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OSCILLATION ANALYSIS FOR NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS OF SECOND ORDER WITH SEVERAL DELAYS

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Abstract. In this work, oscillatory and asymptotic behavior of solutions of a class of nonlinear neutral differential equations of second-order with several delays of the form

$$(E)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\left[a(t)\frac{\mathrm{d}}{\mathrm{d}t}\big[x(t)+p(t)x(t-\tau)\big]\right]+\sum_{i=1}^{m}q_i(t)H\big(x(t-\sigma_i)\big)=0,\ \ t\geq t_0,$$

are studied, for various ranges of the bounded neutral coefficient p, under the assumptions $\int_0^\infty \frac{\mathrm{d}\eta}{a(\eta)}=\infty$ and $\int_0^\infty \frac{\mathrm{d}\eta}{a(\eta)}<\infty$. Also, an attempt is made to discuss existence of bounded positive solutions of (E). Further, some illustrative examples, showing the applicability of the new results, are included.

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Key words. Oscillation, nonoscillation, nonlinear, delay, neutral differential equations, existence of positive solution, contraction principle.

1. INTRODUCTION

In the last decades, the study of the asymptotic and oscillatory behavior of solutions of neutral differential equations is of importance in applications. This is due to the fact that such equations appear in various phenomena including networks containing lossless transmission lines in high speed computers which are used to interconnect switching circuits, in the study of vibrating masses attached to an elastic bar, as the Euler equations for the minimization of functionals involving a time delay in some variational problems and in the theory of automatic control (see Boe and Chang [1], Driver [3] and Hale [7]). The construction of these models, using delays, is complemented by the mathematical investigation of nonlinear equations. Moreover, the neutral delay differential equations play an important role in modelling virtually physical, technical or biological processes, from celestial motion, to bridge design, to interactions between neurons.

There have been many investigations into the oscillation and nonoscillation of second order nonlinear neutral delay differential equations (see e.g. [2, 4], [8–15], [18–22]).

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However, the study of oscillatory and asymptotic behavior of solutions of (E) has received much less attention, which is mainly due to the technical difficulties arising in its analysis. In what follows, we provide some background details that motivated this study. In [16], Santra has considered

$$(E_1) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\big[x(t)+p(t)x(t-\tau)\big]+\sum_{i=1}^{m} q_i(t)H\big(x(t-\sigma_i)\big)=f(t),$$

and

$$(E_2) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\big[x(t)+p(t)x(t-\tau)\big]+\sum_{i=1}^{m} q_i(t)H\big(x(t-\sigma_i)\big)=0.$$

He has established sufficient conditions for oscillation and nonoscillation of solutions of (E_1) and (E_2), for $|p(t)|<+\infty$, when H is linear, sublinear and superlinear.

In an another paper [17], Santra has studied necessary and sufficient conditions of (E_2), for various ranges of the bounded neutral coefficient p. Many references to some applications of the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big[x(t)+p(t)x(t-\tau)\big]+q(t)H\big(x(t-\sigma)\big)=0, \quad t\geq t_0,$$

can be found in [5] and [7].

There is also some work on the equation of the form (E) for single delay (see e.g. [2], [4], [8–15], [18–22]. All of them established sufficient conditions for the oscillation of solutions of the equation (E), only under the assumption $\int_0^\infty \frac{\mathrm{d}\eta}{a(\eta)}=\infty$ and only for $0\leq p(t)\leq 1$. Hence, in this work, an attempt is made to study the oscillatory and asymptotic behavior of solutions of a class of nonlinear neutral second order delay differential equations of the form

$$(1) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left[a(t)\frac{\mathrm{d}}{\mathrm{d}t}\big[x(t)+p(t)x(t-\tau)\big]\right]+\sum_{i=1}^{m} q_i(t)H\big(x(t-\sigma_i)\big)=0,$$

where

$$\tau,\sigma_i\in\mathbb{R}_+=(0,+\infty),\ p\in C([0,\infty),\mathbb{R}),\ q_i,a\in C(\mathbb{R}_+,\mathbb{R}_+),\ i=1,2,\ldots,m,$$

and H is nondecreasing with

$$H\in C(\mathbb{R},\mathbb{R}), \quad uH(u)>0, \quad u\neq 0.$$

This investigation on the oscillatory and asymptotic behavior of solutions of (1) depends on various ranges of the bounded neutral coefficient p and on the following two possible conditions

(C_1) $\int_0^\infty \frac{\mathrm{d}\eta}{a(\eta)}=\infty$,
(C_2) $\int_0^\infty \frac{\mathrm{d}\eta}{a(\eta)}<\infty$.

