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RINGS WHOSE UNITS COMMUTE WITH NILPOTENT ELEMENTS

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Abstract. Rings with the property in the title are studied under the name of "uni" rings. These are compared with other known classes of rings and, since commutative rings and reduced rings trivially have this property, conditions which added to uni rings imply commutativity or reduceness are found.

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1. INTRODUCTION

In any unital ring R, the three sets $U(R)$, $Id(R)$, and $N(R)$, which denote, respectively, the unit group, the set of idempotents, and the set of nilpotent elements in R are of utmost importance. In the last four decades, an additive theory has emerged in the study of these three sets. In this note we investigate classes of rings in which elements of two of these sets commute. As usually $Z(R)$ denotes the center of the ring R.

We can immediately discard two possibilities: rings whose *idempotents commute with nilpotents* or *idempotents commute with units* turn out (see [10, Ex. 12.7]) to be precisely the so called Abelian rings (i.e., rings with only central idempotents).

In the sequel we study the remaining possible commutation. We need the following definitions.

A ring will be called *uni* ring [**un**it-**ni**lpotent] if units commute with nilpotents. In order to place such rings in some known environment recall that a ring is *unit-central* if $U(R) \subseteq Z(R)$ and *nilpotent-central* if $N(R) \subseteq Z(R)$, the analogues (by definition) of Abelian rings. The first class was studied in [9] and the second in [7] (as *CN*-rings) and (rediscovered as *central reduced* rings) in [13]. We just mention that there is a great deal of work on central units for integral group rings (see [8] for a comprehensive bibliography).

Finally, rings *with commuting units* and rings *with commuting nilpotents* (used in [4] and [9]) may be considered. The first were studied by various authors as rings with Abelian group of units.

Recall that for every nilpotent element t in a ring with identity R, $1 + t$ is a unit which we call *unipotent*. Via unipotent elements, the following chart is

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readily checked (DF is for Dedekind finite)

$$\begin{array}{ccccccc} & & \text{commuting} - \text{units} & & & & \\ & \nearrow & & \searrow & & & \\ \text{unit} - \text{central} & \rightarrow & \text{nilp} - \text{central} & \rightarrow & \text{uni} & \rightarrow & \text{commuting} - \text{nilp} \\ & & \downarrow & & & & \downarrow \\ & & \text{Abelian} & & \longrightarrow & & \text{DF} \end{array}$$

In the above chart, classes not united by arrows are independent and none of the arrows is reversible. This is shown by examples in the first section.

It is readily seen that none of these commutation properties passes to matrix rings. Therefore none of this properties are Morita invariant.

However, corners, products, centers, quotients (properties preserved by homomorphisms) are studied in the second section.

All these definitions are generalizations of commutative or reduced rings. Notice that unit central rings and rings with commuting units may not be reduced; however, reduced rings are obviously nilpotent central.

Finally an important matter is to recapture commutativity or the lack of nonzero nilpotent elements, by combining this generalization (the uni rings) with several known conditions. This is done in sections 3 and 4. Two open questions are also stated.

2. EXAMPLES

We first use an example in [9], for a *uni ring or ring with commuting nilpotents which is not Abelian.* Let $R = \begin{bmatrix} \mathbf{F}_2 & V \\ 0 & \mathbf{F}_2 \end{bmatrix}$ where V is any nonzero $\mathbf{F}_2$-vector space. Then R is a semiprimary ring with commuting units which is not Abelian (indeed, for two vectors $v, w \in V$, $\begin{bmatrix} 1 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & w \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & w \\ 0 & 0 \end{bmatrix} \neq 0_2 = \begin{bmatrix} 0 & w \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & v \\ 0 & 0 \end{bmatrix}$) and so not nilpotent central nor unit central. More, it is uni (indeed, with previous notations, $\begin{bmatrix} 1 & v \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & w \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & w \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & w \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & v \\ 0 & 1 \end{bmatrix}$), and so with commuting nilpotents.

This way we have also covered, *commuting units which is not unit central and uni which is not nilpotent central.*

Next, consider the ring

$$R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a \equiv d \pmod 2, b \equiv c \equiv 0 \pmod 2 \right\}$$

of integral matrices. Since it has only trivial idempotents, *R is Abelian*. On the other hand, $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \in N(R)$, $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \in U(R)$ do not commute so R is *not uni*. More, $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \in N(R)$ do not commute.

