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ON DIFFERENCES OF SEMICONTINUOUS FUNCTIONS AND PERFECT CLASSES

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Abstract. Let K be a metric space and $f : K \to \mathbb{R}$ be a bounded function. H. Rosenthal and others showed in a series of papers that f can be written as a difference of two bounded semicontinuous functions on K if and only if its transfinite oscillations are bounded on K. We provide a generalization of this characterization to an arbitrary Hausdorff topological space. As an application, we provide an alternative proof of the result obtained by J. Saint Raymond that the class of differences of semicontinuous functions is perfect.

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1. INTRODUCTION

In a series of papers by Rosenthal and others (see e.g. [2, 5, 7, 8]), the space of differences of semicontinuous functions and the space of differences of bounded semicontinuous functions were extensively studied. Among others, the authors provided the following characterization of differences of bounded semicontinuous functions in metric spaces, using the transfinite oscillations defined below.

Definition 1.1 (Transfinite oscillations). Let T be a Hausdorff topological space and let f be a real-valued function on T. The *α-th oscillation* of f, in the sense of H. Rosenthal (cf. [2, Section 5]), is defined by transfinite induction. Let $\operatorname{osc}_0 f = 0$ on T and let $\beta > 0$ be a given ordinal. If $\operatorname{osc}_\alpha f$ has been defined for each $\alpha < \beta$, then, for $x \in T$ and $\beta = \alpha + 1$, we set (the limes superior is considered to be non-exclusive)

$$\widetilde{\operatorname{osc}}_\beta f(x) = \limsup_{y \to x} \big(|f(y) - f(x)| + \operatorname{osc}_\alpha f(y) \big)$$

and, for β a limit ordinal, we set

$$\widetilde{\operatorname{osc}}_\beta f(x) = \sup_{\alpha < \beta} \operatorname{osc}_\alpha f(x).$$

Finally, we set

$$\operatorname{osc}_\beta f(x) = \limsup_{y \to x} \ \widetilde{\operatorname{osc}}_\beta f(y).$$

Theorem 1.2 (Rosenthal). *Let K be a metric space and $f : K \to \mathbb{R}$ be a bounded function. Then there exists a countable ordinal α such that $\operatorname{osc}_\beta f = \operatorname{osc}_\alpha f$, for any $\beta > \alpha$. Let τ be the least such α. Then f is a difference of bounded semicontinuous functions on T if and only if $osc_\tau f$ is bounded.*

If the metric space is separable, the proof can be found in [7, Theorem 3.5]. In the unpublished paper [8], the proof is presented for an arbitrary metric space (see [8, Theorem 3.2]).

In this paper, we extend Theorem 1.2 to an arbitrary Hausdorff topological space, using Rosenthal's ideas .

Our paper is organized as follows. In Section 2, we provide a summary of the notations used in the paper. Section 3 provides a proof of our main result. Then, we present an application in Section 4. We show a different proof for a result that was originally achieved by J. Saint Raymond (see e.g. [3, Lemma 3]), that differences of (unbounded) semicontinuous functions form a so-called perfect class.

2. NOTATION

Throughout the article, we use the following notations. We denote by $\mathbb{N}$ the set of natural numbers and by $\mathbb{R}$ the set of real numbers equipped with the Euclidean topology. We say that a function is *real-valued* if its range lies in $\mathbb{R}$. We denote by $C(T)$ the space of all continuous functions on a topological space T.

For $f : T \to [-\infty, +\infty]$ and $x \in T$, we define the *limes inferior* of f at $x \in T$ by

$$\liminf_{y \to x} f(y) := \sup\{\inf f(V) : V \text{ is a neighborhood of } x\}$$

and the *limes superior* by

$$\limsup_{y \to x} f(y) := \inf\{\sup f(V) : V \text{ is a neighborhood of } x\}.$$

We say that a function $f : T \to [-\infty, +\infty]$ is *lower semicontinuous*, if the set $\{x \in T : f(x) > a\}$ is open, for each $a \in \mathbb{R}$. This is equivalent to $f(x) = \liminf_{y \to x} f(y)$, for each $x \in T$. We say that a function $f : T \to [-\infty, +\infty]$ is *upper semicontinuous*, if $-f$ is lower semicontinuous.

3. DIFFERENCES OF BOUNDED SEMICONTINUOUS FUNCTIONS AND TRANSFINITE OSCILLATIONS

This section is devoted to the proof of our main result.