By a solution of (1), we understand a function $x \in C([-\rho,\infty),\mathbb{R})$ such that $\big(x(t)+p(t)x(t-\tau)\big)$ is twice continuously differentiable, $\big(a(t)(x(t)+p(t)x(t-\tau))'\big)$ is once continuously differentiable and equation (1) is satisfied for $t \geq 0$, where $\rho = \max\{\tau, \sigma_i\}$, for $i = 1, 2, \ldots, m$, and $\sup\{|x(t)| : t \geq t_0\} > 0$, for every $t_0 \geq 0$. A solution of (1) is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory.

2. SUFFICIENT CONDITIONS FOR OSCILLATION

In this section, sufficient conditions are obtained for the oscillatory and a-symptotic behavior of solutions for nonlinear second order neutral differential equations with several delays of the form (1). In the sequel, we need the following conditions for this work.

(A_1) there exists $\lambda > 0$ such that $H(u) + H(v) \geq \lambda H(u + v)$, for $u, v \geq 0$;
(A_2) $H(uv) = H(u)H(v)$, for $u, v \in \mathbb{R}$;
(A_3) $\int_\tau^\infty \sum_{i=1}^m Q_i(\eta)\mathrm{d}\eta = \infty$, where $Q_i(t) = \min\{q_i(t), q_i(t-\tau)\}$;
(A_4) $\int_0^\infty \sum_{i=1}^m q_i(\eta)\mathrm{d}\eta = \infty$;
(A_5) $\int_T^\infty \frac{1}{a(\eta)} \int_{T_1}^\eta \sum_{i=1}^m Q_i(\zeta)H\big(A(\zeta-\sigma_i)\big)\mathrm{d}\zeta\mathrm{d}\eta = \infty$, for $T, T_1 > 0$;
(A_6) $\int_T^\infty \frac{1}{a(\eta)} \int_{T_1}^\eta \sum_{i=1}^m q_i(\zeta)H\big(A(\zeta-\sigma_i)\big)\mathrm{d}\zeta\mathrm{d}\eta = \infty$, for $T, T_1 > 0$;
(A_7) $\int_0^\infty \frac{1}{a(\eta)} \int_0^\eta \sum_{i=1}^m q_i(\zeta)\mathrm{d}\zeta\mathrm{d}\eta = \infty$.

REMARK 2.1 ([16]). Assumption (A_2) implies that H is an odd function. Indeed, $H(1)H(1) = H(1)$ and $H(1) > 0$ imply that $H(1) = 1$. Further, $H(-1)H(-1) = H(1) = 1$ implies that $(H(-1))^2 = 1$. Since $H(-1) < 0$, we conclude that $H(-1) = -1$. Hence, $H(-u) = H(-1)H(-u) = -H(-u)$. On the other hand, $H(uv) = H(u)H(v)$, for $u > 0$ and $v > 0$, and $H(-u) = H(u)$ imply that $H(xy) = H(x)H(y)$, for every $x, y \in \mathbb{R}$.

REMARK 2.2 ([16]). We may note that if $x(t)$ is a solution of (1), then $y(t) = -x(t)$ is also a solution of (1), provided that H satisfies (A_2).

2.1. OSCILLATION UNDER THE CONDITION (C_1)

THEOREM 2.3. *Let* $0 \leq p(t) \leq p < \infty$, $t \in \mathbb{R}_+$. *Assume that* (C_1) *and* (A_1)*–*(A_3) *hold. Then every solution of the equation* (1) *is oscillatory.*

Proof. Suppose that $x(t)$ is a nonoscillatory solution of equation (1). Then there exists $t_0 \geq \rho$ such that $x(t) > 0$ or $x(t) < 0$, for $t \geq t_0$. Assume that $x(t) > 0$ and $x(t-\sigma) > 0$, for $t \geq t_0$. Let

$$z(t) = x(t) + p(t)x(t-\tau), \quad t \geq t_0. \tag{2}$$

From (1), it follows that

$$\big[a(t)z'(t)\big]' = -\sum_{i=1}^m q_i(t)H\big(x(t-\sigma_i)\big) < 0, \tag{3}$$

for $t \geq t_1 > t_0$. Consequently, $a(t)z'(t)$ is nonincreasing and monotonic on $[t_2, \infty)$, $t_2 > t_1$. We claim that $a(t)z'(t) > 0$, for $t \geq t_2$. If not, let $a(t)z'(t) < 0$, for $t \geq t_2$. Then we can find $\varepsilon > 0$ and $t_3 > t_2$ such that $a(t)z'(t) \leq -\varepsilon$, for $t \geq t_3$. Integrating the relation $z'(t) \leq -\frac{\varepsilon}{a(t)}$, $t \geq t_3$ from t_3 to $t(> t_3)$, we obtain

$$z(t) - z(t_3) \leq -\varepsilon \int_{t_3}^{t} \frac{\mathrm{d}\eta}{a(\eta)},$$

that is,

$$z(t) \leq z(t_3) - \varepsilon \left[\int_{t_3}^{t} \frac{\mathrm{d}\eta}{a(\eta)} \right] \to -\infty, \text{ as } t \to \infty,$$