Rings R with $U(R) = \{1\}$ (and so $N(R) = \{0\}$) are trivially included in all the classes in the above chart. Every division ring which is not a field *is trivially nilpotent central (and so uni) but is not unit central (and so has not commuting units)*.

Further, any matrix ring over any stably finite ring is Dedekind finite but may not have only commuting nilpotents. Indeed,

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

In searching for examples notice that the following three conditions are equivalent for any ring:

(i) nilpotent elements commute;

(ii) unipotent elements commute with nilpotent elements;

(iii) unipotent elements commute.

If any of these holds, $N(R)$ is a subring of R.

Recall (see [1]) that a ring is UU if every unit is unipotent.

As a by-product of this equivalence notice that *for UU rings, commuting units, uni and commuting nilpotents are equivalent conditions.*

Hence for an example of *ring with commuting nilpotents which is not uni*, units which are not unipotent must be used. Consider upper triangular 2×2 matrices over a reduced ring of characteristic $\neq 2$, $\mathcal{T}_2(R)$. Then $N(\mathcal{T}_2(R)) = \begin{bmatrix} 0 & R \\ 0 & 0 \end{bmatrix}$ is a zero-square ring and so trivially with *commuting nilpotents.* However it is *not uni*:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

In closing this section we state the following open

QUESTION. *Are uni Abelian rings, nilpotent-central* ?

3. UNITS COMMUTE WITH NILPOTENTS

A ring with identity R is called *uni* if for every $u \in U(R)$ and every $t \in N(R) : ut = tu$. In such rings every sum $t + u$ is a unit, i.e., $N(R) + U(R) = U(R)$. Moreover $U(R)N(R) = N(R)U(R) = N(R)$. Since $1 + N(R) \subseteq U(R)$, in any uni ring *nilpotents commute* and so $N(R)$ forms a subring.

Since $1 + J(R) \subseteq U(R)$, in any uni ring, *nilpotents commute with the elements in the Jacobson radical.*

The following properties follow from definitions. A nonzero ring was called *fine* (see [2]) if every nonzero element is a sum of a unit and a nilpotent element.

Proposition 3.1. (i) *Commutative rings and reduced rings (e.g., domains, division rings) are uni.*

(ii) *Matrix rings (and so split extensions of uni rings) are not uni. Hence uni is not a Morita invariant property.*

(iii) *Products of uni rings are uni.*

(iv) *Fine uni rings are division rings.*

(v) *A UU-ring is uni iff nilpotents commute.*

(vi) *Any uni ring (additively) generated by units is nilpotent central.*

Also immediate is the following

Remark 3.2. Let R be a ring such that $R-U(R)=N(R)$ (i.e. a local ring with nil radical). Then

(i) if R is uni then R is nilpotent-central;

(ii) if R has commuting units then R is unit-central.

Proposition 3.3. *Let R be a ring and $e \in R$ an idempotent. If R is uni, then the corner ring eRe is uni.*

Proof. Suppose R is uni, with $ere \in U(eRe)$ and $ese \in N(eRe)$. Then $ere+(1-e)$ is contained in $U(R)$ (indeed, if $(ere)v=v(ere)=e$ with $v \in eRe$, then $(ere+\overline{e})(v+\overline{e})=(v+\overline{e})(ere+\overline{e})=e+\overline{e}=1$), so it commutes with $ese \in N(R)$, and hence ere and ese commute. □

Example 3.4. *The homomorphic image of a uni ring need not be uni.*

Let D be a division ring, $R=D[x,y]$ and $I=<x^2,y^2>$ where $xy \neq yx$. Since R is a domain, R is reduced and so uni. On the other hand, $x+I$ is a nilpotent element of R/I which is not central (not commuting with $y+I$). Hence R/I is not nilpotent central. More, it is not uni since $x+I$ does not commute with $1+y+I$, which is unipotent in R/I.

