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REGULAR MATRIX TRANSFORMATION ON TRIPLE SEQUENCE SPACES-I

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Abstract. The main aim of this article is to establish necessary and sufficient conditions for a six dimensional matrix mapping to be regular on triple sequence spaces.

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Key words. Triple sequence, regular matrix transformation, divergent triple series, convergent sequence.

1. INTRODUCTION AND PRELIMINARIES

Several definitions for giving a value to a divergent simple series, as for example the Cesaro and Holder means, can be expressed by means of a linear transformation defined by an infinite matrix of numbers. Two types of transformations are given, as follows, one by a triangular matrix and the other by a square matrix:

$$T\colon \begin{pmatrix} a_{1,1} & & & & & & \\ a_{2.1} & a_{2,2} & & & & & \\ a_{3,1} & a_{3,2} & a_{3,3} & & & & \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

$$S\colon \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \cdots & \cdots & \cdots \\ a_{2.1} & a_{2,2} & a_{2,3} & a_{2,4} & \cdots & \cdots & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & \cdots & \cdots & \cdots \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

For any given sequence (x_n) a new sequence (y_n) is defined as follows: $y_n = \sum_{k=1}^{n} a_{n,k}x_k$ for the matrix T, and $y_n = \sum_{k=1}^{\infty} a_{n,k}x_k$ for the matrix S,

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provided (y_n) has a meaning, in the later case. If to any matrix of type T we adjoin the elements $a_{n,k} = 0, k > n$ (all n), we obtain a matrix of type S. Since this addition does not affect the transformation, any transformation of the type T may be considered as a special case of transformation of type S. If for either transformation $\lim_{n\to\infty} y_n$ exists, the limit is called the generalized value of the sequence (x_n) given by the corresponding transformation. If (y_n) converges to the same value whenever (x_n) converges, then the transformation is said to be regular. The criterion for the regularity of these transformations is stated as follows.

THEOREM 1.1 ([9]). *A necessary and sufficient condition for the transformation T to be regular is that:*

(a) $\lim_{n\to\infty} a_{n,k} = 0$, *for every* k,
(b) $\lim_{n\to\infty} \sum_{k=1}^{n} a_{n,k} = 1$,
(c) $\sum_{k=1}^{n} |a_{n,k}| < A$, *for all* n.

THEOREM 1.2 ([9]). *A necessary and sufficient condition for the transformation S to be regular is that:*

(a) $\lim_{n\to\infty} a_{n,k} = 0$, *for every* k,
(b) $\sum_{k=1}^{\infty} |a_{n,k}|$ *converges for each* n,
(c) $\sum_{k=1}^{\infty} |a_{n,k}| < A$, *for all* n,
(d) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = 1$.

Regarding the summability of a single series, Robison [9] gave certain definitions for the value of a divergent double series, by considering the double sequence of the series, and established some conditions for regularity of the linear transformations of double sequence spaces.

A triple sequence (real or complex) can be defined as a function $x : N \times N \times N \to R(C)$, where N, R and C denote the sets of natural numbers, real numbers and complex numbers, respectively. Various notions of triple sequences and series and corresponding properties have been studied firstly by Sahiner and Tripathy [13], Savas and Esi [14] and Dutta et. al. [1]. Later on, further generalizations have been studied by Tripathy and Goswami [17], Subramaniam and Esi [15], Esi and Savas [6] and many others. Recently, Debnath et. al. [4] have given a condition for the regularity of a triangular matrix transformation on triple sequence space. As an extension of their work, in this paper we give a necessary and sufficient condition for a square type matrix transformation (S) to be regular.

2. MAIN RESULT

We define the following enumeration for the value of a triple series. Let the series be enumerated as follows:

