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A SPECTRAL METHOD FOR FOURTH-ORDER BOUNDARY VALUE PROBLEMS

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Abstract. Boundary-value problems for fourth-order partial differential equations are studied in this paper; more precisely, vibrational phenomena of plates in an incompressible non-viscous fluid along the edge are mathematically analyzed. The spectral method via the variational formulation is used to prove existence, uniqueness and regularity theorems for the strong solution. We discuss also a discrete variational formulation for the considered problem.

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1. INTRODUCTION

The spectral methods (of approximation) are used to approximate solutions of partial differential equations by means of truncated series of orthogonal functions (eigenfunctions). They are particularly attractive, since the distance between the solution of the problem and its spectral approximation depends only on the smoothness of the solution - see [4, 1]. The spectral method has been applied successfully in numerical simulations in science and engineering. It is well known that many topics in mathematical physics require the investigation of the eigenvalues and eigenfunctions of the linear boundary value problems. In the presence of nonlinearities, linearization approaches are often used to convert the given original problem to an equivalent linear integro-differential equation which implies all boundary condition (see e.g. [6, 2]). The spectral methods enjoy a variety of well known virtues for the solution of the ordinary and partial differential equations.

The aim of this paper is to develop a spectral method for fourth order problems with mixed boundary conditions. We shall consider the following inhomogeneous boundary value problem:

$$(1)\qquad (\mathcal{P}_0)\begin{cases} \Delta^2 u + \alpha\partial_{tt}u - \beta\partial_{tt}\Delta u = f, & \text{in} \quad]0,T[\times\Omega, \\ u(.,0) = u_0,\ \partial_t u(.,0) = u_1 & \text{in} \quad \Omega, \\ u = \partial_n u = 0 & \text{in} \quad]0,T[\times\Gamma, \end{cases}$$

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where Ω is an open bounded and convex set in $\mathbb{R}^2$ with polygonal boundary Γ, $f \in L^2(]0,T[\,;L^2(\Omega))$, $T > 0$, and ∂_n is the normal derivative on Γ with $\alpha \geq 0, \beta \geq 0$. Δ^2 is the biLaplacian; it appears in various problems of linear elasticity, for example when looking at small displacements of a plate. $H^p((]0,T[\,;L^2(\Omega))$ is the Sobolev space of $L^2(\Omega)$-valued functions on $]0,T[$ with p-summable weak derivative, $p \in [0,+\infty[$. $C^q(]0,T[;L^2(\Omega))$ is the space of continuously $q-$differentiable functions of $L^2(\Omega)$-valued functions on $]0,\ T[$, $q \in \mathbb{N}$.

When Γ is smooth, the existence and uniqueness of the solution of nonlinear problems governed by the biharmonic equations in the plane is usually obtained by using Green's formula [5]. If Γ is nonsmooth, the situation is more complicated and brings certain additional difficulties. $(\mathcal{P}_0)$ can be solved by Hilbertian decomposition. For this purpose it is very convenient to choose an orthonormal basis of the basic Hilbert space consisting of the eigenfunctions of a perturbation of Δ^2 with Dirichlet and Neumann conditions on Γ. More precisely, one of the proposed approaches to solve the problem $(\mathcal{P}_0)$ is to define the solution u with respect to the basis of eigenfunctions of the system (1). For this, we compute the eigenvalues and eigenvectors of the system (1), we set $f = 0$ and we look for a solution of the form $u(x,t) = \psi(x)\phi(t)$, in view of separation of variables x and t. We are therefore led to find the spectral parameters $\lambda \in \mathbb{R}$, for which there exists a non-zero eigenfunction $w : \Omega \to \mathbb{R}$ satisfying the stationary mixed clamped and buckled plate eigenvalue problem for Δ^2

$$(2) \qquad \begin{cases} \Delta^2 w = \lambda(\alpha w - \beta \Delta w), & \text{ae in} \quad \Omega, \\ w = 0 & \text{on} \quad \Gamma, \\ \partial_n w = 0 & \text{on} \quad \Gamma. \end{cases}$$

