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NOTE ON THE MINIMUM NUMBER OF CRITICAL POINTS OF REAL OR CIRCULAR FUNCTIONS

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Abstract. The aim of this note is to estimate the minimum number of critical points of circular functions, which are defined on a product of two manifolds, in terms of the minimum number of critical points of circular functions, defined on the two factors. Some special attention is paid to the class of circular Morse functions. Such estimations were previously done for real functions in [2].

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Key words. Circular function, φ-category, circular φ-category, Morse function.

1. INTRODUCTION

Let M, N be two smooth manifolds, such that $\dim(M) = m$, $\dim(N) = n$ and let $\mathcal{F} \subseteq C^\infty(M, N)$ be a family of smooth mappings. Then we can define the $\varphi_{\mathcal{F}}$-category of the pair (M, N) as

$$\varphi_{\mathcal{F}}(M, N) = \min\{\mu(f) |\ f \in \mathcal{F}\},$$

where $\mu(f) = \operatorname{card}(C(f))$ represents the cardinality of the set of critical points of the function f.

Note that $0 \leq \varphi_{\mathcal{F}} \leq \infty$ and $\varphi_{\mathcal{F}} = 0$ if and only if $\mathcal{F}$ contains immersions, submersions or local diffeomorphisms.

Recall that

(1) $\varphi_{\mathcal{F}}(M, \mathbb{R})$ is denoted by $\varphi(M)$, if $\mathcal{F} = C^\infty(M, \mathbb{R})$;
(2) $\varphi_{\mathcal{F}}(M, S^1)$ is denoted by $\varphi_{S^1}(M)$ if $\mathcal{F} = C^\infty(M, S^1)$;
(3) $\varphi_{\mathcal{F}}(M, \mathbb{R})$ is denoted by $\gamma(M)$, if

$$\mathcal{F} = \{f \in C^\infty(M, \mathbb{R})\ :\ f - \text{Morse function}\};$$

(4) $\varphi_{\mathcal{F}}(M, S^1)$ is denoted by $\gamma_{S^1}(M)$, if

$$\mathcal{F} = \{f \in C^\infty(M, S^1) : f - \text{Morse function}\}.$$

In other words,

$$\begin{aligned}
\varphi(M) &= \min\{\mu(f) : f \in C^\infty(M, \mathbb{R})\}\\
\varphi_{S^1}(M) &= \min\left\{\mu(f) : f \in C^\infty(M, S^1)\right\}\\
\gamma(M) &= \min\{\mu(f) : f : M \to \mathbb{R} \text{ Morse function}\}\\
\gamma_{S^1}(M) &= \min\left\{\mu(f) : f : M \to S^1 \text{ Morse function}\right\}.
\end{aligned}$$

If $f : M^m \to \mathbb{R}$ is a Morse function, recall that $\mu(f) = \sum_{k=1}^m \mu_k(f)$, where

$$\mu_k(f) = \{x \in C(f) : \operatorname{index}(x) = k\},\ 0 \leq k \leq m.$$

For the above cardinalities, we have the following labels [1, 2, 3]:

(1) $\varphi(M)$ is called the *φ-category of M*;
(2) $\varphi_{S^1}(M)$ is called the *circular φ-category of M*;
(3) $\gamma(M)$ is called the *Morse-Smale characteristic* of M;
(4) $\gamma_{S^1}(M)$ is called the *Morse-Smale characteristic* of M.

According to Takens [7], the following inequalities hold:

$$\mathrm{cat}(M) \leq \varphi(M) \leq \dim(M) + 1,$$

where $\mathrm{cat}(M)$ is a homotopical invariant, which provides the smallest number of open contractible subsets that cover X such that the inclusion map is null-homotopic, called the *Lusternik-Schnirelmann category* or *LS-category.* On the other hand, the inequalities

$$\varphi(M) \leq \gamma(M), \quad \varphi_{S^1}(M) \leq \gamma_{S^1}(M) \tag{1}$$

are obvious. We shall justify here the submultiplicativity properties of φ_{S^1} and γ_{S^1}, as the submultiplicativity properties of γ and φ have been earlier proved by Andrica [1, pp. 109, 131]. Note however that the submultiplicativity property of φ_{S^1} appears in [4] in a more general setting. Some consequences of these submultiplicativity properties will be also pointed out.

2. THE SUBMULTIPLICATIVITY PROPERTY OF φ_{S^1}

Let M, N be two manifolds and $f : M \to G$ and $g : N \to G$ be given maps. We then define the product map

$$f \odot g : M \times N \to G, \ (f \odot g)(x, y) = f(x)g(y).$$

Proposition 2.1 ([4]). *Let M, N be smooth manifolds with $\dim(M) = m$, $\dim(N) = n$, and let $(G, \cdot)$ be a Lie group of dimension $\dim(G) \leq \min(m, n)$. Then, for two smooth maps $A : M \to G$ and $B : N \to G$, we have*

$$C(A \odot B) \subseteq C(A) \times C(B).$$

Corollary 2.2. *For two manifold M, N, the following inequality holds*

$$\varphi_{S^1}(M \times N) \leq \varphi_{S^1}(M)\varphi_{S^1}(N). \tag{2}$$

Moreover, if $\chi(M), \chi(N) \neq 0$, then $\varphi_{S^1}(M \times N) \geq 1$.