$$
\begin{aligned}
&u_{1,1,1}+u_{1,1,2}+u_{1,1,3}+u_{1,1,4}+u_{1,1,5}+\dots\\
&+u_{1,2,1}+u_{1,2,2}+u_{1,2,3}+u_{1,2,4}+\dots\\
&+u_{1,3,1}+u_{1,3,2}+u_{1,3,3}+\dots\\
&+\dots\\
&+u_{2,1,1}+u_{2,1,2}+u_{2,1,3}+u_{2,1,4}+u_{2,1,5}+\dots\\
&+u_{2,2,1}+u_{2,2,2}+u_{2,2,3}+u_{2,2,4}+\dots\\
&+u_{2,3,1}+u_{2,3,2}+u_{2,3,3}+\dots\\
&+\dots\\
&+u_{3,1,1}+u_{3,1,2}+u_{3,1,3}+u_{3,1,4}+u_{3,1,5}+\dots\\
&+u_{3,2,1}+u_{3,2,2}+u_{3,2,3}+u_{3,2,4}+\dots\\
&+u_{3,3,1}+u_{3,3,2}+u_{3,3,3}+\dots\\
&+\dots\\
&+u_{4,1,1}+u_{4,1,2}+u_{4,1,3}+u_{4,1,4}+u_{4,1,5}+\dots\\
&+u_{4,2,1}+u_{4,2,2}+u_{4,2,3}+u_{4,2,4}+\dots\\
&+u_{4,3,1}+u_{4,3,2}+u_{4,3,3}+\dots\\
&+\dots
\end{aligned}
$$

Then the triple sequence of partial sums $(x_{l,m,n})$ for this series is given by the equality $x_{l,m,n}=\sum_{p=1,q=1,r=1}^{l,m,n} u_{p,q,r}$. Thus we have the following recurrence relations:

$$
\begin{aligned}
u_{l,m,n} &= (x_{l,m,n}+x_{l,m-1,n-1}-x_{l,m,n-1}-x_{l,m-1,n})\\
&\quad -(x_{l-1,m,n}+x_{l-1,m-1,n-1}-x_{l-1,m,n-1}-x_{l-1,m-1,n}) \quad (l,m,n>1);\\
u_{l,m,1} &= (x_{l,m,1}-x_{l,m-1,1})-(x_{l-1,m,1}-x_{l-1,m-1,1}) \quad (l,m>1);\\
u_{l,1,n} &= (x_{l,1,n}-x_{l,1,n-1})-(x_{l-1,1,n}-x_{l-1,1,n-1}) \quad (l,n>1);\\
u_{1,m,n} &= (x_{1,m,n}-x_{1,m,n-1})-(x_{1,m-1,n}-x_{1,m-1,n-1}) \quad (m,n>1);\\
u_{l,1,1} &= x_{l,1,1}-x_{l-1,1,1} \quad (l>1);\\
u_{1,m,1} &= x_{1,m,1}-x_{1,m-1,1} \quad (m>1);\\
u_{1,1,n} &= x_{1,1,n}-x_{1,1,n-1} \quad (n>1);\\
u_{1,1,1} &= x_{1,1,1}.
\end{aligned}
$$

Now we define a new sequence by the relation

$$
y_{l,m,n}=\sum_{p=1}^{l}\sum_{q=1}^{m}\sum_{r=1}^{n} a_{l,m,n,p,q,r}x_{p,q,r}.
$$

We say that this transformation and its matrix $A : (a_{l,m,n,p,q,r})$ are of type T. Taking the limit to infinity in the above, we get:

$$y_{l,m,n} = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{r=1}^{\infty} a_{l,m,n,p,q,r}x_{p,q,r}.$$

We say that this transformation and its matrix $A : (a_{l,m,n,p,q,r})$ are of type S, where p, q and r take all positive integral values. Any transformation of type T may be considered as a special case of a transformation of type S, by adding the elements:

(i) $a_{l,m,n,p,q,r} = 0$, $1 \le p \le l, 1 \le q \le m, n < r$, for all l, m and n,
(ii) $a_{l,m,n,p,q,r} = 0$, $1 \le p \le l, m < q, 1 \le r \le n$, for all l, m and n,
(iii) $a_{l,m,n,p,q,r} = 0$, $1 \le p \le l, m < q, n < r$, for all l, m and n,
(iv) $a_{l,m,n,p,q,r} = 0$, $l < p, 1 \le q \le m, 1 \le r \le n$, for all l, m and n,
(v) $a_{l,m,n,p,q,r} = 0$, $l < p, 1 \le q \le m, n < r$, for all l, m and n,
(vi) $a_{l,m,n,p,q,r} = 0$, $l < p, m < q, 1 \le r \le n$, for all l, m and n,
(vii) $a_{l,m,n,p,q,r} = 0$, $l < p, m < q, n < r$, for all l, m and n.