Problem (2) arises in the study of the vibration modes of a free elastic plate subject to lateral tension (represented by the parameters α and β), whose total mass is concentrated at the boundary. This concentration phenomenon is described by the spectrum of the eigenvalue problem. This, combined with the variational formulation, allows us to prove existence, uniqueness and regularity theorems of the strong solution and conclude that the problem (1) is well posed. The paper is organized as follows. In Section 2, we formulate the nonlinear boundary problem and present some preliminary results for the boundary value problems for the biharmonic equation. Section 3 contains the core materials for the basic boundary integral equations. Theorem 3 in Section 3 contains the main result concerning existence, uniqueness and regularity of the solution of the problem (1). In Section 4, we discuss the discrete variational formulation of the problem.

2. FORMULATION OF THE STATIONARY PROBLEM

We first recall some spectral analysis results that can be found in [3]. We particularly characterize some properties of the variational equations that guarantee the existence of a unique solution. Then, in every concrete situation, one has only to check these properties. Let V and H be two real Hilbert spaces such that

$$V \subset H, \quad V \text{ is dense in } H. \tag{3}$$

Furthermore, let there be given two linear forms, $a(.,.)$ defined on $V \times V$ and $b(.,.)$ defined on $H \times H$ such that

$$\begin{cases} a(.,.) & \text{is symetric, continuous and } V\text{-elliptical,} \\ b(.,.) & \text{is symetric, continuous and } H\text{-elliptical.} \end{cases} \tag{4}$$

The existence and uniqueness of the solutions of the variational equations $a(w,v) = \lambda b(w,v)$, $\lambda \in \mathbb{R}$, is ensured by a general version of the Lax-Milgram theorem [7]. Indeed, there exist a real increasing sequence $(\lambda_n)_{n\geq 1}$ such that, $0 \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \leq ....$ and a basis $(w_n)_{n\geq 1}$ of H, where each w_n is an eigenvector of V and $(w_n)_{n\geq 1}$ is orthonormalized in H with respect to the scalar product defined by the form $b(.,.)$:

$$\begin{cases} a(w_n, v) = \lambda_n b(w_n, v), \forall v \in V, \forall n \geq 1, \\ b(w_n, w_m) = \delta_{nm}, \forall n, m \geq 1. \end{cases} \tag{5}$$

We remark that, if we put

$$v_n = \frac{1}{\sqrt{\lambda_n}} w_n, \forall n \geq 1, \tag{6}$$

then $a(v_n, v_m) = \delta_{nm}$, for all $n, m \geq 1$. This means that $(v_n)_{n\geq 1}$ is an orthonormal basis of V with respect to the scalar product defined by the form $a(.,.)$. We also note that, for every $w \in H$ and $v \in V$,

$$w = \sum_{n\geq 1} \alpha_n w_n \text{ and } v = \sum_{n\geq 1} \beta_n v_n,$$

where $\alpha_n, \beta_n \in \mathbb{R}$, for all $n \geq 1$.

Now, we formulate the weakly stationary problem (2). Taking into account the boundary conditions, we apply Green's formula twice to obtain, for all $v \in H^4(\Omega)$ and $w \in H_0^2(\Omega)$,

$$\begin{aligned} \int_\Omega \left(\Delta^2 v\right) w \mathrm{d}x &= \int_\Gamma \partial_n \left(\Delta v\right) w \mathrm{d}s - \int_\Omega \nabla(\Delta v) \nabla w \mathrm{d}x \\ &= -\int_\Gamma \Delta v \partial_n w \mathrm{d}s + \int_\Omega \Delta v \Delta w \mathrm{d}x \\ &= \int_\Omega \Delta v \Delta w \mathrm{d}x \end{aligned} \tag{7}$$

and

$$\int_{\Omega} \Delta v w \mathrm{d}x = -\int_{\Omega} \nabla v \nabla w \mathrm{d}x. \tag{8}$$

