m > 0$, $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{m-1} < 0$.

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THEOREM 1.2. *Let* $a_{rj} > 0$ $(r = 1, 2, \dots, n,\ j = 1, 2, \dots, m)$ *and* $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m > 0$ *be such that* $a_{1j}^{\lambda_j} - \sum\limits_{r=2}^{n} a_{rj}^{\lambda_j} > 0$ $(j = 1, 2, \dots, m)$. *Also, let* $m \geq 2$, $n \geq 2$, *and* $\rho = \min\left\{\sum\limits_{j=1}^{m} \frac{1}{\lambda_j}, 1\right\}$. *Then*

$$\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} \leq n^{1-\rho}\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj} - \frac{a_{11}a_{12}\dots a_{1m}}{2\max\left\{\lambda_1, \frac{m-1}{2}\right\}}\sum_{j=1}^{m-1}\left[\sum_{r=2}^{n}\left(\frac{a_{rj}^{\lambda_j}}{a_{1j}^{\lambda_j}} - \frac{a_{r(j+1)}^{\lambda_{j+1}}}{a_{1(j+1)}^{\lambda_{j+1}}}\right)\right]^2.$$

In the same paper [18], the authors proved the following important theorems.

THEOREM 1.3. *Let* $B_j > 0$, $(j = 1, 2, \dots, m)$, $\lambda_m > 0$, $(\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{m-1} < 0)$ *and let* f_j $(j = 1, 2, \dots, m)$ *be positive integrable functions defined on* $[a, b]$ *such that* $\sum\limits_{j=1}^{m} \frac{1}{\lambda_j} = 1$, $m \geq 2$, *and* $B_j^{\lambda_j} - \int\limits_a^b f_j^{\lambda_j}(u)\mathrm{d}u > 0$. *Then*

$$\prod_{j=1}^{m}\left(B_j^{\lambda_j} - \int_a^b f_j^{\lambda_j}(u)\mathrm{d}u\right)^{\frac{1}{\lambda_j}} \geq \prod_{j=1}^{m} B_j - \int_a^b \prod_{j=1}^{m} f_j(u)\mathrm{d}u - \frac{\prod\limits_{j=1}^{m} B_j}{2\lambda_1}\sum_{j=1}^{m-1}\left[\int_a^b\left(\frac{f_j^{\lambda_j}(u)}{B_j^{\lambda_j}} - \frac{f_{j+1}^{\lambda_{j+1}}(u)}{B_{j+1}^{\lambda_{j+1}}}\right)\mathrm{d}u\right]^2.$$

THEOREM 1.4. *Let* $B_j > 0$, $(j = 1, 2, \dots, m)$, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m > 0$ *and let* f_j $(j = 1, 2, \dots, m)$ *be positive integrable functions defined on* $[a, b]$ *such that* $\sum\limits_{j=1}^{m} \frac{1}{\lambda_j} = 1$, $m \geq 2$, *and* $B_j^{\lambda_j} - \int\limits_a^b f_j^{\lambda_j}(u)\mathrm{d}u > 0$. *Then,*

$$\prod_{j=1}^{m}\left(B_j^{\lambda_j} - \int_a^b f_j^{\lambda_j}(u)\mathrm{d}u\right)^{\frac{1}{\lambda_j}} \leq \prod_{j=1}^{m} B_j - \int_a^b \prod_{j=1}^{m} f_j(u)\mathrm{d}u - \frac{B_1B_2\dots B_m}{2\max\left\{\lambda_1, \frac{m-1}{2}\right\}}\sum_{j=1}^{m-1}\left[\int_a^b\left(\frac{f_j^{\lambda_j}(u)}{B_j^{\lambda_j}} - \frac{f_{j+1}^{\lambda_{j+1}}(u)}{B_{j+1}^{\lambda_{j+1}}}\right)\mathrm{d}u\right]^2.$$

In this paper, using the Riemann-Liouville integral operator, we present recent fractional integral results related to the Aczél inequality. Our results are related to the interesting paper [18]. Theorems 3.1 and 3.2 in [18] can be deduced as particular cases of our results.