Given any matrix of type T we can obtain a matrix of type S such that the resulting transformation is identical with the original one. For, either type of transformation $(y_{l,m,n})$ possesses a limit, which is called the generalized value of the sequence $(x_{l,m,n})$ given by the transformation.

It is well known that if a simple series converges, the corresponding sequence is bounded. G. M. Robison (1926) proved that the above result does not hold for a double series. In this paper we prove this for a triple sequence, i.e. if a triple series converges, the corresponding triple sequence may not be bounded. To prove this, we consider the following example.

Consider the series $u_{1,1,n} = 1, u_{2,1,n} = -1$ and $u_{l,m,n} = 0$ $(l > 2, m > 1)$. This series converges, but the corresponding sequence is not bounded. Thus the convergent triple series may be divided into two classes according to whether the corresponding sequences are bounded or not. The following definition of the regularity of a transformation is constructed with regard to a convergent bounded sequence; even if a transformation is regular, it does not need to be given by an unbounded convergent sequence, the value to which it converges.

A transformation of a triple sequence is regular, if whenever $(x_{l,m,n})$ is a bounded convergent sequence to L, $(y_{l,m,n})$ also converges to L.

Lemma 2.1. *If $\sum_{p=1,q=1,r=1}^{\infty,\infty,\infty} a_{l,m,n}$ is not absolutely convergent, it is possible to find a sequence $(x_{l,m,n})$ which is bounded and convergent to zero and $\sum_{p=1,q=1,r=1}^{\infty,\infty,\infty} a_{l,m,n}x_{l,m,n}$ diverges to $+\infty$.*

Proof. The proof is simple. □

In this article we establish necessary and sufficient conditions for a six dimensional matrix map to be regular on triple sequence spaces, as follows.

Theorem 2.2. *The transformation S is regular if and only if the following conditions hold:*

(a) $\lim_{l,m,n\to\infty} a_{l,m,n,p,q,r} = 0$, *for every p, q and r;*
(b) $\sum_{p=1,q=1,r=1}^{\infty,\infty,\infty} |a_{l,m,n,p,q,r}|$ *converges, for every l, m and n;*
(c) $\lim_{l,m,n\to\infty} \sum_{p=1,q=1,r=1}^{\infty,\infty,\infty} a_{l,m,n,p,q,r} = 1$*;*
(d) $\lim_{l,m,n\to\infty} \sum_{p=1}^{\infty} | a_{l,m,n,p,q,r} |= 0$ *for every q and r;*
(e) $\lim_{l,m,n\to\infty} \sum_{q=1}^{\infty} | a_{l,m,n,p,q,r} |= 0$ *for every p and r;*
(f) $\lim_{l,m,n\to\infty} \sum_{r=1}^{\infty} | a_{l,m,n,p,q,r} |= 0$ *for every p and q;*
(g) $\sum_{p=1,q=1,r=1}^{\infty,\infty,\infty} | a_{l,m,n,p,q,r} |\le A$, *where A is a constant.*

Proof. Proof of necessity:

(a) To show the necessity of condition (a), consider a sequence $(x_{l,m,n})$ as follows:

$$x_{l,m,n} = \begin{cases} 1, & \text{if } l = i, m = j, n = k, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\lim_{l,m,n\to\infty} x_{l,m,n} = 0$ and $y_{l,m,n} = a_{l,m,n,i,j,k}$. Therefore, in order to have $\lim_{l,m,n\to\infty} y_{l,m,n} = 0$, it is necessary that $\lim_{l,m,n\to\infty} a_{l,m,n,i,j,k} = 0$, for every i, j and k. Thus condition (a) is necessary.

(b) Assume $\sum_{p=1,q=1,r=1}^{\infty,\infty,\infty} |a_{l,m,n,p,q,r}|$ diverges for some fixed values of l, m and n. Then there exist a sequence $(x_{l,m,n})$ which is bounded and converges to zero (by Lemma 2.1). Thus $\sum_{p=1,q=1,r=1}^{\infty,\infty,\infty} a_{l,m,n,p,q,r}x_{p,q,r}$ diverges. This contradicts our assumption. Hence condition (b) is necessary.