due to (C_1), which is in contradiction with the fact that $z(t) > 0$, for $t \geq t_1$. So, our claim holds. Hence, $a(t)z'(t) > 0$, for $t \geq t_2$. As a result, $z(t)$ is nondecreasing on $[t_2, \infty)$, $t_2 > t_1$. So, there exists $\varepsilon > 0$ and $t_3 > t_2$ such that $z(t) \geq \varepsilon$, for $t \geq t_3$. We note that $\lim_{t\to\infty}[a(t)z'(t)]$ exists. Using (1), it follows that

$$\big[a(t)z'(t)\big]' + \sum_{i=1}^{m} q_i(t) H\big(x(t-\sigma_i)\big) + H(p)\big[a(t-\tau)z'(t-\tau)\big]' + H(p) \sum_{i=1}^{m} q_i(t-\tau) H\big(x(t-\tau-\sigma_i)\big) = 0.$$

Using (A_1) and (A_2), the last equation becomes

$$\begin{aligned} 0 &\geq \big[a(t)z'(t)\big]' + H(p)\big[a(t-\tau)z'(t-\tau)\big]' \\ &+ \sum_{i=1}^{m} Q_i(t)\big[H\big(x(t-\sigma_i)\big) + H\big(px(t-\tau-\sigma_i)\big)\big] \\ &\geq \big[a(t)z'(t)\big]' + H(p)\big[a(t-\tau)z'(t-\tau)\big]' \\ &+ \lambda \sum_{i=1}^{m} Q_i(t) H\big[x(t-\sigma_i) + px(t-\tau-\sigma_i)\big] \\ &\geq \big[a(t)z'(t)\big]' + H(p)\big[a(t-\tau)z'(t-\tau)\big]' + \lambda \sum_{i=1}^{m} Q_i(t) H\big(z(t-\sigma_i)\big), \end{aligned} \tag{4}$$

where $z(t) \leq x(t) + px(t-\tau)$. Consequently, there exists $t_4 > t_3$ such that

$$\big[a(t)z'(t)\big]' + H(p)\big(a(t-\tau)z'(t-\tau)\big)' + \lambda H(\varepsilon) \sum_{i=1}^{m} Q_i(t) \leq 0, \tag{5}$$

for $t \geq t_4$. Integrating (5) from t_4 to $t\,(> t_4)$, we get

$$\big[a(\eta)z'(\eta)\big]_{t_4}^{t} + H(p)[a(\eta-\tau)z'(\eta-\tau)]_{t_4}^{t} + \lambda H(\varepsilon) \left[\int_{t_4}^{t} \sum_{i=1}^{m} Q_i(\eta) \mathrm{d}\eta \right] \leq 0,$$

that is

$$\lambda H(\varepsilon)\left[\int_{t_4}^{t}\sum_{i=1}^{m} Q_i(\eta)\mathrm{d}\eta\right] \leq -\big[a(\eta)z'(\eta) + H(p)\big(a(\eta-\tau)z'(\eta-\tau)\big)\big]_{t_4}^{t}$$
$$< \infty, \quad \text{as } t \to \infty,$$

which is in contradiction with assumption (A_3). If $x(t) < 0$, for $t \geq t_0$, then we set $y(t) = -x(t)$, for $t \geq t_0$, in (1) and, using (A_2), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[a(t)\frac{\mathrm{d}}{\mathrm{d}t}\big[y(t) + p(t)y(t-\tau)\big]\right] + \sum_{i=1}^{m} q_i(t)H\big(y(t-\sigma_i)\big) = 0.$$

Then, proceeding as above, we get the same contradiction. This completes the proof of the theorem. □

REMARK 2.4. Indeed, we do not need (A_1), if we restrict $0 \leq p(t) < 1$, $t \in \mathbb{R}_+$. When $z(t)$ is nondecreasing, it happens that

$$\begin{aligned} z(t) - p(t)z(t-\tau) &= x(t) + p(t)x(t-\tau) - p(t)x(t-\tau) \\ &\quad - p(t-\tau)p(t)x(t-2\tau) = x(t) - p(t)p(t-\tau)x(t-2\tau) < x(t), \end{aligned}$$

that is, $(1 - p(t))z(t) < x(t)$. Therefore, (1) can be written as

$$\big(a(t)z'(t)\big)' + \sum_{i=1}^{m} q_i(t)H\big(1 - p(t-\sigma_i)\big)H\big(z(t-\sigma_i)\big) \leq 0,$$

due to (A_2). Hence, we have proved the following theorem.