THEOREM 3.1. *Let T be a Hausdorff topological space and $f : T \to \mathbb{R}$ be a bounded function. Then there exists an ordinal α such that $\operatorname{osc}_\beta f = \operatorname{osc}_\alpha f$, for any $\beta > \alpha$. Let τ be the least such α. Then f is a difference of bounded semicontinuous functions on T if and only if $osc_\tau f$ is bounded.*

Proof. We prove the theorem in the following nine steps.

The first step is a simple observation.

(1) *The function $\operatorname{osc}_\alpha f$ is an upper semicontinuous function with values in $[0, +\infty]$. For every ordinals $\alpha \leq \beta$, we have $\operatorname{osc}_\alpha f \leq \operatorname{osc}_\beta f$.*

This first assertion is obvious, while the second one follows easily by transfinite induction.

Now, we show the existence of the ordinal τ.

(2) *There exists an ordinal number α such that* $\operatorname{osc}_\beta f = \operatorname{osc}_\alpha f$, *for all* $\beta > \alpha$.

Fix $\mathcal{B}$ a base of the topology of T. The generalized sequence $(\operatorname{osc}_\alpha f)$, indexed by ordinal numbers, is nondecreasing, by step (1). For a contradiction, let us assume that, if $\alpha + 1$ is a successor of α, then we always have $\operatorname{osc}_{\alpha+1} f \neq \operatorname{osc}_\alpha f$. Then we may pick $x_\alpha \in T$ such that

$$\operatorname{osc}_{\alpha+1} f(x_\alpha) > \operatorname{osc}_\alpha f(x_\alpha). \qquad (*)$$

By the upper semicontinuity of $\operatorname{osc}_\alpha f$, there exists $U_\alpha \in \mathcal{B}$, which contains x_α, such that

$$\sup_{y \in U_\alpha} \operatorname{osc}_\alpha f(y) < \operatorname{osc}_{\alpha+1} f(x_\alpha).$$

If α is an ordinal strictly greater than the cardinality of a base $\mathcal{B}$ of T and equal at least to ω_1, then we may choose an uncountable subset Γ of α such that

$$U_{\gamma_1} = U_{\gamma_2} =: U, \qquad \text{for all } \gamma_1, \gamma_2 \in \Gamma.$$

We claim that

$$\sup_{y \in U} \operatorname{osc}_{\beta_1} f(y) < \sup_{y \in U} \operatorname{osc}_{\beta_2} f(y), \qquad \beta_1 < \beta_2, \quad \beta_1, \beta_2 \in \Gamma.$$

Indeed, consider x_{β_1} defined above by $(*)$. Then $x_{\beta_1} \in U_{\beta_1} = U$ and, therefore,

$$\sup_{y \in U_{\beta_1}} \operatorname{osc}_{\beta_1} f(y) < \operatorname{osc}_{\beta_1+1} f(x_{\beta_1}) \leq \operatorname{osc}_{\beta_2} f(x_{\beta_1}) \leq \sup_{y \in U_{\beta_2}} \operatorname{osc}_{\beta_2} f(y).$$

Since Γ is uncountable, this is not possible. Indeed, for $\gamma \in \Gamma$, let

$$\lambda_\gamma := \sup_{y \in U_\gamma} \operatorname{osc}_\gamma f(y) \text{ and } \Lambda := \{\lambda_\gamma : \gamma \in \Gamma\}.$$

The set Γ is well ordered and there exists a natural isomorphism between Γ and Λ. Thus Λ is well-ordered with respect to the ordering induced by this isomorphism. However, it is easy to see that this induced ordering is identical with the standard ordering on real numbers. Hence, we have an uncountable subset Γ of $\mathbb{R}$, ordered by the standard ordering on $\mathbb{R}$, that is well ordered. But, this is known to be impossible.

Next, we shall prove, in steps (3)–(5), that, if $osc_\tau f$ is bounded, then f is a difference of bounded semicontinuous functions.