(c) We consider the sequence $(x_{l,m,n})$ defined by $x_{l,m,n} = 1$, for all l, m and n. Then the sequence $(y_{l,m,n})$ becomes $y_{l,m,n} = \sum_{p=1,q=1,r=1}^{\infty,\infty,\infty} a_{l,m,n,p,q,r}$. Since $\lim_{l,m,n\to\infty} y_{l,m,n} = 1$, condition (b) is necessary.

(d) To prove the necessity of condition (c), we assume that condition (a) is satisfied and (c) is not, in order to get a contradiction. Since we are assuming that, for $r = r_0$ (some fixed integer), the sequence $\sum_{p=1}^{\infty} |a_{l,m,n,p,q,r}|$ does not approach zero, for some preassigned constant $k > 0$, there exists a subsequence of this sequence, such that each element of it is greater than k. Choose arbitrarily l_1, m_1, n_1 and r_1. Now choose $l_2 > l_1$, $m_2 > m_1$ and $n_2 > n_1$ such that

$$\sum_{p=1}^{r_1} |a_{l_1,m_1,n_1,p,q,r_0}| \le k/8.$$

By condition (a), we have

$$\sum_{p=1}^{\infty} |a_{l_1,m_1,n_1,p,q,r_0}| \ge k.$$

Further, for $r_2 > r_1$, we get

$$\sum_{p=r_2+1}^{\infty} |a_{l_1,m_1,n_1,p,q,r_0}| \le k/8,$$

from condition (b). Further, choose $l_t > l_{t-1}$, $m_t > m_{t-1}$ and $n_t > n_{t-1}$ such that

$$\sum_{p=1}^{l_{t-1}} |a_{l_t,m_t,n_t,p,q,r_0}| \leq k/8.$$

From condition (a), we have

$$\sum_{p=1}^{\infty} |a_{l_t,m_t,n_t,p,q,r_0}| \geq k. \tag{1}$$

For $r_t > r_{t-1}$, we get

$$\sum_{p=r_{t-1}+1}^{\infty} |a_{l_1,m_1,n_1,p,q,r_0}| \geq k/8, \tag{2}$$

from condition (b). From conditions (1) and (2) we get the following:

$$\sum_{p=r_{t-1}+1}^{r_t} |a_{l_1,m_1,n_1,p,q,r_0}| \geq 3k/8. \tag{3}$$

Consider the sequence $(x_{l,m,n})$ defined as follows:

$$\begin{aligned} x_{l,m,n} &= 0, n \neq r_0; \\ x_{l,m,n} &= sgn\, a_{l_1,m_1,n_1,p,q,r_0}, l \leq l_1; \\ x_{l,m,n} &= sgn\, a_{l_2,m_2,n_2,p,q,r_0}, l_1 < l \leq l_2; \\ &\dots \\ x_{l,m,n} &= sgn\, a_{l_t,m_t,n_t,p,q,r_0}, l_{t-1} < l \leq l_t \end{aligned} \tag{4}$$

Sine $\lim_{l,m,n\to\infty} x_{l,m,n} = 0$, we have, from (1) and (2):

$$\left|\sum_{p=1}^{r_{t-1}} a_{l_t,m_t,n_t,p,q,r_0} x_{p,q,r_0}\right| \leq \sum_{p=1}^{r_{t-1}} |a_{l_t,m_t,n_t,p,q,r_0}| \leq k/8.$$

Now, from (3) and (4), it follows that

$$\sum_{p=r_{t-1}+1}^{r_t} a_{l_t,m_t,n_t,p,q,r_0} x_{p,q,r_0} = \sum_{p=r_{t-1}+1}^{r_t} |a_{l_t,m_t,n_t,p,q,r_0}| \geq 3k/8.$$

Therefore,

$$\begin{aligned} |y_{l_t,m_t,n_t}| &= \left|\sum_{p=1}^{r_{t-1}} a_{l_t,m_t,n_t,p,q,r_0} x_{p,q,r_0}\right| \\ &\geq \sum_{p=r_{t-1}+1}^{r_t} |a_{l_t,m_t,n_t,p,q,r_0} x_{p,q,r_0}| - \left|\sum_{p=1}^{r_{t-1}} a_{l_t,m_t,n_t,p,q,r_0} x_{p,q,r_0}\right| \end{aligned}$$

$$- \left| \sum_{p=r_t+1}^{\infty} a_{l_t,m_t,n_t,p,q,r_0} x_{p,q,r_0} \right|$$
$$\geq \frac{3k}{8} - \frac{k}{8} - \frac{k}{8} = \frac{k}{2}.$$

Thus $(y_{l,m,n})$ does not have the limit equal to zero, from which we deduce that the condition (d) is necessary.