2. RIEMANN-LIOUVILLE INTEGRATION

In this section, we recall the Riemann-Liouville integral operator with some of its properties that will be used throughout this paper.

DEFINITION 2.1. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function h on $[a, b]$, is defined by

$$J^{\alpha}[h(t)] = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} h(\tau) \mathrm{d}\tau; \quad \alpha > 0, a < t \leq b,$$
$$J^0[h(t)] = h(t),$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-u} u^{\alpha-1} \mathrm{d}u$.

For $t = b$, we have:

$$J^{\alpha}[h(b)] = \frac{1}{\Gamma(\alpha)} \int_a^b (b-\tau)^{\alpha-1} h(\tau) \mathrm{d}\tau.$$

We present the following properties:

$$J^{\alpha} J^{\beta}[h(t)] = J^{\alpha+\beta}[h(t)], \alpha \geq 0, \beta \geq 0,$$

and

$$J^{\alpha} J^{\beta}[h(t)] = J^{\beta} J^{\alpha}[h(t)].$$

For more details on fractional integration, we refer to [8].

3. MAIN RESULTS

We begin by proving the following theorem.

THEOREM 3.1. *For $i = 1, \ldots, m$, we consider $B_i > 0, \lambda_m > 0, \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{m-1} < 0$, and we suppose that f_i are m positive continuous functions on $[a, b]$ such that $\sum_{i=1}^{m} \frac{1}{\lambda_i} = 1$ and $B_i^{\lambda_i} - J_a^{\alpha} f_i^{\lambda_i}(b) > 0$. Then, for every $\alpha > 0$, the following inequality holds:*

$$\prod_{j=1}^{m} \left(B_i^{\lambda_i} - J_a^{\alpha} f_i^{\lambda_i}(b) \right)^{\frac{1}{\lambda_i}} \geq \prod_{i=1}^{m} B_i - J_a^{\alpha} \left(\prod_{i=1}^{m} f_i \right)(b)$$
$$- \frac{1}{2\lambda_1} \prod_{i=1}^{m} B_i \sum_{i=1}^{m-1} \left[\frac{J_a^{\alpha} f_i^{\lambda_i}(b)}{B_i^{\lambda_i}} - \frac{J_a^{\alpha} f_{i+1}^{\lambda_{i+1}}(b)}{B_{i+1}^{\lambda_{i+1}}} \right]^2.$$

Proof. To prove this theorem, we use some ideas inspired from Theorem 1.3. For every positive integer n, we consider $(x_k)_{k=0,1,..,n}$, such that $x_0 < x_1 < x_2 < \ldots < x_k < \ldots < x_{n-1} < x_n$, where, $x_k = a + k\frac{(b-a)}{n}$, for all $k = 0, 1, \ldots, n$. Thanks to the second hypothesis on B_i, for every $i = 1, \ldots, m$, we can write:

$$B_i^{\lambda_i} - \int_a^b \frac{(b-u)^{\alpha-1}}{\Gamma(\alpha)} f_i^{\lambda_i}(u) \mathrm{d}u > 0.$$

Hence $B_i^{\lambda_i} - \int\limits_a^b h_i^{\lambda_i}(u)\mathrm{d}u > 0$, with

$$h_i(u) = \left[\frac{(b-u)^{\alpha-1}}{\Gamma(\alpha)}\right]^{\frac{1}{\lambda_i}} f_i(u). \tag{2}$$

Therefore

$$B_i^{\lambda_i} - \lim_{n\to\infty}\sum_{k=1}^{n} h_i^{\lambda_i}(x_k)\frac{b-a}{n} > 0, i = 1, 2, \ldots, m.$$