(3) *If α is an ordinal such that $\operatorname{osc}_\alpha f = \operatorname{osc}_{\alpha+1} f$, then $\operatorname{osc}_\beta f = \operatorname{osc}_\alpha f$, for any $\beta > \alpha$.*
We proceed by transfinite induction.
Let β be a successor ordinal, that is, $\beta = \gamma + 1$, and $\operatorname{osc}_\gamma f = \operatorname{osc}_\alpha f$. Fix $x \in T$. We have

$$\begin{aligned}\widetilde{\operatorname{osc}}_\beta f(x) &= \widetilde{\operatorname{osc}}_{\gamma+1} f(x)\\ &= \limsup_{y\to x}\big(|f(y)-f(x)| + \operatorname{osc}_\gamma f(y)\big)\\ &= \limsup_{y\to x}\big(|f(y)-f(x)| + \operatorname{osc}_\alpha f(y)\big)\\ &= \widetilde{\operatorname{osc}}_{\alpha+1} f(x) \le \operatorname{osc}_{\alpha+1} f(x) = \operatorname{osc}_\alpha f(x).\end{aligned}$$

Due to the fact that $\operatorname{osc}_\gamma f$ is upper semicontinuous, we have

$$\widetilde{\operatorname{osc}}_\beta f(x) = \widetilde{\operatorname{osc}}_{\gamma+1} f(x) = \limsup_{y\to x}\big(|f(y)-f(x)| + \operatorname{osc}_\gamma f(y)\big)$$

$$\ge \limsup_{y\to x} \operatorname{osc}_\gamma f(y) = \operatorname{osc}_\gamma f(x) = \operatorname{osc}_\alpha f(x).$$

Thus, we get

$$\widetilde{\operatorname{osc}}_\beta f(x) = \operatorname{osc}_\alpha f(x).$$

Since $\operatorname{osc}_\beta f$ is an upper semicontinuous regularization of $\widetilde{\operatorname{osc}}_\beta f$, which (being equal to $\operatorname{osc}_\alpha f$) is already an upper semicontinuous function, we have

$$\operatorname{osc}_\beta f = \widetilde{\operatorname{osc}}_\beta f(x) = \operatorname{osc}_\alpha f.$$

Let β be a limit ordinal. Then, by definition and by the previous part of the proof, for each $x \in T$, we have

$$\widetilde{\operatorname{osc}}_\beta f(x) = \sup_{\gamma<\beta} \operatorname{osc}_\gamma f(x) = \sup_{\alpha<\gamma<\beta} \operatorname{osc}_\gamma f(x) = \operatorname{osc}_\alpha f(x)$$

The rest of the proof follows as above.

(4) *If α is an ordinal number such that $\operatorname{osc}_\alpha f = \operatorname{osc}_{\alpha+1} f$, then the functions $\operatorname{osc}_\alpha f + f$ and $\operatorname{osc}_\alpha f - f$ are upper semicontinuous.*
Let $x \in T$. Then

$$\limsup_{y\to x}\big(f(y) - f(x) + \operatorname{osc}_\alpha f(y)\big)$$

$$\le \limsup_{y\to x}\big(|f(y)-f(x)| + \operatorname{osc}_\alpha f(y)\big)$$

$$= \widetilde{\operatorname{osc}}_{\alpha+1} f(x) = \operatorname{osc}_\alpha f(x).$$

This implies

$$\limsup_{y\to x}\big(f(y) + \operatorname{osc}_\alpha f(y)\big) \le f(x) + \operatorname{osc}_\alpha f(x)$$

and thus $f + \operatorname{osc}_\alpha f$ is upper semicontinuous at x. We also observe that, by definition,

$$\operatorname{osc}_\beta f = \operatorname{osc}_\beta(-f),$$

for any ordinal β. Hence, we have

$$\begin{aligned}&\limsup_{y\to x}\big(f(x) - f(y) + \operatorname{osc}_\alpha(-f)(y)\big)\\ &\le \limsup_{y\to x}\big(|f(x) - f(y)| + \operatorname{osc}_\alpha f(y)\big)\\ &= \widetilde{\operatorname{osc}}_{\alpha+1} f(x) = \operatorname{osc}_\alpha f(x)\end{aligned}$$

and therefore

$$\limsup_{y\to x}\big(\operatorname{osc}_\alpha f(y) - f(y)\big) \le \operatorname{osc}_\alpha f(x) - f(x).$$

It follows that the function $\operatorname{osc}_\alpha f - f$ is upper semicontinuous at x as well.

(5) It follows directly that, if there exists an ordinal τ such that $\operatorname{osc}_\tau f$ is bounded and $\operatorname{osc}_{\tau+1} f = \operatorname{osc}_\tau f$, then

$$2f = (\operatorname{osc}_\tau f + f) - (\operatorname{osc}_\tau f - f)$$

and therefore f is a difference of bounded upper semicontinuous functions.