(e) The above proof can be used to show the necessity of condition (e).

(f) In a similar way, one proves the necessity of condition (f).

(g) Suppose that conditions (a) and (b) are satisfied and condition (f) is not. We choose arbitrarily l_1 , m_1, n_1, u_1, v_1 and w_1. Further, choose $l_2 > l_1$, $m_2 > m_1$ and $n_2 > n_1$ such that

$$\sum_{p=1,q=1,r=1}^{l_1,m_1,n_1} |a_{l_2,m_2,n_2,p,q,r}| \leq 2.$$

From condition (a), we have

$$\sum_{p=1,q=1,r=1}^{l_2,m_2,n_2} |a_{l_2,m_2,n_2,p,q,r}| \geq 2^4.$$

Choose $u_2 > u_1$, $v_2 > v_1$ and $w_2 > w_1$ such that

$$\sum_{p=u_2+1,q=v_2+1,r=w_2+1}^{\infty,\infty,\infty} |a_{l_2,m_2,n_2,p,q,r}| + \sum_{p=1,q=v_2+1,r=w_2+1}^{u_2,\infty,\infty} |a_{l_2,m_2,n_2,p,q,r}|$$
$$+ \sum_{p=u_2+1,q=1,r=w_2+1}^{\infty,v_2,\infty} |a_{l_2,m_2,n_2,p,q,r}| + \sum_{p=u_2+1,q=v_2+1,r=1}^{\infty,\infty,w_2} |a_{l_2,m_2,n_2,p,q,r}| \leq 2^2,$$

taking into account condition (b). Choose $l_3 > l_2$, $m_3 > m_2$ and $n_3 > n_2$ and note that

$$\sum_{p=1,q=1,r=1}^{l_2,m_2,n_2} |a_{l_3,m_3,n_3,p,q,r}| \leq 2^2, \quad \sum_{p=1,q=1,r=1}^{\infty,\infty,\infty} |a_{l_3,m_3,n_3,p,q,r}| \geq 2^6.$$

Choose $u_3 > u_2$, $v_3 > v_2$ and $w_3 > w_2$ and note that

$$\sum_{p=u_3+1,q=v_3+1,r=w_3+1}^{\infty,\infty,\infty} |a_{l_3,m_3,n_3,p,q,r}| + \sum_{p=1,q=v_3+1,r=w_3+1}^{u_3,\infty,\infty} |a_{l_3,m_3,n_3,p,q,r}|$$
$$+ \sum_{p=u_3+1,q=1,r=w_3+1}^{\infty,v_3,\infty} |a_{l_3,m_3,n_3,p,q,r}| + \sum_{p=u_3+1,q=v_3+1,r=1}^{\infty,\infty,w_3} |a_{l_3,m_3,n_3,p,q,r}| \leq 2^4.$$

Choose arbitrarily $l_t > l_{t-1}$, $m_t > m_{t-1}$ and $n_t > n_{t-1}$ such that

$$\sum_{p=1,q=1,r=1}^{l_{t-1},m_{t-1},n_{t-1}} |a_{l_t,m_t,n_t,p,q,r}| \leq 2^{t-1}, \quad \sum_{p=1,q=1,r=1}^{\infty,\infty,\infty} |a_{l_t,m_t,n_t,p,q,r}| \geq 2^{2t}. \tag{5}$$

Choose $u_t > u_{t-1}$, $v_t > v_{t-1}$ and $w_t > w_{t-1}$ and note that

$$\begin{aligned} &\sum_{p=u_t+1,q=v_t+1,r=w_t+1}^{\infty,\infty,\infty} |a_{l_t,m_t,n_t,p,q,r}| \\ &+ \sum_{p=1,q=v_t+1,r=w_t+1}^{u_t,\infty,\infty} |a_{l_t,m_t,n_t,p,q,r}| \\ &+ \sum_{p=u_t+1,q=1,r=w_t+1}^{\infty,v_t,\infty} |a_{l_t,m_t,n_t,p,q,r}| \\ &+ \sum_{p=u_t+1,q=v_t+1,r=1}^{\infty,\infty,w_t} |a_{l_t,m_t,n_t,p,q,r}| \leq 2^{2t-2}. \end{aligned} \tag{6}$$