So, we can state that there exists $N \in \mathbb{N}$ such that, for all $n > N$ and $i = 1, 2, \ldots, m$, we have $B_i^{\lambda_i} - \sum_{k=1}^{n} h_i^{\lambda_i}(x_k)(\frac{b-a}{n}) > 0$. Theorem 1.3 yields that

$$\prod_{i=1}^{m}\left(B_i^{\lambda_i} - \sum_{k=1}^{n} h_i^{\lambda_i}(x_k)\left(\frac{b-a}{n}\right)\right)^{\frac{1}{\lambda_j}} \geq \prod_{i=1}^{m} B_i - \sum_{k=1}^{n}\prod_{i=1}^{m} h_i(x_k)\left(\frac{b-a}{n}\right)^{\frac{1}{\lambda_i}}$$
$$-\frac{1}{2\lambda_1}\prod_{i=1}^{m} B_i \sum_{i=1}^{m-1}\left[\sum_{k=1}^{n}\left(\frac{h_i^{\lambda_i}(x_k)\left(\frac{b-a}{n}\right)}{B_i^{\lambda_i}} - \frac{h_{i+1}^{\lambda_{i+1}}(x_k)\left(\frac{b-a}{n}\right)}{B_{i+1}^{\lambda_{i+1}}}\right)\right]^2.$$

On the other hand, since $\sum\limits_{i=1}^{m}\frac{1}{\lambda_i} = 1$, we obtain $\prod\limits_{i=1}^{m}\left(\frac{b-a}{n}\right)^{\frac{1}{\lambda_i}} = \frac{b-a}{n}$.

Therefore

$$\prod_{i=1}^{m}\left(B_i^{\lambda_i} - \sum_{k=1}^{n} h_i^{\lambda_i}(x_k)\left(\frac{b-a}{n}\right)\right)^{\frac{1}{\lambda_j}} \geq \prod_{i=1}^{m} B_i - \sum_{k=1}^{n}\prod_{i=1}^{m} h_i(x_k)\left(\frac{b-a}{n}\right)$$
$$-\frac{1}{2\lambda_1}\prod_{i=1}^{m} B_i \sum_{i=1}^{m-1}\left[\sum_{k=1}^{n}\left(\frac{h_i^{\lambda_i}(x_k)\left(\frac{b-a}{n}\right)}{B_i^{\lambda_i}} - \frac{h_{i+1}^{\lambda_{i+1}}(x_k)\left(\frac{b-a}{n}\right)}{B_{i+1}^{\lambda_{i+1}}}\right)\right]^2.$$

Consequently,

$$\prod_{i=1}^{m}\left(B_i^{\lambda_i} - \int_a^b h_i^{\lambda_i}(u)\mathrm{d}u\right)^{\frac{1}{\lambda_j}} \geq \prod_{i=1}^{m} B_i - \int_a^b \prod_{i=1}^{m} h_i(u)\mathrm{d}u$$
$$-\frac{1}{2\lambda_1}\prod_{i=1}^{m} B_i \sum_{i=1}^{m-1}\left[\frac{\int\limits_a^b h_i^{\lambda_i}(u)\mathrm{d}u}{B_i^{\lambda_i}} - \frac{\int\limits_a^b h_{i+1}^{\lambda_{i+1}}(u)\mathrm{d}u}{B_{i+1}^{\lambda_{i+1}}}\right]^2.$$

By (2), we get

$$\prod_{i=1}^{m}\left(B_i^{\lambda_i} - \frac{1}{\Gamma(\alpha)}\int_a^b (b-u)^{\alpha-1} f_i^{\lambda_i}(u)\mathrm{d}u\right)^{\frac{1}{\lambda_j}}$$

$$\geq \prod_{i=1}^{m} B_i - \int_a^b \prod_{i=1}^{m} \left[\frac{(b-u)^{\alpha-1}}{\Gamma(\alpha)}\right]^{\frac{1}{\lambda_i}} \prod_{i=1}^{m} f_i(u)\mathrm{d}u$$
$$-\frac{1}{2\lambda_1}\prod_{i=1}^{m} B_i \sum_{i=1}^{m-1}\left[\frac{1}{B_i^{\lambda_i}\Gamma(\alpha)}\int_a^b (b-u)^{\alpha-1} f_i^{\lambda_i}(u)\mathrm{d}u\right.$$
$$\left.-\frac{1}{B_{i+1}^{\lambda_{i+1}}\Gamma(\alpha)}\int_a^b (b-u)^{\alpha-1} f_{i+1}^{\lambda_{i+1}}(u)\mathrm{d}u\right]^2.$$