Finally, we shall prove, in steps (6)-(9), the converse implication that, if f is a difference of bounded semicontinuous functions, then $osc_\tau f$ is bounded.

(6) *If α is an ordinal such that functions $\operatorname{osc}_\alpha f + f$ and $\operatorname{osc}_\alpha f - f$ are upper semicontinuous, then $\operatorname{osc}_\alpha f = \operatorname{osc}_{\alpha+1} f$.*
Assume that $\operatorname{osc}_\alpha f \neq \operatorname{osc}_{\alpha+1} f$. Then there exists $x \in T$, such that $\widetilde{\operatorname{osc}}_{\alpha+1} f(x) > \widetilde{\operatorname{osc}}_\alpha f(x)$, because otherwise we would have

$$\widetilde{\operatorname{osc}}_\alpha f(x) \ge \widetilde{\operatorname{osc}}_{\alpha+1} f(x) \ge \operatorname{osc}_\alpha f(x) \ge \widetilde{\operatorname{osc}}_\alpha f(x),$$

thus $\widetilde{\operatorname{osc}}_{\alpha+1} f(x) = \operatorname{osc}_\alpha f(x)$ and, therefore, $\operatorname{osc}_{\alpha+1} f(x) = \operatorname{osc}_\alpha f(x)$. From

$$\widetilde{\operatorname{osc}}_{\alpha+1} f(x) > \widetilde{\operatorname{osc}}_\alpha f(x)$$

and by the definition of the $(\alpha+1)$-th oscillation, we get that

$$\limsup_{y\to x}\big(|f(y) - f(x)| + \operatorname{osc}_\alpha f(y)\big) > \operatorname{osc}_\alpha f(x),$$

which implies that at least one of the two following inequalities holds

$$\limsup_{y\to x}\big(f(y) - f(x) + \operatorname{osc}_\alpha f(y)\big) > \operatorname{osc}_\alpha f(x),$$

$$\limsup_{y\to x}\big(f(x) - f(y) + \operatorname{osc}_\alpha f(y)\big) > \operatorname{osc}_\alpha f(x).$$

If we rewrite them in the following way

$$\limsup_{y\to x}\,(\operatorname{osc}_\alpha f(y)+f(y)) > \operatorname{osc}_\alpha f(x)+f(x),$$

$$\limsup_{y\to x}\,(\operatorname{osc}_\alpha f(y)-f(y)) > \operatorname{osc}_\alpha f(x)-f(x),$$

then it follows that either $\operatorname{osc}_\alpha f+f$ or $\operatorname{osc}_\alpha f-f$ is not upper semicontinuous.

(7) *If f is upper or lower semicontinuous, then* $\operatorname{osc}_\alpha f = \operatorname{osc}_1 f$*, for each ordinal* α*.*
First, if f is an upper semicontinuous function, then $\operatorname{osc}_1 f+f$ is clearly upper semicontinuous.
We want to prove that also $\operatorname{osc}_1 f - f$ is upper semicontinuous. We have

$$\widetilde{\operatorname{osc}}_1 f(x) = \limsup_{y\to x}|f(y)-f(x)|$$

$$= \max\left\{\limsup_{y\to x}\big(f(y)-f(x)\big),\ \limsup_{y\to x}\big(f(x)-f(y)\big)\right\}.$$

By the upper semicontinuity of f, we have

$$\limsup_{y\to x}\big(f(y)-f(x)\big)=0$$

and also

$$\begin{aligned}0=\limsup_{y\to x}0 &= \limsup_{y\to x}\big(f(y)-f(y)\big)\\ &\le \limsup_{y\to x} f(y)+\limsup_{y\to x}(-f(y))\\ &= f(x)+\limsup_{y\to x}\big(-f(y)\big)=\limsup_{y\to x}\big(f(x)-f(y)\big).\end{aligned}$$

Thus

$$\widetilde{\operatorname{osc}}_1 f(x) = \limsup_{y\to x}\big(f(x)-f(y)\big) = f(x)+\limsup_{y\to x}(-f(y)),$$

so

$$\widetilde{\operatorname{osc}}_1 f(x) - f(x) = \limsup_{y\to x}(-f(y)),$$

which yields that the function $\widetilde{\operatorname{osc}}_1 f - f$ is an upper semicontinuous regularization of the function $-f$, in particular, an upper semicontinuous function. Hence $\widetilde{\operatorname{osc}}_1 f$ is upper semicontinuous, therefore $\operatorname{osc}_1 f = \widetilde{\operatorname{osc}}_1 f$ and, finally, $\operatorname{osc}_1 f - f$ is upper semicontinous, since $\operatorname{osc}_1 f - f = \widetilde{\operatorname{osc}}_1 f - f$.
We have just proved that both functions $\operatorname{osc}_1 f \pm f$ are upper semicontinuous. Now, the assertion that f is upper semicontinuous follows immediately from the step (6).