Now, from inequalities (5) and (6), we obtain

$$\begin{aligned} &\sum_{p=u_{t-1}+1,q=v_{t-1}+1,r=w_{t-1}+1}^{u_t,v_t,w_t} |a_{l_t,m_t,n_t,p,q,r}| \\ &+ \sum_{p=1,q=v_{t-1}+1,r=w_{t-1}+1}^{u_{t-1},v_t,w_t} |a_{l_t,m_t,n_t,p,q,r}| \\ &+ \sum_{p=u_{t-1}+1,q=1,r=w_{t-1}+1}^{u_t,v_{t-1},w_t} |a_{l_t,m_t,n_t,p,q,r}| \\ &+ \sum_{p=u_{t-1}+1,q=v_{t-1}+1,r=1}^{u_t,v_t,w_{t-1}} |a_{l_t,m_t,n_t,p,q,r}| \geq 2^{2t-1}. \end{aligned} \tag{7}$$

Consider the sequence $(x_{l,m,n})$ defined as follows:

$$\begin{aligned} &x_{l,m,n} = \operatorname{sgn} a_{l_1,m_1,n_1,p,q,r}, \quad p \leq l_1, q \leq m_1, r \leq n_1; \\ &x_{l,m,n} = 1/2 \operatorname{sgn} a_{l_2,m_2,n_2,p,q,r}, \quad 1 \leq p \leq l_1, 1 \leq q \leq m_1, 1 \leq r \leq n_1, \\ &\qquad l_1 < p \leq l_2, m_1 < q \leq m_2, n_1 < r \leq n_2; \\ &\quad \dots \\ &x_{l,m,n} = 1/2^{t-1} \operatorname{sgn} a_{l_t,m_t,n_t,p,q,r}, \quad 1 \leq p \leq l_{t-1}, 1 \leq q \leq m_{t-1}, \\ &\qquad 1 \leq r \leq n_{t-1}, l_{t-1} < p \leq l_t, m_{t-1} < q \leq m_t, n_{t-1} < r \leq n_t. \end{aligned} \tag{8}$$

Now, from inequalities (5), (6), (7) and (8), we obtain

$$\left|\sum_{p=1,q=1,r=1}^{l_{t-1},m_{t-1},n_{t-1}} a_{l_t,m_t,n_t,p,q,r}x_{p,q,r}\right| \leq \sum_{p=1,q=1,r=1}^{l_{t-1},m_{t-1},n_{t-1}} |a_{l_t,m_t,n_t,p,q,r}| \leq 2^{t-1}, \tag{9}$$

$$\begin{aligned}
&\sum_{p=u_{t-1}+1,q=v_{t-1}+1,r=w_{t-1}+1}^{l_t,m_t,n_t} a_{l_t,m_t,n_t,p,q,r}x_{p,q,r} \\
&\quad + \sum_{p=1,q=v_{t-1}+1,r=w_{t-1}+1}^{l_{t-1},m_t,n_t} a_{l_t,m_t,n_t,p,q,r}x_{p,q,r} \\
&\quad + \sum_{p=u_{t-1}+1,q=1,r=w_{t-1}+1}^{l_t,m_{t-1},n_t} a_{l_t,m_t,n_t,p,q,r}x_{p,q,r} \\
&\quad + \sum_{p=u_{t-1}+1,q=v_{t-1}+1,r=1}^{l_t,m_t,n_{t-1}} a_{l_t,m_t,n_t,p,q,r}x_{p,q,r} \\
&= \frac{1}{2^{t-1}}\left[\sum_{p=u_{t-1}+1,q=v_{t-1}+1,r=w_{t-1}+1}^{l_t,m_t,n_t} |a_{l_t,m_t,n_t,p,q,r}|\right. \\
&\quad + \sum_{p=1,q=v_{t-1}+1,r=w_{t-1}+1}^{l_{t-1},m_t,n_t} |a_{l_t,m_t,n_t,p,q,r}| \\
&\quad + \sum_{p=u_{t-1}+1,q=1,r=w_{t-1}+1}^{l_t,m_{t-1},n_t} |a_{l_t,m_t,n_t,p,q,r}| \\
&\quad \left. + \sum_{p=u_{t-1}+1,q=v_{t-1}+1,r=1}^{l_t,m_t,n_{t-1}} |a_{l_t,m_t,n_t,p,q,r}|\right] \geq \frac{1}{2^{t-1}}2^{2t-1} = 2^t,
\end{aligned} \tag{10}$$