Therefore, the weak formulation of the given problem reads as follows. Find $w \in H_0^2(\Omega)$ such that

$$\int_{\Omega} \Delta v \Delta w \mathrm{d}x = \lambda \left[\alpha \int_{\Omega} v w \mathrm{d}x + \beta \int_{\Omega} \nabla v \nabla w \mathrm{d}x \right] \quad \text{for all } v \in H_0^2(\Omega). \tag{Var}$$

Let us put

$$V = H_0^2(\Omega) \text{ and } H = \begin{cases} H_0^1(\Omega) \text{ if } \beta > 0 \\ L^2(\Omega) \text{ if } \beta = 0. \end{cases}$$

and the associated forms for $v \in V$

$$\begin{cases} a(v,w) = \int_{\Omega} \Delta v \Delta w \mathrm{d}x, \\ b(v,w) = \alpha \int_{\Omega} v w \mathrm{d}x + \beta \int_{\Omega} \nabla v \nabla w \mathrm{d}x. \end{cases} \tag{9}$$

The variational equation (Var) can be written in the abstract form

$$a(w,v) = \lambda b(w,v), \text{ for all } v \in V.$$

Using the Friedrichs inequality, we remark that the previous hypothesis are satisfied. From the above result, we deduce the existence of two sequences $(\lambda_n)_{n\geq 1} \in \mathbb{R}_+^*$ and $(w_n)_{n\geq 1} \in V$ orthonormal in H such that

$$\begin{cases} 0 \leq \lambda_n \leq \lambda_{n+1}, \text{ for all } n \geq 1, \\ a(w_n, v) = \lambda_n b(w_n, v), \text{ for all } n \geq 1 \text{ and } v \in V, \\ b(w_n, w_m) = \delta_{nm}, \text{ for all } n, m \geq 1, \\ v_n = \frac{1}{\sqrt{\lambda_n}} w_n \in V, \text{ for all } n \geq 1, \\ a(v_n, v_m) = \delta_{nm}, \text{ for all } n, m \geq 1. \\ \text{For all } w \in H \text{ and } v \in V, \text{ there exists } (\alpha_n)_{n\geq 1} \text{ and} \\ (\beta_n)_{n\geq 1} \text{ in } \mathbb{R}, \text{ such that } w = \sum_{n\geq 1} \alpha_n w_n \text{ and } v = \sum_{n\geq 1} \beta_n v_n. \end{cases} \tag{10}$$

Thus,

$$\Delta^2 w_n = \lambda_n (\alpha w_n - \beta \Delta w_n), \text{ for all } n \geq 1,$$

almost everywhere on Ω.

3. EXISTENCE, UNICITY AND REGULARITY OF THE NON-STATIONARY PROBLEM

Let us now proceed to discussing some solvability results of the problem $(\mathcal{P}_0)$, where $f \in L^2(]0,T[; L^2(\Omega))$, $u_0 \in H_0^2(\Omega)$ and $u_1 \in H_0^1(\Omega)$.

THEOREM 3.1. *For $u = u(t)$, $t \in \,]0, T[\,$, sufficiently regular, the weak formulation of the problem $(\mathcal{P}_0)$ is expressed by*

$$(11)\qquad a(u(t), v) + \frac{d^2}{dt^2} b(u(t), v) = \int\limits_{\Omega} f(t) v \mathrm{d}\Omega, \forall v \in H_0^2(\Omega), \quad \forall t \in \,]0, T[\,,$$

where $u(t) = u(x, t)$, $x \in \Omega$ and $t \in]0, T[$.