THEOREM 2.5. *Let $0 \leq p(t) \leq p < 1$, $t \in \mathbb{R}_+$. Assume that (C_1), (A_2) and (A_4) hold. Then the conclusion of Theorem 2.3 is true.*

THEOREM 2.6. *Let $-1 \leq p(t) \leq 0$, $t \in \mathbb{R}_+$. If (C_1), (A_2) and(A_4) hold, then every unbounded solution of (1) oscillates.*

Proof. Suppose that $x(t)$ is an unbounded solution of (1)) on $[t_0, \infty)$, $t_0 > \rho$. Proceeding as in Theorem 2.3, we conclude that $a(t)z'(t)$ is nonincreasing on $[t_2, \infty)$. Since $z(t)$ is monotonic, there exists $t_3 > t_2$ such that $z(t) > 0$ or < 0, for $t \geq t_3$. We claim that $z(t) > 0$, for $t \geq t_3$. If not, let $z(t) < 0$, for $t \geq t_3$. Then $x(t) < x(t-\tau) < x(t-2\tau) < x(t-3\tau) < \ldots$ and thus $x(t) < t_3$, that is, $x(t)$ is bounded, which is absurd. So, our claim holds. Hence, $z(t) > 0$, for $t \geq t_3$.

Suppose now that $a(t)z'(t) > 0$, for $t \geq t_3$. Clearly, $z(t) \leq x(t)$ implies that

$$\big[a(t)z'(t)\big]' + \sum_{i=1}^{m} q_i(t)H\big(z(t-\sigma_i)\big) \leq 0, \tag{6}$$

for $t \geq t_3$. On the other hand, since $z(t)$ is nondecreasing, there exist $\varepsilon > 0$ and $t_4 > t_3$ such that $z(t) \geq \varepsilon$, for $t \geq t_4$. Consequently, for $t_5 > t_4 + \sigma$, it follows from (6) that

$$\big[a(t)z'(t)\big]' + H(\varepsilon)\sum_{i=1}^{m} q_i(t) \le 0, \quad t \ge t_5.$$

Integrating the last inequality from t_5 to t ($> t_5$), we have

$$\big[a(\eta)z'(\eta)\big]_{t_5}^{t} + H(\varepsilon)\left[\int_{t_5}^{t}\sum_{i=1}^{m} q_i(\eta)\mathrm{d}\eta\right] \le 0,$$

that is,

$$H(\varepsilon)\left[\int_{t_5}^{t}\sum_{i=1}^{m} q_i(\eta)\mathrm{d}\eta\right] \le -\big[a(\eta)z'(\eta)\big]_{t_5}^{t} < \infty, \quad \text{as} \quad t \to \infty,$$

which is in contradiction with (A_4). Hence, $a(t)z'(t) < 0$, for $t \ge t_3$. The rest of the theorem follows from Theorem 2.3. Thus, the proof of the theorem is complete. □

THEOREM 2.7. *Let $-1 < -p \le p(t) \le 0$, $t \in \mathbb{R}_+$ and $p > 0$. If all the assumptions of Theorem* 2.6 *hold, then every solution of* (1) *either oscillates, or converges to zero as $t \to \infty$.*

Proof. Proceeding as in the proof of Theorem 2.3, we obtain (3) and thus $a(t)z'(t)$ and $z(t)$ have constant sign on $[t_2, \infty)$, $t_2 > t_1$. So, we have four cases, namely:

(i) $z(t) > 0$, $a(t)z'(t) > 0$;
(ii) $z(t) > 0$, $a(t)z'(t) < 0$;
(iii) $z(t) < 0$, $a(t)z'(t) > 0$;
(iv) $z(t) < 0$, $a(t)z'(t) < 0$.

Using the same arguments as those in the proofs of Theorem 2.3 and Theorem 2.6, we get contradictions with (C_1) and (A_4), for the cases (ii) and (i), respectively. Since $z(t) < 0$ implies that $x(t)$ is bounded, that is, $z(t)$ is bounded, case (iv) is not possible, due to Theorem 2.3 (because $a(t)z'(t) < 0$ implies that $z(t)$ is unbounded).

Consequently, case (iii) holds, for $t \ge t_3$. In this case, $\lim_{t\to\infty} z(t)$ exists. As a result,

$$\begin{aligned}
0 &\ge \lim_{t\to\infty} z(t) = \limsup_{t\to\infty} z(t)\\
&= \limsup_{t\to\infty}\big(x(t) + p(t)x(t-\tau)\big)\\
&\ge \limsup_{t\to\infty}\big(x(t) - px(t-\tau)\big)\\
&\ge \limsup_{t\to\infty} x(t) + \liminf_{t\to\infty}\big(-px(t-\tau)\big)\\
&= (1-p)\limsup_{t\to\infty} x(t)
\end{aligned}$$

implies that $\limsup_{t\to\infty} x(t) = 0$ (because $1 - p > 0$) and thus we have $\liminf_{t\to\infty} x(t) = 0$. Hence, $\lim_{t\to\infty} x(t) = 0$.