Let now f be lower semicontinuous. Then $-f$ is upper semicontinuous and it follows, as above, that $\operatorname{osc}_1(-f) \pm (-f)$ are upper semicontinuous. Since $\operatorname{osc}_1(-f) = \operatorname{osc}_1 f$, by the definition of the 1-th oscillation, we have that the functions $\operatorname{osc}_1 f \mp f$ are upper semicontinuous and the rest of the proof follows, again, from step (6).

(8) *If u, v are semicontinuous, then* $\operatorname{osc}_\alpha(u-v) \le \operatorname{osc}_\alpha(u+v) = \operatorname{osc}_1(u+v)$, *for each ordinal α.*
Without any loss of generality, we may assume that u and v are lower semicontinuous. For $\alpha = 1$, the assertion is trivial. Next, let $\alpha = \beta + 1$ and assume that the assertion is true for each ordinal less than α. Fix $x \in T$. By definition,

$$\widetilde{\operatorname{osc}}_\alpha(u-v)(x) = \limsup_{y \to x} \Big[\big| (u(y) - v(y)) - (u(x) - v(x)) \big| + \operatorname{osc}_\beta \big(u(y) - v(y) \big) \Big].$$

By the triangle inequality, we get

$$\begin{aligned} &\limsup_{y \to x} \Big[\big| (u(y) - v(y)) - (u(x) - v(x)) \big| + \operatorname{osc}_\beta \big(u(y) - v(y) \big) \Big] \\ &\le \limsup_{y \to x} \Big[\big| u(y) - u(x) \big| + \big| v(y) - v(x) \big| + \operatorname{osc}_\beta \big(u(y) - v(y) \big) \Big] \\ &\le \limsup_{y \to x} \Big[\big| u(y) - u(x) \big| + \big| v(y) - v(x) \big| + \operatorname{osc}_\beta \big(u(y) + v(y) \big) \Big]. \end{aligned}$$

Let (y_γ) be a net that converges to y. We may assume that also the nets $(u(y_\gamma))$, $(v(y_\gamma))$ and $\big(\operatorname{osc}_\beta \big(u(y_\gamma) + v(y_\gamma) \big) \big)$ converge. Then

$$\begin{aligned} &\limsup_{y \to x} \Big[\big| u(y) - u(x) \big| + \big| v(y) - v(x) \big| + \operatorname{osc}_\beta \big(u(y) + v(y) \big) \Big] \\ &= \sup_{y_\gamma \to y} \lim_\gamma \Big[\big| u(y_\gamma) - u(x) \big| + \big| v(y_\gamma) - v(x) \big| + \operatorname{osc}_\beta \big(u(y_\gamma) + v(y_\gamma) \big) \Big]. \end{aligned}$$

By the lower semicontinuity of u, we get

$$\limsup_{y \to x} (u(x) - u(y)) = u(x) - \liminf_{y \to x} u(y) = 0$$

and

$$\limsup_{y \to x} u(y) \ge \liminf_{y \to x} u(y) = u(x).$$

Therefore

$$\begin{aligned} &\lim_\gamma |u(y_\gamma) - u(x)| \\ &= \max \Big\{ \lim_\gamma \big(u(y_\gamma) - u(x) \big),\ \lim_\gamma \big(u(x) - u(y_\gamma) \big) \Big\} \\ &= \lim_\gamma \big(u(y_\gamma) - u(x) \big). \end{aligned}$$

Using the same ideas for v, we get

$$\sup_{y_\gamma \to y} \lim_\gamma \Big[\big|u(y_\gamma) - u(x)\big| + \big|v(y) - v(x)\big| + \operatorname{osc}_\beta \big(u(y) + v(y)\big) \Big]$$
$$= \sup_{y_\gamma \to y} \lim_\gamma \Big[\big(u(y) - u(x) + v(y) - v(x)\big) + \operatorname{osc}_\beta \big(u(y) + v(y)\big) \Big]$$
$$= \sup_{y_\gamma \to y} \lim_\gamma \Big[\big|u(y) - u(x) + v(y) - v(x)\big| + \operatorname{osc}_\beta \big(u(y) + v(y)\big) \Big]$$
$$= \widetilde{\operatorname{osc}}_\alpha (u+v)(x) \le \operatorname{osc}_\alpha (u+v)(x).$$