$$\begin{aligned}
&\left|\sum_{p=l_t+1,q=m_t+1,r=n_t+1}^{\infty,\infty,\infty} a_{l_t,m_t,n_t,p,q,r}x_{p,q,r}\right. \\
&\quad + \sum_{p=1,q=m_t+1,r=n_t+1}^{l_t,\infty,\infty} a_{l_t,m_t,n_t,p,q,r}x_{p,q,r} \\
&\quad + \sum_{p=l_t+1,q=1,r=n_t+1}^{\infty,m_t,\infty} a_{l_t,m_t,n_t,p,q,r}x_{p,q,r} \\
&\quad \left. + \sum_{p=l_t+1,q=m_t+1,r=1}^{\infty,\infty,n_t} a_{l_t,m_t,n_t,p,q,r}x_{p,q,r}\right|
\end{aligned}$$

$$\leq \frac{1}{2^t}\left[\sum_{p=l_t+1,q=m_t+1,r=n_t+1}^{\infty,\infty,\infty}|a_{l_t,m_t,n_t,p,q,r}| + \sum_{p=1,q=m_t+1,r=n_t+1}^{l_t,\infty,\infty}|a_{l_t,m_t,n_t,p,q,r}| + \sum_{p=l_t+1,q=1,r=n_t+1}^{\infty,m_t,\infty}|a_{l_t,m_t,n_t,p,q,r}| + \sum_{p=l_t+1,q=m_t+1,r=1}^{\infty,\infty,n_t}|a_{l_t,m_t,n_t,p,q,r}|\right] \leq \frac{1}{2^t}2^{2t-2} = 2^{t-2}. \tag{11}$$

Now, we consider the sequence (y_{l_t,m_t,n_t}) and, applying the results from (10) and (11), we get the following

$$|y_{l_t,m_t,n_t}| = \left|\sum_{p=1,q=1,r=1}^{\infty,\infty,\infty} a_{l_t,m_t,n_t,p,q,r}x_{p,q,r}\right| + \left[\sum_{p=u_{t-1}+1,q=v_{t-1}+1,r=w_{t-1}+1}^{l_t,m_t,n_t} a_{l_t,m_t,n_t,p,q,r}x_{p,q,r} + \sum_{p=1,q=v_{t-1}+1,r=w_{t-1}+1}^{l_{t-1},m_t,n_t} a_{l_t,m_t,n_t,p,q,r}x_{p,q,r} + \sum_{p=u_{t-1}+1,q=1,r=w_{t-1}+1}^{l_t,m_{t-1},n_t} a_{l_t,m_t,n_t,p,q,r}x_{p,q,r} + \sum_{p=u_{t-1}+1,q=v_{t-1}+1,r=1}^{l_t,m_t,n_{t-1}} a_{l_t,m_t,n_t,p,q,r}x_{p,q,r}\right] - \left[\sum_{p=l_t+1,q=m_t+1,r=n_t+1}^{\infty,\infty,\infty} a_{l_t,m_t,n_t,p,q,r}x_{p,q,r} + \sum_{p=1,q=m_t+1,r=n_t+1}^{l_t,\infty,\infty} a_{l_t,m_t,n_t,p,q,r}x_{p,q,r} + \sum_{p=l_t+1,q=1,r=n_t+1}^{\infty,m_t,\infty} a_{l_t,m_t,n_t,p,q,r}x_{p,q,r} + \sum_{p=l_t+1,q=m_t+1,r=1}^{\infty,\infty,n_t} a_{l_t,m_t,n_t,p,q,r}x_{p,q,r}\right]$$

$$+\left|\sum_{p=1,q=1,r=1}^{l_t,m_t,n_t} a_{l_t,m_t,n_t,p,q,r}x_{p,q,r}\right| \geq 2^t-2^{t-1}-2^{t-2}=2^{t-2}[4-2-1]=2^{t-2}.$$

Therefore $\lim_{t\to\infty}|y_{l_t,m_t,n_t}| = \infty$. Since this subsequence of the sequence $(y_{l,m,n})$ does not converge, the sequence $(y_{l,m,n})$ has no finite limit and thus we have proved that condition (g) is necessary.