The case $x(t) < 0$ follows similarly. This completes the proof. □

THEOREM 2.8. *Let $-\infty < -p_1 \le p(t) \le -p_2 < -1$, $p_1, p_2 > 0$ and $t \in \mathbb{R}_+$. If (C_1), (A_2) and (A_4) hold, then every bounded solution of* (1) *either oscillates, or converges to zero as $t \to \infty$.*

Proof. Suppose that $x(t)$ is a solution of (1) which is bounded on $[t_0, \infty)$, $t_0 > \rho$. Using the same argument as in the proof of Theorem 2.3, we have that $a(t)z'(t)$ and $z(t)$ have the same sign on $[t_2, \infty)$ and we have four possible cases as in the proof of Theorem 2.7. By Theorem2.3, (iv) is not possible, because of (C_1) and $z(t)$ is bounded ($a(t)z'(t) < 0$ implies $z(t)$ is unbounded). Also, same is true for case (ii) ($a(t)z'(t) < 0$ implies $z(t)$ is negative). Case (i) follows from the proof of Theorem 2.6.

Consider now case (iii). In this case, $\lim_{t\to\infty} z(t)$ exists. Let $\lim_{t\to\infty} z(t) = \beta$, $\beta \in (-\infty, 0]$. Assume that $-\infty < \beta < 0$. Then there exists $\alpha < 0$ and $t_3 > t_2$ such that $z(t + \tau - \sigma_i) < \alpha$, for $t \ge t_3$ and $i = 1, 2, \ldots, m$. Hence, $z(t) \ge p(t)x(t-\tau) \ge -p_1 x(t-\tau)$ implies that$x(t-\sigma_i) \ge -p_1^{-1}\alpha > 0$ for $t \ge t_3$ and $i = 1, 2, \ldots, m$. Consequently, (1) becomes

$$\big[a(t)z'(t)\big]' + H(-p_1^{-1}\alpha)\sum_{i=1}^{m} q_i(t) \le 0, \tag{7}$$

for $t \ge t_3$. Integrating (7) from t_3 to t ($> t_3$), we get

$$\big[a(\eta)z'(\eta)\big]_{t_3}^{t} + H(-p_1^{-1}\alpha)\left[\int_{t_3}^{t}\sum_{i=1}^{m} q_i(\eta)\mathrm{d}\eta\right] \le 0,$$

that is,

$$H(-p_1^{-1}\alpha)\left[\int_{t_3}^{t}\sum_{i=1}^{m} q_i(\eta)\mathrm{d}\eta\right] \le -\big[a(\eta)z'(\eta)\big]_{t_3}^{t} < \infty,\ \textit{as } t \to \infty,$$

which is in contradiction with (A_4).

Finally, let $\beta = 0$. Then

$$\begin{aligned} 0 &= \lim_{t\to\infty} z(t) = \liminf_{t\to\infty} z(t) \\ &\le \liminf_{t\to\infty}\big(x(t) - p_2 x(t-\tau)\big) \\ &\le \limsup_{t\to\infty} x(t) + \liminf_{t\to\infty}\big(-p_2 x(t-\tau)\big) \\ &= (1 - p_2)\limsup_{t\to\infty} x(t) \end{aligned}$$

implies that $\limsup_{t\to\infty} x(t) = 0$ (because $1-p_2 < 0$). Thus, $\liminf_{t\to\infty} x(t) = 0$ and hence $\lim_{t\to\infty} x(t) = 0$. Therefore, any solution $x(t)$ of (1) converges to zero. The case $x(t) < 0$ is similar. Thus the proof of the theorem is complete. □

2.2. Oscillation Under the Condition (C_2)

Remark 2.9. If we denote $A(t)=\int_t^\infty \frac{\mathrm{d}\eta}{a(\eta)}$, then (C_2) implies that $A(t)\to 0$, as $t\to\infty$, and $A(t)$ is nonincreasing.

Theorem 2.10. *Let* $0\le p(t)\le p<\infty$, $t\in\mathbb{R}_+$. *Assume that* (C_2), (A_1)–(A_3) *and* (A_5) *hold. Then the same conclusion as in Theorem* 2.3 *is true, where* $Q_i(t)$ *is defined in Theorem* 2.3.

Proof. We proceed as in the proof of Theorem 2.3 to obtain (3), for $t\ge t_1$, and to show that $a(t)z'(t)$ is nonincreasing on $[t_2,\infty)$, $t_2>t_1$. The case $a(t)z'(t)>0$, for $t\ge t_0$, can be treated in the same way as in the proof of Theorem 2.3 and this gives a contradiction with (A_3). Suppose that $a(t)z'(t)<0$, for $t\ge t_2$. Then, for $s\ge t>t_2$, $a(s)z'(s)\le a(t)z'(t)$ implies that $z'(s)\le \frac{a(t)z'(t)}{a(s)}$. Consequently, $z(s)\le z(t)+a(t)z'(t)\int_t^s \frac{\mathrm{d}\zeta}{a(\zeta)}$. Since $a(t)z'(t)$ is nonincreasing, we can find a constant $\varepsilon>0$ such that $a(t)z'(t)\le-\varepsilon$, for $t\ge t_2$. As a consequence, $z(s)\le z(t)-\varepsilon\int_t^s\frac{\mathrm{d}\zeta}{a(\zeta)}$, for $s\ge t>t_2$. Letting $s\to\infty$, we get $0\le z(t)-\varepsilon A(t)$, for $t\ge t_2$. Using the above fact in (4), we get