Proof. Multiply equation (1) by a function $v \in H_0^2(\Omega)$ and, using the Green formula and taking into account (7) and (8), deduce formula (11). □

THEOREM 3.2. *Under the conditions $f \in L^2(]0, T[; L^2(\Omega))$, $u_0 \in H_0^2(\Omega)$ and $u_1 \in H_0^1(\Omega)$, the problem (11) admits a unique solution $u \in C^0(]0, T[\,; H_0^2(\Omega)) \cap C^1(]0, T[\,; H_0^1(\Omega))$.*

Proof. Since $u(t) \in C^0(]0, T[\,; H_0^2(\Omega))$, we have

$$(12)\qquad u(t) = \sum_{m \geq 1} \xi_m(t) w_m,$$

where $\xi_m(t) = b(u(t), w_m)$ and

$$(13)\qquad a(w_m, v) = \lambda_m b(w_m, v).$$

Let, for all $m \geq 1$,

$$(14)\qquad \phi_m(t) = \int\limits_{\Omega} f(t, x) w_m(x) \mathrm{d}x.$$

We obtain from (11), that for all $n \geq 1$ and $t \in \,]0, T[$,

$$(15)\qquad \sum_{m \geq 1} \xi_m(t) a(w_m, w_n) + \sum_{m \geq 1} \frac{\mathrm{d}^2 \xi_m}{\mathrm{d}t^2}(t) b(w_m, w_n) = \phi_n(t).$$

Using (10), we have:

$$(16)\qquad \frac{\mathrm{d}^2 \xi_n}{\mathrm{d}t^2}(t) + \lambda_n \xi_n(t) = \phi_n(t), \text{ for all } n \geq 1 \text{ and } t \in \,]0, T[\,.$$

On the other hand, according to (10), we have

$$(17)\qquad u_0(x) = \sum_{m \geq 1} \xi_{0m} w_m(x) \text{ and } u_1(x) = \sum_{m \geq 1} \xi_{1m} w_m(x),$$

where $\xi_{0m}(t) = b(u_0, w_m)$ and $\xi_{1m}(t) = b(u_1, w_m)$.

The relation (11) can be reduced to the system

$$(p_0) \begin{cases} \frac{\mathrm{d}^2 \xi_n}{\mathrm{d}t^2}(t) + \lambda_n \xi_n(t) = \phi_n(t), \text{ for all } n \geq 1 \text{ and } t \in \,]0, T[\,, \\ \xi_n(0) = \xi_{0n}, \ \frac{\mathrm{d}\xi}{\mathrm{d}t}(0) = \xi_{1n}. \end{cases}$$

The Cauchy problem (p_0) can be easily solved by the method of variation of parameters, using the two linearly independent solutions $\cos(\sqrt{\lambda_n} t)$ and

$\frac{1}{\sqrt{\lambda_n}}\sin(\sqrt{\lambda_n}t)$ of the homogenous part and the initial conditions, the explicit solution of this system is given by

$$\xi_n(t) = b(u_0, w_n)\cos(\sqrt{\lambda_n}t) + \frac{1}{\sqrt{\lambda_n}}b(u_1, w_n)\sin(\sqrt{\lambda_n}t) + \frac{1}{\sqrt{\lambda_n}}A_n(t), \tag{18}$$

with

$$A_n(t) = \int_0^t \phi_n(s)\sin(\sqrt{\lambda_n}(t-s))\mathrm{d}s, \tag{19}$$

where $\phi_n(t)$ is given by the relation (14).

According to the explicit expression of $\xi_n(t)$, we find that, if the function u is the solution of the problem (11), it is given by

$$u(t) = \sum_{n\geq 1}\left\{b(u_0, w_n)\cos(\sqrt{\lambda_n}t) + \frac{1}{\sqrt{\lambda_n}}b(u_1, w_n)\sin(\sqrt{\lambda_n}t) + \frac{1}{\sqrt{\lambda_n}}A_n(t)\right\}w_n, \tag{20}$$

we deduce the uniqueness of the solution u.