Hence

$$\prod_{j=1}^{m}\left(B_i^{\lambda_i} - J_a^\alpha f_i^{\lambda_i}(b)\right)^{\frac{1}{\lambda_i}} \geq \prod_{i=1}^{m} B_i - \int_a^b \prod_{i=1}^{m} \left[\frac{(b-u)^{\alpha-1}}{\Gamma(\alpha)}\right]^{\frac{1}{\lambda_i}} \prod_{i=1}^{m} f_i(u)\mathrm{d}u$$
$$-\frac{1}{2\lambda_1}\prod_{i=1}^{m} B_i \sum_{i=1}^{m-1}\left[\frac{J_a^\alpha f_i^{\lambda_i}(b)}{B_i^{\lambda_i}} - \frac{J_a^\alpha f_{i+1}^{\lambda_{i+1}}(b)}{B_{i+1}^{\lambda_{i+1}}}\right]^2.$$

Using the fact that

$$\prod_{i=1}^{m}\left[\frac{(b-u)^{\alpha-1}}{\Gamma(\alpha)}\right]^{\frac{1}{\lambda_i}} = \left[\frac{(b-u)^{\alpha-1}}{\Gamma(\alpha)}\right]^{\sum_{i=1}^{m}\frac{1}{\lambda_i}} = \frac{(b-u)^{\alpha-1}}{\Gamma(\alpha)},$$

we deduce that

$$\prod_{i=1}^{m}\left(B_i^{\lambda_i} - J_a^\alpha f_i^{\lambda_i}(b)\right)^{\frac{1}{\lambda_i}} \geq \prod_{i=1}^{m} B_i - J_a^\alpha\left(\prod_{i=1}^{m} f_i\right)(b)$$
$$-\frac{1}{2\lambda_1}\prod_{i=1}^{m} B_i \sum_{i=1}^{m-1}\left[\frac{J_a^\alpha f_i^{\lambda_i}(b)}{B_i^{\lambda_i}} - \frac{J_a^\alpha f_{i+1}^{\lambda_{i+1}}(b)}{B_{i+1}^{\lambda_{i+1}}}\right]^2.$$

Theorem 3.1 is thus proved. □

REMARK 3.2. In the above theorem, if we take $\alpha = 1$, we obtain Theorem 1.3 (see [18, Theorem 3.1]).

Changing the conditions on B_i, we present to the reader the following theorem.

THEOREM 3.3. *For $i = 1, \ldots, m$, we consider $B_i > 0, \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m > 0$. Suppose also that f_i are m positive continuous functions on $[a, b]$ such that*

$\sum_{i=1}^{m} \frac{1}{\lambda_i} = 1$, $m \geq 2$. *If* $B_i^{\lambda_i} - J_a^\alpha f_i^{\lambda_i}(b) > 0$, *then, for every* $\alpha > 0$, *we have*

$$\prod_{i=1}^{m} \left(B_i^{\lambda_i} - J_a^\alpha f_i^{\lambda_i}(b) \right)^{\frac{1}{\lambda_i}} \leq \prod_{i=1}^{m} B_i - J_a^\alpha \left(\prod_{i=1}^{m} f_i \right)(b)$$
$$- \frac{\prod_{i=1}^{m} B_i}{2 \max \left\{ \lambda_1, \frac{m-1}{2} \right\}} \sum_{i=1}^{m-1} \left[\frac{J_a^\alpha f_i^{\lambda_i}(b)}{B_i^{\lambda_i}} - \frac{J_a^\alpha f_{i+1}^{\lambda_{i+1}}(b)}{B_{i+1}^{\lambda_{i+1}}} \right]^2 .$$

Proof. We use the same arguments as in the proof of Theorem 3.1. □

The third main result is the following.

Theorem 3.4. *Let* $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m > 0$ *and let* f_i, $i = 1, \ldots, m$, *be* m *positive continuous functions on* $[a, b]$ *with* $\sum_{i=1}^{m} \frac{1}{\lambda_i} = 1$. *Then the inequality*

$$J_a^\alpha \left(\prod_{i=1}^{m} f_i \right)(b) \leq \prod_{i=1}^{m} \left[J_a^\alpha f_i^{\lambda_i}(b) \right]^{\frac{1}{\lambda_i}}$$

holds for all $\alpha > 0$.