$$\big[a(t)z'(t)\big]'+H(p)\big[a(t-\tau)z'(t-\tau)\big]'+\lambda H(\varepsilon)\sum_{i=1}^m Q_i(t)H\big(A(t-\sigma_i)\big)\le 0, \tag{8}$$

for $t\ge t_3>t_2$. Integrating (8) from t_3 to $t(>t_3)$, we obtain

$$\big[a(\eta)z'(\eta)\big]_{t_3}^t+H(p)\big[a(\eta-\tau)z'(\eta-\tau)\big]_{t_3}^t + \lambda H(\varepsilon)\left[\int_{t_3}^t\sum_{i=1}^m Q_i(\eta)H\big(A(\eta-\sigma_i)\big)\mathrm{d}\eta\right]\le 0,$$

that is

$$\begin{aligned}\lambda H(\varepsilon)\left[\int_{t_3}^t\sum_{i=1}^m Q_i(\eta)H\big(A(\eta-\sigma_i)\big)\mathrm{d}\eta\right] &\le -\big[a(\eta)z'(\eta)+H(p)\big(a(\eta-\tau)z'(\eta-\tau)\big)\big]_{t_3}^t\\ &\le -\big[a(t)z'(t)+H(p)\big(a(t-\tau)z'(t-\tau)\big)\big]\\ &\le -\big(1+H(p)\big)a(t)z'(t)\end{aligned}$$

implies that

$$\frac{\lambda H(\varepsilon)}{1+H(p)}\frac{1}{a(t)}\left[\int_{t_3}^t\sum_{i=1}^m Q_i(\eta)H\big(A(\eta-\sigma_i)\big)\mathrm{d}\eta\right]\le -z'(t).$$

Integrating again the last inequality, we obtain that

$$\frac{\lambda H(\varepsilon)}{1+H(p)}\int_{t_3}^t\frac{1}{a(\eta)}\left[\int_{t_3}^\eta\sum_{i=1}^m Q_i(\zeta)H\big(A(\zeta-\sigma_i)\big)\mathrm{d}\zeta\right]\mathrm{d}\eta\le -\big[z(\eta)\big]_{t_3}^t\le z(t_3).$$

Since $z(t)$ is bounded and monotonic, then

$$\int_{t_3}^{\infty} \frac{1}{a(\eta)} \Big[\int_{t_3}^{\eta} \sum_{i=1}^{m} Q_i(\zeta) H\big(A(\zeta-\sigma_i)\big) \mathrm{d}\zeta \Big] \mathrm{d}\eta < \infty,$$

a contradiction to (A_5). The case $x(t) < 0$ can be treated similarly. This completes the proof of the theorem. □

THEOREM 2.11. *Let $-1 \leq p(t) \leq 0$, $t \in \mathbb{R}_+$. Assume that (C_2), (A_2), (A_4) and (A_6) hold. Then every unbounded solution of* (1) *oscillates.*

Proof. The proof of the theorem follows from the proofs of Theorem 2.6 and Theorem 2.10 and hence we omit the details. □

THEOREM 2.12. *Let $-1 < -p \leq p(t) \leq 0$, $t \in \mathbb{R}_+$ and $p > 0$. If allthe conditions of Theorem 2.11 are satisfied, then everysolution of* (1) *either oscillates or converges to zero as$t \to \infty$.*

Proof. The proof of the theorem follows from the proof of Theorem 2.7 and Theorem 2.11. Hence, we omit the proof. □

THEOREM 2.13. *Let $-\infty < -p_1 \leq p(t) \leq -p_2 < -1$, $t \in \mathbb{R}_+$ and $p_1, p_2 > 0$. Assume that (C_2), (A_2), (A_4), (A_6) and (A_7) hold. Then every bounded solution of* (1) *either oscillates, or converges to zero, as $t \to \infty$.*

Proof. Proceeding as in the proof of the Theorem 2.8, we have four possible cases for $t \geq t_2$. Case (i) and case (iii) can be treated similarly as in the proof of Theorem 2.8.