It follows that

$$\widetilde{\operatorname{osc}}_\alpha (u-v) \le \operatorname{osc}_\alpha (u+v)$$

and, by the upper semicontinuity of $\operatorname{osc}_\alpha(u+v)$,

$$\operatorname{osc}_\alpha (u-v) \le \operatorname{osc}_\alpha (u+v).$$

By step (7), we obtain

$$\operatorname{osc}_\alpha (u+v) = \operatorname{osc}_1 (u+v)$$

and thus the proof of step (8) is complete.

(9) Now, if $f = u - v$, where u, v are bounded semicontinuous functions, and there exists an ordinal τ such that $\operatorname{osc}_\beta f = \operatorname{osc}_\tau f$, for each ordinal $\beta > \tau$, then, by step (8), we have

$$\operatorname{osc}_\tau f(x) \le \operatorname{osc}_1 (u+v)(x)$$
$$= \limsup_{y \to x} |u(y) - u(x) + v(y) - v(x)| \le 2|u(x)| + 2|v(x)|.$$

Hence, $\operatorname{osc}_\tau f$ is bounded and the proof is complete.

□

4. APPLICATION TO PERFECT CLASSES OF FUNCTIONS

If $\mathfrak{C}$ denotes a class of functions and T is a topological space, the symbol $\mathfrak{C}(T)$ stands for the set of all functions on T that belong to $\mathfrak{C}$.

Definition 4.1 (*Perfect classes of functions)*. We say that a class $\mathfrak{C}$ of functions is *perfect* if, given X and Y compact spaces (which need not to be metrizable) and a continuous surjective function $\varphi : X \to Y$, we have $g \circ \varphi \in \mathfrak{C}(X)$ if and only if $g \in \mathfrak{C}(Y)$.

The perfect classes of functions were studied, for example, in [9] and [10]. It is proved in [6, Corollary 5.27] that the following real-valued functions form classes that are perfect: lower and upper semicontinuous functions, Baire functions of class α, for $\alpha \in (0, \omega_1)$, Borel functions and universally measurable functions. It is proved in [3, Lemma 3] that the same property holds for the differences of (unbounded) semicontinuous functions. In the following, we present a proof, different from the one given in [3], based on transfinite oscillations.

Theorem 4.2. *Each of the following classes of real-valued functions is perfect:*

(a) *the class of differences of semicontinuous functions,*
(b) *the class of bounded functions which are differences of semicontinuous functions,*
(c) *the class of differences of bounded semicontinuous functions.*

Since a composition of continuous function and a difference of (bounded) semicontinuous functions is again a difference of (bounded) semicontinuous functions, one implication is obvious in each part of (a)-(c).

For the converse implication, we prove first part (c) and then continue with part (a). Part (b) obviously follows from part (a).

Part (c) is derived from Theorem 4.5 and thus the proof of part (a) is completed by Theorem 4.7.

Next, we use the technique of transfinite oscillations developed above. Given compact spaces X and Y and a continuous surjective function $\varphi : X \to Y$, the map φ is closed and thus it is a quotient mapping (see [1, Proposition 2.4.3]). Therefore, it is natural to expect that the oscillations of any real-valued function g on Y control the oscillations of $g \circ \varphi$ on X and vice versa, in some sense. In Lemma 4.3, we shall prove a result of this kind, adequate for our purposes.

We shall adopt one more definition, before proving our first key lemma. Let X and Y be (Hausdorff) topological spaces. We say that a given mapping $\varphi : X \to Y$ is *inversely cluster preserving*, if, for any net (x_α) in X whose image $(\varphi(x_\alpha))$ converges to a point $y \in Y$, the inverse fiber $\varphi^{-1}(y)$ contains at least one cluster point of the net (x_α). This means that we can choose a subnet (x_γ) of (x_α) such that (x_γ) converges to x, while $(\varphi(x_\gamma))$ converges to $\varphi(x) = y$.