Proof. Taking $B_i = \left[2 J_a^\alpha f_i^{\lambda_i}(b) \right]^{\frac{1}{\lambda_i}}$, for $i = 1, 2, \ldots, m$, in Theorem 3.1, we obtain

$$\begin{aligned} \prod_{i=1}^{m} \left(J_a^\alpha f_i^{\lambda_i}(b) \right)^{\frac{1}{\lambda_i}} &\leq \prod_{i=1}^{m} 2^{\frac{1}{\lambda_i}} \prod_{i=1}^{m} \left(J_a^\alpha f_i^{\lambda_i}(b) \right)^{\frac{1}{\lambda_i}} - J_a^\alpha \left(\prod_{i=1}^{m} f_i \right)(b) \\ &= 2 \prod_{i=1}^{m} \left(J_a^\alpha f_i^{\lambda_i}(b) \right)^{\frac{1}{\lambda_i}} - J_a^\alpha \left(\prod_{i=1}^{m} f_i \right)(b). \end{aligned}$$

Therefore

$$J_a^\alpha \left(\prod_{i=1}^{m} f_i \right)(b) \leq \prod_{i=1}^{m} \left(J_a^\alpha f_i^{\lambda_i}(b) \right)^{\frac{1}{\lambda_i}} .$$

This ends the proof. □

Remark 3.5. In the above theorem, if we take $\alpha = 1$, we obtain Theorem 1.4 (see [18, Theorem 3.3]).

We also present to the reader the following result.

Theorem 3.6. *Let* $\lambda_m > 0, \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{m-1} < 0$. *If* f_i, $i = 1, 2, \ldots, m$, *are* m *positive continuous functions on* $[a, b]$ *such that* $\sum_{i=1}^{m} \frac{1}{\lambda_i} = 1$. *Then, for all* $\alpha > 0$, *we have*

$$J_a^\alpha \left(\prod_{i=1}^{m} f_i \right)(b) \geq \prod_{i=1}^{m} \left[J_a^\alpha f_i^{\lambda_i}(b) \right]^{\frac{1}{\lambda_i}} .$$

Proof. The proof of this theorem is similar to the proof of Theorem 3.2. □

Based on Theorems 3.2 and 3.4, we prove the following result.

THEOREM 3.7. *Let $m \geq 2$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m > 0$ and let f_i, $i = 1, 2, \ldots, m-1$ be $m-1$ positive continuous functions defined on $[a, b]$ with $\sum_{i=1}^{m-1} \frac{1}{\lambda_i} < 1$ and $(b-a)^\alpha \geq \Gamma(\alpha+1), \alpha > 0$. Then*

$$\prod_{i=1}^{m-1} \left[\left(J_a^\alpha \frac{1}{f_i^{\lambda_i}}(b) \right) \left(J_a^\alpha f_i^{\lambda_i}(b) \right) \right]^{\frac{1}{\lambda_i}} \geq 1.$$

Proof. Let $0 < \lambda_i \leq \lambda_{i-1}, i = 2, \ldots, m$, such that $\sum_{i=1}^{m} \frac{1}{\lambda_i} = 1$ and let $f_1, \ldots, f_{m-1}$ be positive continuous function defined on $[a, b]$. Thanks to 3.4, for $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m > 0$, we have

$$J_a^\alpha \left(\prod_{i=1}^{m} f_i \right)(b) \leq \prod_{i=1}^{m} \left[J_a^\alpha f_i^{\lambda_i}(b) \right]^{\frac{1}{\lambda_i}}. \tag{3}$$

If we take $\mu_i = -\lambda_i$ $(i = 1, 2, \ldots, m-1)$ and $\mu_m = \frac{\lambda_m}{2\lambda_m - 1}$, we can observe that

$$\sum_{i=1}^{m} \frac{1}{\mu_i} = \sum_{i=1}^{m-1} \frac{-1}{\lambda_i} + \frac{2\lambda_m - 1}{\lambda_m} = -\sum_{i=1}^{m-1} \frac{1}{\lambda_i} + 2 - \frac{1}{\lambda_m} = 1.$$

ALso, we have $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_{m-1}$ and $\frac{1}{\lambda_m} = 1 - \sum_{i=1}^{m-1} \frac{1}{\lambda_i} \in]0, 1[$. Hence $\lambda_m > 1$ and $\mu_m \in]0, 1[$.