Case (iii) follows from Theorem 2.10 and thus we get a contradiction with (A_6). Hence, we consider only case (iv). Using the same type of reasoning as in case (iii) of Theorem 2.8, we get (7) and hence

$$H(-p_1^{-1}\alpha) \Big[\int_{t_3}^{t} \sum_{i=1}^{m} q_i(\eta) \mathrm{d}\eta \Big] \leq -a(t) z'(t).$$

Therefore,

$$H(-p_1^{-1}\alpha) \int_{t_3}^{t} \frac{1}{a(\eta)} \Big[\int_{t_3}^{\eta} \sum_{i=1}^{m} q_i(\zeta) \mathrm{d}\zeta \Big] \mathrm{d}\eta \leq -\big[z(t)\big]_{t_3}^{t} \leq -z(t) < \infty,$$

since $z(t)$ is bounded and monotonic, which give a contradiction with (A_7). The rest of the proof for this case follows from Theorem 2.8.

The case $x(t) < 0$ is similar. Thus the proof of the theorem is complete. □

3. EXISTENCE OF POSITIVE SOLUTIONS

THEOREM 3.1. *Let $0 \leq p(t) \leq p_1 < 1$, $t \in \mathbb{R}_+$ and H be Lipschitzian on the interval $[a, b]$, where $0 < a < b < \infty$. If*

(A_8) $\int_0^{\infty} \frac{1}{a(\eta)} \big[\int_0^{\infty} \sum_{i=1}^{m} q_i(\zeta) \mathrm{d}\zeta \big] \mathrm{d}\eta < \infty$,

then equation (1) *admits a positive bounded solution.*

Proof. Due to (A_8), it is possible to find $t_1 > 0$ such that

$$\int_{t_1}^{\infty} \frac{1}{a(\eta)} \left[\int_{t_1}^{\infty} \sum_{i=1}^{m} q_i(\zeta) \mathrm{d}\zeta \right] \mathrm{d}\eta < \frac{1-p_1}{2L},$$

where $L = \max\{L_1, H(1)\}$, L_1 being a Lipschitz constant on $[\frac{1-p_1}{2}, 1]$. For $t_2 > t_1$, we set $X = BC([t_2, \infty), \mathbb{R})$, the space of real-valued bounded continuous functions on $[t_2, \infty)$. Clearly, X is a Banach space with respect to the supremum norm defined by $||x|| = \sup\{|x(t)| : t \geq t_2\}$. We define

$$S = \left\{ u \in X : \frac{1-p_1}{2} \leq u(t) \leq 1, \ t \geq t_2 \right\}.$$

Clearly, S is a closed and convex subset of X. Let $T : S \to S$ be defined by

$$Tx(t) = \begin{cases} Tx(t_2 + \rho), \quad t \in [t_2, t_2 + \rho] \\ -p(t)x(t-\tau) \\ \qquad - \int_t^{\infty} \frac{1}{a(\eta)} \left[\int_{\eta}^{\infty} \sum_{i=1}^{m} q_i(\zeta) H\big(x(\zeta - \sigma_i)\big) \mathrm{d}\zeta \right] \mathrm{d}\eta + 1, \ t \geq t_2 + \rho. \end{cases}$$

For every $x \in S$, $Tx(t) < 1$ and

$$\begin{aligned} Tx(t) &\geq -p(t)x(t-\tau) - H(1) \int_{t_1}^{\infty} \frac{1}{a(\eta)} \left[\int_{t_1}^{\infty} \sum_{i=1}^{m} q_i(\zeta) \mathrm{d}\zeta \right] \mathrm{d}\eta + 1 \\ &\geq -p_1 - \frac{1-p_1}{2} + 1 = \frac{1-p_1}{2} \end{aligned}$$

implies that $Tx \in S$. Now, for $y_1, y_2 \in S$,

$$\begin{aligned} |Ty_1(t) - Ty_2(t)| &\leq |p(t)||y_1(t-\tau) - y_2(t-\tau)| \\ &\quad + L_1 \int_{t_1}^{\infty} \frac{1}{a(\eta)} \left[\int_{t_1}^{\infty} \sum_{i=1}^{m} q_i(\zeta) |y_1(\zeta - \sigma_i) - y_2(\zeta - \sigma_i)| \mathrm{d}\zeta \right] \mathrm{d}\eta, \end{aligned}$$

that is,

$$\begin{aligned} |Ty_1(t) - Ty_2(t)| &\leq p_1 ||y_1 - y_2|| + L_1 ||y_1 - y_2|| \int_{t_1}^{\infty} \frac{1}{a(\eta)} \left[\int_{t_1}^{\infty} \sum_{i=1}^{m} q_i(\zeta) \mathrm{d}\zeta \right] \mathrm{d}\eta \\ &< \left(p_1 + \frac{1-p_1}{2} \right) ||y_1 - y_2||, \end{aligned}$$

which implies that

$$||Ty_1 - Ty_2|| \leq \mu ||y_1 - y_2||,$$

that is T is a contraction mapping, where $\mu = \frac{1+p_1}{2} < 1$. Since S is complete and T is a contraction on S, by the Banach fixed point theorem, T has a unique fixed point on $\left[\frac{1-p_1}{2}, 1\right]$. Hence $Tx = x$ and

$$x(t) = \begin{cases} x(t_2+\rho), \quad t \in [t_2, t_2+\rho] \\ -p(t)x(t-\tau) \\ \qquad\qquad - \int_t^\infty \frac{1}{a(\eta)} \left[\int_\eta^\infty \sum_{i=1}^m q_i(\zeta) H\big(x(\zeta-\sigma_i)\big) \mathrm{d}\zeta\right] \mathrm{d}\eta + 1, \ t \geq t_2+\rho \end{cases}$$

is a bounded positive solution of the equation (1) on $\left[\frac{1-p_1}{2}, 1\right]$. This completes the proof of the theorem. $\square$

Remark 3.2. Theorems similar to Theorems 3.1 can be proved for other ranges of $p(t)$.