Proposition 3.5. *Let R be a uni ring. If I is a nil ideal of R, then R/I is uni.*

Proof. First observe that in any ring, since I is nil, $t+I \in N(R/I)$ iff $t \in N(R)$. Further, if the ring is DF, $u+I \in U(R/I)$ iff $u \in U(R)$. Indeed, there exists $v \in R$ such that $uv-1 \in I$, that is, $uv=1+t_1$ with $t_1 \in I \subseteq N(R)$. Therefore uv is unipotent and so there is $w \in R$ such that $uvw=1$. Finally, since uni rings are DF, $u \in U(R)$. Hence R/I is uni whenever R is so. □

Example 3.6. *Notice that for a ring R and an ideal I, if R/I is uni, then R need not be uni.*

Let $R=\begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$, where F is any field with at least three elements and $0 \neq u \neq 1$. Since $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is nilpotent but $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix}=\begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix} \neq$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, R is not uni. Now consider the ideal $I = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$ of R. Then R/I is commutative and so uni.

Since nilpotents and units in a ring R are also nilpotents and units in the polynomial ring $R[X]$, if $R[X]$ is uni (or nilpotent-central) so is the ring R. In [13], it was proved that R is nilpotent-central iff $R[X]$ is nilpotent-central. Hence if R is nilpotent-central $R[X]$ is also uni.

The following remains an open

QUESTION. *$R[X]$ is uni whenever R is uni?*

Let R be a *ring with commuting nilpotents*. It is then easy to see that $N(R)$ is a subring of R (i.e. it is a NR ring) and is locally nilpotent, and so any polynomial in $N(R)[X]$ is nilpotent. The converse seems unlikely. However, a counterexample cannot be given in the example we gave in section 2: $R = \begin{bmatrix} \mathbf{F}_2 & V \\ 0 & \mathbf{F}_2 \end{bmatrix}$.

PROPOSITION 3.7. *A ring R is uni iff the Dorroh extension $D(R, \mathbf{Z})$ of R is uni.*

Proof. It is readily seen that $N(D(R, \mathbf{Z})) = \{(r, k) : r \in N(R), k = 0\}$ and (via quasiregular elements) that $U(D(R, \mathbf{Z})) = \{(r, k) : 1 + r \in U(R), k \in \{\pm 1\}\}$. Hence the claim follows. □

EXAMPLE 3.8. *Let R be a uni ring and M an (R, R)-bimodule. Then the trivial extension $T(R, M)$ of R by M need not be a uni ring.*

Let $\mathbf{H}$ be the division ring of quaternions over the real numbers. Then $\mathbf{H}$ is a reduced ring and so uni. The nilpotent element $\begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}$ of $T(\mathbf{H}, \mathbf{H})$ and the unit $\begin{bmatrix} j & 0 \\ 0 & j \end{bmatrix}$ do not commute. Hence $T(\mathbf{H}, \mathbf{H})$ is not uni.

However we can prove

PROPOSITION 3.9. *Let R be a ring and M an (R, R)-bimodule. The trivial extension $T(R, M)$ of R by M is a uni ring iff R is uni and $um = mu$ for all $u \in U(R)$ and $m \in M$.*

Proof. Straightforward. □

Recall that a ring R is called *weakly semicommutative* [11], if for any $a, b \in R$, $ab = 0$ implies arb is a nilpotent element for each $r \in R$. Then it is proved in [13] that *any nilpotent-central ring is weakly semicommutative.*

For uni rings we can prove an analogue

PROPOSITION 3.10. *In a uni ring R let $ab = 0$ for $a, b \in R$. Then for any unit $u \in U(R)$, aub is nilpotent.*

The converse fails: we can use Example 2.14 in [13]. This is a weakly semicommutative ring which is not uni.

4. BACK TO COMMUTATIVE

PROPOSITION 4.1. *Let R be a ring with commuting units. If R is fine, then it is commutative.*

Proof. Let $a \in R$. Since R is fine, there are a unit u and a nilpotent t exist in R such that $a = u + t$. Since commuting units implies uni and commuting nilpotents, $ab = ba$ for any $a, b \in R$. □

This fails for uni rings: any division ring which is not a field is trivially uni and fine but not commutative.