Let us put

$$M(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Then, from equation (18), we can write

$$\begin{aligned}\begin{pmatrix} \sqrt{\lambda_n}\xi_n(t) \\ \frac{\mathrm{d}\xi_n}{\mathrm{d}t}(t)\end{pmatrix} &= M(\sqrt{\lambda_n}t)\begin{pmatrix} \sqrt{\lambda_n}b(u_0, w_n) \\ b(u_1, w_n)\end{pmatrix} \\ &\quad + \int_0^t M(\sqrt{\lambda_n}(t-s))\begin{pmatrix} 0 \\ \phi_n(s)\end{pmatrix}.\end{aligned} \tag{21}$$

For the existence of the solution, we construct an approximate solution u_m of the problem (11) on a subspace V_m of V:

$$u_m(t) = \sum_{i=1}^{m}\xi_i(t)w_i. \tag{22}$$

Using equation (21) and the approximate problem of (11), we can show that $u_m(t)$ is a Cauchy sequence in $C^0(]0,T[\,;H_0^2(\Omega))$ and $C^1(]0,T[\,;H_0^1(\Omega))$. Since these two spaces are complete, we deduce that the sequence $(u_m)_m$ converges to a function u in $C^0(]0,T[\,;H_0^2(\Omega)) \cap C^1(]0,T[\,;H_0^1(\Omega))$. This limit satisfies problem (11). $\square$

4. DISCRETE VARIATIONAL FORMULATION

In order to obtain a numerical solution of the problem $(\mathcal{P}_0)$ in H, we shall approximate the operator $L = a(u(t), v) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}b(u(t), v) = \int_\Omega f(t)v\mathrm{d}x$ by a family of "discrete" operators L_N, $N \in \mathbb{N}$. Every "discrete" operator will be defined on a finite dimensional subspace V_h of V, in which the solution will be searched and its co-domain is a subspace W_h of H.

Under the hypotheses (3) and (4), let us consider

$$\left\{ \begin{array}{l} V_h \text{ a finite-dimensional subspace of } V = H_0^2(\Omega), \\ n = \dim V_h,\ (\varphi_{h_i})_{1\le i\le n}, \text{ is a basis of } V_h. \end{array} \right. \tag{23}$$

Under the assumptions (3), (4) and (23), there exists an increasing finite sequence of eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \ldots \lambda_n \tag{24}$$

and eigenvectors $\left(X_h^i\right)_{1\le i\le n}$ of the pair (A, B)

$$X_h^i \in \mathbb{R}^n,\ X_h^i \neq 0,\ 1 \le i \le n, \tag{25}$$

were the matrices A and B are defined in $\mathbb{R}^{n\times n}$ by

$$\left\{ \begin{array}{l} A_{ij} = a(\varphi_{h_i}, \varphi_{h_j}),\ 1 \le i, j \le n \\ B_{ij} = b(\varphi_{h_i}, \varphi_{h_j}),\ 1 \le i, j \le n. \end{array} \right. \tag{26}$$

Assumptions (4) ensure that A and B are symmetric and positive definite matrices. Using the variational formulation of Section 2 and the results obtained in Section 3, we establish the following theorem.

THEOREM 4.1. *The problem*

$$\left\{ \begin{array}{l} \text{Find } u_h \in V_h \text{ such that} \\ a(u_h, v_h) = \lambda_h b(u_h, v_h),\ \text{for all } v_h \in V_h, \end{array} \right.$$

is equivalent to the matrix spectral problem:

$$AX_h^i = \lambda_i BX_h^i,\ 1 \le i \le n. \tag{27}$$

We can then use the Cholesky decomposition and put $B = M^t M$, where M is upper triangular with positive diagonal elements.

We replace equation (27) by

$$\begin{aligned} AX_h^i &= \lambda_i BX_h^i \Leftrightarrow AX_h^i = \lambda_i M^t M X_h^i \\ &\Leftrightarrow (M^t)^{-1} A M^{-1} M X_h^i = \lambda_i M X_h^i \\ &\Leftrightarrow F\eta_i = \lambda_i \eta_i\ , \end{aligned}$$

where $F = (M^t)^{-1} A M^{-1}$ is a symmetric positive definite matrix and $\eta_i = MX_h^i$.

CONCLUSION

In this note, we present a method based on spectral tools (this method was used for the heat and the wave problem in the Laplacian case) and we generalize it for the fourth order operator.

In future work, we will test numerically this method on a 2D rectangular domain. We aim also to extend this study for 3D domains.

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