4. FINAL COMMENTS AND EXAMPLES

In this section, we give two simple remarks and some examples to close the paper.

Remark 4.1. In Theorems 2.3–2.13, H is allowed to be linear, sublinear or superlinear.

Remark 4.2. A prototype of the function H satisfying (A_1) and (A_2) is

$$(1+\alpha|u|^\beta)|u|^\gamma \mathrm{sgn}(u), \quad \text{for } u \in \mathbb{R}, \tag{9}$$

where $\alpha \geq 1$ or $\alpha = 0$ and $\beta, \gamma > 0$. For verifying (A_6), we use the well-known inequality ([6, p. 292])

$$u^p + v^p \geq h(p)(u+v)^p, \quad \text{for } u, v > 0,$$

where $h(p) := \begin{cases} 1, & 0 \leq p \leq 1, \\ \dfrac{1}{2^{p-1}}, & p \geq 1. \end{cases}$.

Example 4.3. Consider

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big[x(t) + x(t-\pi)\big] + x(t-2\pi) + x(t-2\pi) = 0, \tag{10}$$

for $t \geq 0$, where $p(t) = 1$, $a(t) = 1$, $q_1(t) = 1 = q_2(t)$, $\tau = \pi$, $m = 2$, $\sigma_1 = 2\pi$ and $\sigma_2 = 3\pi$. Clearly, (C_1),(A_1), (A_2) and $\int_\pi^\infty [Q_1(t) + Q_2(t)]\mathrm{d}t = \infty$ hold, where $Q_1(t) = Q_2(t) = 1$. Hence, Theorem 2.3 can be applied to (10), that is every solution of (10) oscillates. Indeed, $x(t) = \sin t$ is such a solution of (10).

Example 4.4. Consider

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[a(t)\frac{d}{dt}\big[x(t) - \mathrm{e}^{-2\pi}x(t-2\pi)\big]\right] + q_1(t)x(t-2\pi) + q_2(t)x(t-3\pi) = 0, \tag{11}$$

for $t \geq 0$, where $a(t) = \mathrm{e}^{-t}$, $-1 < p(t) = -\mathrm{e}^{-2\pi} \leq 0$, $q_1(t) = \mathrm{e}^{t-2\pi}$, $q_2(t) = \mathrm{e}^{t-3\pi}$ and $m = 2$. Clearly, all the assumptions of the Theorem 2.7 hold true.

Hence, by Theorem 2.7, every solution of (11) either oscillates, or converges to zero, as $t \to \infty$. Indeed, $x(t) = \mathrm{e}^{-t} \sin t$ is such a solution of (11).

EXAMPLE 4.5. Consider

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^t \frac{\mathrm{d}}{\mathrm{d}t}\big[x(t) + x(t-\pi)\big]\right] + \mathrm{e}^t x(t-2\pi) + \mathrm{e}^t x(t-3\pi) = 0, \quad t \geq 0, \tag{12}$$

where $p(t) = 1$, $a(t) = \mathrm{e}^t$, $A(t) = \mathrm{e}^{-t}$, $Q_1(t) = \mathrm{e}^{t-\pi} = Q_2(t)$ and $m = 2$. Clearly, the assumptions (C_2)–(A_3) and (A_5) hold true. Hence, by Theorem 2.10, every solution of (12) oscillates. Thus, in particular, $x(t) = \sin t$ is an oscillatory solution.

EXAMPLE 4.6. Consider

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^t \frac{d}{dt}\big[x(t) - \mathrm{e}^{-2\pi} x(t-2\pi)\big]\right] + \mathrm{e}^{t-2\pi} x(t-2\pi) + \mathrm{e}^{t-3\pi} x(t-3\pi) = 0, \tag{13}$$

for $t \geq 0$, where $-1 < p(t) = -\mathrm{e}^{-2\pi} < 0$, $a(t) = \mathrm{e}^t$, $A(t) = \mathrm{e}^{-t}$ and $m = 2$. Clearly, all the assumptions of Theorem2.12 hold true. Hence, by Theorem 2.12, every solution of (13) either oscillates, or tends to zero, as $t \to \infty$. Indeed, $x(t) = \mathrm{e}^{-t} \sin t$ is such a solution of (13).

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