It was proved in [4, Theorem 3.1] that φ is inversely cluster preserving if and only if it is compact and closed. In particular, if X and Y are compact spaces and $\varphi : X \to Y$ is a continuous surjective function, then φ is closed and compact and thus inversely cluster preserving.

Lemma 4.3. *Let X and Y be Hausdorff topological spaces and $\varphi : X \to Y$ be a continuous surjective function that is inversely cluster preserving. Let $g : Y \to \mathbb{R}$ be an arbitrary function and $f = g \circ \varphi$. If there exists $M > 0$ such that $\operatorname{osc}_\alpha f(x) \le M$, for each $x \in X$ and for each ordinal α, then $\operatorname{osc}_\alpha g(y) \le M$, for each $y \in Y$ and for each ordinal α.*

Proof. If α is a limit ordinal, the assertion is trivial. Next, let $\alpha = \beta + 1$ be a successor ordinal. Fix $y \in Y$. Whenever a net (y_γ) in Y tends to y, there exists a subnet such that the nets $g(y_\gamma)$ and $\operatorname{osc}_\beta(y_\gamma)$ converge. There exists also $x_\gamma \in X$ such that $\varphi(x_\gamma) = y_\gamma$. Since φ is inversely cluster preserving, there exists $x \in \varphi^{-1}(y)$ and a subnet of (x_γ) that tends to $x \in X$. Thus, without any loss of generality, we can assume that (x_γ) tends to x and the nets $f(x_\gamma)$ and $\operatorname{osc}_\beta f(x_\gamma)$ converge. Therefore

$$\begin{aligned}
\widetilde{\operatorname{osc}}_\alpha g(y) &= \limsup_{z\to y}\big(|g(z)-g(y)| + \operatorname{osc}_\beta g(z)\big)\\
&= \sup_{y_\gamma\to y}\lim_\gamma\big(|g(y_\gamma)-g(y)| + \operatorname{osc}_\beta g(y_\gamma)\big)\\
&\le \sup_{x_\gamma\to x}\lim_\gamma\big(|g(\varphi(x_\gamma))-g(\varphi(x))| + \operatorname{osc}_\beta g(\varphi(x_\gamma))\big)\\
&= \sup_{x_\gamma\to x}\lim_\gamma\big(|f(x_\gamma)-f(x)| + \operatorname{osc}_\beta f(x_\gamma)\big)\\
&= \widetilde{\operatorname{osc}}_\alpha f(x) \le \operatorname{osc}_\alpha f(x) \le M
\end{aligned}$$

Hence $\operatorname{osc}_\alpha g(y) \le M$ and thus the proof is complete. □

Corollary 4.4. *Let X and Y be compact spaces and $\varphi : X \to Y$ be a continuous surjective function. Let $g : Y \to \mathbb{R}$ and $f = g \circ \varphi$. If there exists $M > 0$ such that $\operatorname{osc}_\alpha f(x) \le M$, for each $x \in X$ and for each ordinal α, then $\operatorname{osc}_\alpha g(y) \le M$, for each $y \in Y$ and for each ordinal α.*

We are now ready for the proof of part (c) of Theorem 4.2.

Theorem 4.5. *The class of differences of bounded semicontinuous functions is perfect.*

Proof. Let X, Y be compact spaces, $\varphi : X \to Y$ be a continuous surjective function and $g : Y \to \mathbb{R}$.

It is obvious that, if g is a difference of bounded semicontinuous functions, then $g\circ\varphi$ has the same property. Now, let us assume that $g\circ\varphi$ can be written as a difference of bounded semicontinuous functions on X. Then, by Theorem 3.1, $\operatorname{osc}_\tau(g \circ \varphi)$ is bounded on X. By Corollary 4.4, $\operatorname{osc}_\tau(g)$ is bounded on Y and, again by Theorem 3.1, g can be written as a difference of bounded semicontinuous functions on Y. □

We shall now move from the case of differences of bounded semicontinuous functions to the case of differences of semicontinuous functions. It is not obvious whether the idea presented above, that is to write f as a difference of $\operatorname{osc}_\tau f \pm f$, would work. The reason is that $\operatorname{osc}_\tau f \pm f$ may be infinite at some points. For example, let $f(x) = 1/x$ on $(0, 1]$ and $f(0) = 0$. Then, obviously, f is lower semicontinuous on $[0, 1]$ and $\operatorname{osc}_1 f(0) = +\infty$. Therefore, we shall use another approach.