Proof of sufficiency:

Let the limit of the convergent sequence $(x_{l,m,n})$ be x. Then $y_{l,m,n} - x = \sum_{p=1,q=1,r=1}^{\infty,\infty,\infty} a_{l,m,n,p,q,r}x_{p,q,r} - x$. Using condition (c), we can write

$$\sum_{p=1,q=1,r=1}^{\infty,\infty,\infty} a_{l,m,n,p,q,r}x_{p,q,r} + z_{l,m,n} = 1,$$

where $\lim_{l,m,n\to\infty} z_{l,m,n} = 0$. Therefore

$$y_{l,m,n} - x = \sum_{p=1,q=1,r=1}^{\infty,\infty,\infty} a_{l,m,n,p,q,r}(x_{p,q,r} - x) - z_{l,m,n}x,$$

and thus

$$\begin{aligned}
\mid y_{l,m,n} - x \mid \leq& \mid \sum_{p=1,q=1,r=1}^{u,v,w} a_{l,m,n,p,q,r}x_{p,q,r} \mid \\
&+ \mid \sum_{p=1,q=1,r=w+1}^{u,v,\infty} a_{l,m,n,p,q,r}x_{p,q,r} \mid \\
&+ \mid \sum_{p=1,q=v+1,r=1}^{u,\infty,w} a_{l,m,n,p,q,r}x_{p,q,r} \mid \\
&+ \mid \sum_{p=u+1,q=1,r=1}^{\infty,v,w} a_{l,m,n,p,q,r}x_{p,q,r} \mid \\
&+ \mid \sum_{p=1,q=v+1,r=w+1}^{u,\infty,\infty} a_{l,m,n,p,q,r}x_{p,q,r} \mid \\
&+ \mid \sum_{p=u+1,q=1,r=w+1}^{\infty,v,\infty} a_{l,m,n,p,q,r}x_{p,q,r} \mid \\
&+ \mid \sum_{p=u+1,q=v+1,r=1}^{\infty,\infty,w} a_{l,m,n,p,q,r}x_{p,q,r} \mid \\
&+ \mid \sum_{p=u+1,q=v+1,r=w+1}^{\infty,\infty,\infty} a_{l,m,n,p,q,r}x_{p,q,r} \mid + \mid z_{l,m,n}x \mid .
\end{aligned} \tag{12}$$

Since we have $x_{l,m,n} \to x$, we can choose u, v and w so large that, for any preassigned small constant ϵ, we have

$$| x_{l,m,n} - x |< \epsilon/9A, \text{ whenever } p \geq u, q \geq v, r \geq w. \tag{13}$$

Now, let H be the largest number of $| x_{l,m,n} - x |$, for p, q and r. We choose three integers L, M and N such that, whenever $l \geq L$, $m \geq M$, $n \geq N$, the following inequalities are satisfied

$$\begin{aligned}
\sum_{p=1,q=1,r=1}^{u,v,w} | a_{l,m,n,p,q,r} | &< \frac{\epsilon}{9uvwH}, \\
\sum_{p=1}^{\infty} | a_{l,m,n,p,q,r} | &< \frac{\epsilon}{9vwH}, q = 1, 2, \ldots, v; r = 1, 2, \ldots, w, \\
\sum_{q=1}^{\infty} | a_{l,m,n,p,q,r} | &< \frac{\epsilon}{9uwH}, p = 1, 2, \ldots, u; r = 1, 2, \ldots, w, \\
\sum_{r=1}^{\infty} | a_{l,m,n,p,q,r} | &< \frac{\epsilon}{9uvH}, p = 1, 2, \ldots, u; q = 1, 2, \ldots, v, \\
|z_{l,m,n}| &< \frac{\epsilon}{9 \mid x \mid}.
\end{aligned} \tag{14}$$

Hence, whenever $l \geq L$, $m \geq M$, $n \geq N$, we have $|y_{l,m,n} - x| \leq \epsilon$, using the results from (12), (13) and (14). Thus, $\lim_{l,m,n\to\infty} y_{l,m,n} - x = 0$ or, $y_{l,m,n} \to x$. Hence the theorem is proved. □

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