In [9] it is proved (see Th. 2.8) that *a ring with commuting nilpotents does not contain any nonzero nilpotent von Neumann regular element.* This is true also for uni rings, because uni rings are with commuting nilpotents. The result is used in order to show that *any von Neumann regular ring with commuting units is commutative.*

However this fails for uni rings: any regular uni ring is reduced and so strongly regular, but strongly regular rings may not be commutative.

The latter (i.e. a regular uni ring is commutative) is of course true for UU rings, but this also follows from [6]: a UU ring is regular iff it is Boolean.

A generalization from commuting units to uni, from [12], can be proved

LEMMA 4.2. *Let R be a uni ring. If R is semiprime or 2-torsion-free then R is Abelian.*

Proof. Indeed, $a = er\overline{e} \in N(R)$; actually it is zerosquare. Since $1 - 2e \in U(R)$, if R is uni (from $(1 - 2e)a = a(1 - 2e)$) we get $(2e)a = a(2e)$. Then, if R is 2-torsion-free, $2a = 2(ea) = 2ae = 0$ and so $a = 0$.

If R is semiprime, again notice that $b = \overline{e}se \in N(R)$ and since $1 + a \in U(R)$ we obtain $ab = ba$. Hence $ab = (ea)b = eba = 0$. But $ba = 0$ means $s(er - ere) = e(ser) - e(ser)e$ and it follows that $L = \{ex - exe : x \in R\}$ is a left ideal of R. Since $L^2 = 0$ it follows again that $a = 0$. □

However, for uni exchange rings which are either 2-torsion-free or semiprime we cannot derive the commutativity (as this is done for rings with commuting units in [9]). The reason is that for a proof, we need an *Abelian uni ring to be with commuting units.* This fails: any division ring D with $|D| \geq 3$ which is not a field.

5. BACK TO REDUCED

In [3] a nonzero ring R was called *unit-prime* if for any $a, b \in R$, $aU(R)b = 0$ implies $a = 0$ or $b = 0$, and, *unit-semiprime* if for any $a \in R$, $aU(R)a = 0$ implies $a = 0$. The class of unit-semiprime rings is closed under matrix extensions but not for corners.

Uni rings are naturally related to these classes. Indeed, we can prove

Proposition 5.1. *Let R be a ring. Then R is a domain if and only if R is unit-prime and is uni.*

Proof. The conditions are clearly necessary. Conversely, assume R is a unit-prime and uni. Let $a, b \in R$ with $ab = 0$. Then $rab = 0$ for all $r \in R$. Since $(bra)^2 = 0$, bra commutes with units. Let $u \in U(R)$ and $aubraub \in (aub)R(aub)$ for any $r \in R$. Hence $a\underline{ubra}ub = \underline{ab}rau^2b = 0$, and so $(aub)R(aub) = 0$. Since R is (semi)prime, we have $aub = 0$ for all $u \in U(R)$ and so $aU(R)b = 0$. By unit-primeness we get $a = 0$ or $b = 0$, and R is a domain. □

In [4] *prime rings with commuting nilpotents* are considered and sufficient conditions are found which imply that these rings are reduced. These are: every regular element is invertible in $Q_{mr}(R)$ (the maximal quotient ring), or, R has finite right uniform dimension, or else, $Q_{mr}(R)$ is Dedekind finite. Taking a stronger hypothesis instead of commuting nilpotents and a weaker hypothesis instead of prime ring we obtain a characterization.

Proposition 5.2. *Let R be a ring. Then R is reduced if and only if R is unit-semiprime and is uni.*

Proof. Only one way needs verification. Assume that R is a uni ring and $a \in R$ with $a^2 = 0$. Then a commutes with units. If R is unit-semiprime ring and $aU(R)a = a^2U(R) = 0$, it follows that $a = 0$ and so R is reduced. □

As noticed in the previous section, to a well-known characterization we can add

Theorem 5.3. *The following are equivalent for a ring R:*
(i) *R is strongly (von Neumann) regular;*
(ii) *R is (von Neumann) regular and reduced;*
(iii) *R is von Neumann regular and Abelian;*
(iv) *every principal left ideal of R is generated by a central idempotent;*
(v) *R is (von Neumann) regular and uni.*

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