Proposition 4.6. *Let K be a compact space and $f : K \to \mathbb{R}$. Then f is a difference of two real-valued lower semicontinous functions on K if and only if there exists a sequence of compact sets K_n such that $\cup_{n\in\mathbb{N}} K_n = K$, $K_n \subset K_{n+1}$ and $f|_{K_n}$ can be written as a difference of bounded semicontinous functions on K_n.*

Proof. Assume that $f = u - v$, where u, v are real valued and lower semicontinous. Since each lower bounded lower semicontinuous function attains its minimum on the compact K, we may assume, without any loss of generality, that both u and v are nonnegative. For each $n \in \mathbb{N}$, set

$$K_n = \{x \in K : u(x) \leq n\} \cap \{y \in K : v(y) \leq n\}.$$

Obviously, K_n is compact, $\cup_{n=1}^{\infty} K_n = K$, $K_n \subset K_{n+1}$ and both u and v are bounded on K_n. Conversely, assume that there exists an increasing sequence of compact sets K_n such that $\cup_{n\in\mathbb{N}} K_n = K$ and

$$f = u_n - v_n \qquad \text{on } K_n,$$

where u_n and v_n are bounded lower semicontinuous functions. If $K_n = K$, for any $n \in \mathbb{N}$, everything is trivial. So, let us assume that $K_n \neq K$, for each $n \in \mathbb{N}$. Then, without any loss of generality, we may assume that $K_{n+1} \neq K_n$ for each $n \in \mathbb{N}$. We define

$$\tilde{u} = u_n + C_n \qquad \text{on } K_n \setminus K_{n-1},$$
$$\tilde{v} = v_n + C_n \qquad \text{on } K_n \setminus K_{n-1},$$

where C_n are constants that are chosen such that

$$\inf_{K_n \setminus K_{n-1}} u_n + C_n > \sup_{K_{n-1}} u_{n-1} + C_{n-1}$$

and

$$\inf_{K_n \setminus K_{n-1}} v_n + C_n > \sup_{K_{n-1}} v_{n-1} + C_{n-1}.$$

It is now enough to show that $\tilde{u}$ and $\tilde{v}$ are lower semicontinous on K. Fix $x \in K$. Then there exists a unique $n \in \mathbb{N}$ such that $x \in K_n \setminus K_{n-1}$. Since each compact space is regular, there exists a neighborhood V_x of x that does not intersect K_{n-1}. Therefore,

$$\begin{aligned}
\liminf_{y\to x} \tilde{u}(y) &= \sup\{\inf u(V) : V \text{ neighborhood of } x\} \\
&= \sup\{\inf \tilde{u}(V \cap K_n) : V \text{ neighborhood of } x\} \\
&= \sup\{\inf u_n(V \cap K_n) : V \text{ neighborhood of } x\} + C_n \\
&= \sup\{\inf u_n(V) : V \text{ is neighborhood of } x \text{ in } K_n\} + C_n \\
&= \liminf_{y\to x} u_n(y) + C_n = u_n(x) + C_n = \tilde{u}(x).
\end{aligned}$$

Hence $\tilde{u}$ is lower semicontinuous at x. Analogously, the same property holds for $\tilde{v}$. Since $x \in K$ is arbitrary, the proof is complete. □

THEOREM 4.7. *The class of differences of semicontinuous functions is perfect.*

Proof. Let X, Y be compact spaces, $\varphi : X \to Y$ be a continuous surjective function and $g : Y \to \mathbb{R}$. It is obvious that, if g is a difference of semicontinuous functions, then $g \circ \varphi$ has the same property. Now, let us assume that $g \circ \varphi$ can be written as a difference of two (lower) semicontinuous functions. By

Proposition 4.6, there exists a sequence of compact subsets X_n of X such that $\cup_{n\in\mathbb{N}} X_n = X$, $X_n \subset X_{n+1}$ and $(g \circ \varphi)|_{X_n}$ is a difference of two bounded semicontinous functions on X_n. Set $Y_n = \varphi(X_n)$. Then Y_n is a compact subset of Y, $\cup_{n\in\mathbb{N}} Y_n = Y$ and $Y_n \subset Y_{n+1}$. Since $\varphi|_{X_n}$ is a continuous surjective function from X_n onto Y_n, it follows, by Theorem 4.5, that $g|_{Y_n}$ can be written as a difference of two bounded semicontinuous functions on Y_n. But, again by Proposition 4.6, g is a difference of two semicontinous functions. □

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