Hence, by Theorem 3.4, it yields that

$$J_a^\alpha \left(\prod_{i=1}^{m} f_i \right)(b) \geq \prod_{i=1}^{m-1} \left[J_a^\alpha f_i^{-\lambda_i}(b) \right]^{\frac{-1}{\lambda_i}} \left[J_a^\alpha f_i^{\mu_m}(b) \right]^{\frac{1}{\mu_m}}. \tag{4}$$

Combining (3) and (4), we can state that

$$\prod_{i=1}^{m-1} \left[J_a^\alpha f_i^{-\lambda_i}(b) \right]^{\frac{-1}{\lambda_i}} \left[J_a^\alpha f_m^{\mu_m}(b) \right]^{\frac{1}{\mu_m}} \leq \prod_{i=1}^{m} \left[J_a^\alpha f_i^{\lambda_i}(b) \right]^{\frac{1}{\lambda_i}}.$$

Therefore

$$\prod_{i=1}^{m-1} \left[J_a^\alpha f_i^{-\lambda_i}(b) \right]^{\frac{-1}{\lambda_i}} \left[J_a^\alpha f_m^{\mu_m}(b) \right]^{\frac{1}{\mu_m}} \leq \prod_{i=1}^{m-1} \left[J_a^\alpha f_i^{\lambda_i}(b) \right]^{\frac{1}{\lambda_i}} \left[J_a^\alpha f_m^{\lambda_m}(b) \right]^{\frac{1}{\lambda_m}}.$$

So we have

$$\frac{\left[J_a^\alpha f_m^{\mu_m}(b) \right]^{\frac{1}{\mu_m}}}{\left[J_a^\alpha f_m^{\lambda_m}(b) \right]^{\frac{1}{\lambda_m}}} \leq \prod_{i=1}^{m-1} \left[J_a^\alpha f_i^{-\lambda_i}(b) J_a^\alpha f_i^{\lambda_i}(b) \right]^{\frac{1}{\lambda_i}}.$$

Taking $f_m = 1$, we obtain

$$\frac{\left[J_a^\alpha f_i^{\mu_m}(b)\right]^{\frac{1}{\mu_m}}}{\left[J_a^\alpha f_i^{\lambda_m}(b)\right]^{\frac{1}{\lambda_m}}} = \left(J_a^\alpha 1_{|t=b}\right)^{\left(\frac{1}{\mu_m}-\frac{1}{\lambda_m}\right)} = \left(J_a^\alpha 1_{|t=b}\right)^{\left(2-\frac{1}{\lambda_m}-\frac{1}{\lambda_m}\right)}$$

$$= \left(J_a^\alpha 1_{|t=b}\right)^{2\left(1-\frac{1}{\lambda_m}\right)} = \left[\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\right]^{2\sum_{i=1}^{m-1}\frac{1}{\lambda_i}} \geq 1.$$

Consequently, $\prod_{i=1}^{m-1}\left[J_a^\alpha f_i^{-\lambda_i}(b) J_a^\alpha f_i^{\lambda_i}(b)\right]^{\frac{1}{\lambda_i}} \geq 1.$ □

COROLLARY 3.8. *Let $n \geq 1$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > 0$ and g_i and h_i $(i = 1, 2, \ldots, n)$ be positive continuous functions on $[a, b]$ with $\sum_{i=1}^{n}\frac{1}{\lambda_i} < 1$ and $(b-a)^\alpha \geq \Gamma(\alpha+1)$. Then*

$$\prod_{i=1}^{n}\left[\left(J_a^\alpha\left(\frac{h_i}{g_i}\right)^{\lambda_i}(b)\right)\left(J_a^\alpha\left(\frac{g_i}{h_i}\right)^{\lambda_i}(b)\right)\right]^{\frac{1}{\lambda_i}} \geq 1$$

Proof. We apply Theorem 3.5 with $n = m-1$, $f_i = \frac{g_i}{h_i}$. □

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