Proof. The inequality in (2) follows via Proposition 2.1, with the circle S^1 playing the role of the Lie group G. Although inequality (2) appears in [4] in a more general setting, we shall provide here the details of the proof for (2). If the functions $f : M \longrightarrow S^1, \ g : N \longrightarrow S^1$ are such that $\mu(f) = \varphi_{S^1}(M)$ and $\mu(g) = \varphi_{S^1}(N)$, then $\mu(f \odot g) \leq \mathrm{card}(C(f) \times C(g))$, as $C(f \odot g) \subseteq C(f) \times C(g)$. Thus

$$\begin{aligned}\varphi_{S^1}(M \times N) &\leq \mu(f \odot g) \leq \mathrm{card}(C(f) \times C(g)) \\ &= \mathrm{card}(C(f))\mathrm{card}(C(g)) = \varphi_{S^1}(M)\varphi_{S^1}(N).\end{aligned}$$

If $\varphi_{S^1}(M \times N) = 0$, then there is a fibration $F \hookrightarrow M \times N \to S^1$, and thus we obtain $0 \neq \chi(M)\chi(N) = \chi(M \times N) = \chi(F)\chi(S^1) = 0$, which is absurd. $\square$

For $n \in \mathbb{N}^*$, let $SO(n) = \{A \in O(n) : \det(A) = 1\}$ be the special orthogonal group, a subgroup of the orthogonal group $O(n) = \{A \in GL(n) : A^{-1} = A^t\}$, and $Spin(n)$ be the spinor group (or Spin group), described as the universal covering space of $SO(n)$.

EXAMPLE 2.3. Let M be an m-dimensional manifold and let $SO(n)$ and $Spin(n)$ be the groups described above. If $n \geq 2$, then the following inequalities are true

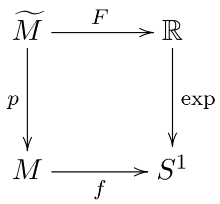
(1) $\varphi_{S^1}(M \times S^n) \leq 2\varphi_{S^1}(M)$. If $\chi(M) \neq 0$, then $\varphi_{S^1}(M \times S^n) \geq 1$.
(2) $\varphi_{S^1}(M \times SO(n)) \leq 2^{n-1}\varphi_{S^1}(M)$.
(3) $\varphi_{S^1}(M \times Spin(n)) \leq 2^n\varphi_{S^1}(M)$.
(4) If $n \geq 3$ and $1 \leq k \leq n-1$, then $\varphi_{S^1}(G_{k,n} \times M) \leq \binom{n+k}{k}\varphi_{S^1}(M)$, where $G_{k,n}$ stands for the Grassmann manifold of all k-dimensional subspaces of $\mathbb{R}^{n+k}$.

3. THE SUBMULTIPLICATIVITY PROPERTY OF γ_{S^1}

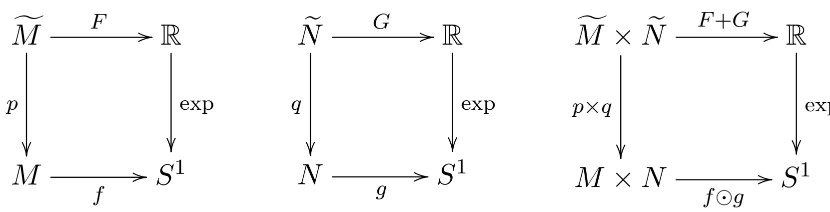
From [6] we have the following description for a function f that lifts to a real-valued Morse function F on $\widetilde{M}$. Let M be a closed smooth manifold, $f : M \to S^1$ be a circular Morse function and $p : \widetilde{M} \to M$ be the infinite cyclic covering induced by the function f from the covering $\exp : \mathbb{R} \to S^1$, where $\exp(t) = e^{2\pi i t}$. Then we have

$$f \circ p = \exp \circ F, \tag{3}$$

or, equivalently, the following diagram is commutative



PROPOSITION 3.1. *Let f, g be circular Morse functions that satisfy* (3)*. Then $f \odot g$ is also a Morse function and the third diagram, describing the lift of $f \odot g$ to $F + G$, is commutative.*



Proof. In order to prove the relation

$$\exp(f(x)+G(y)) = (f\odot g)\circ(p\times q)(x,y)$$

we need to show that

$$\begin{aligned}(f\odot g)\circ(p\times q)(x,y) &= (f\odot g)\,(p(x),g(y))\\ &= f\,(p(x))\cdot g\,(q(x))\\ &= (f\circ p)\,(x)\cdot(g\circ q)\,(y)\\ &= (f\circ p)\odot(g\circ q).\end{aligned}\tag{4}$$

Returning to our result, we have

$$\begin{aligned}\exp(f(x)+G(y)) &= exp(F(x))\cdot exp(F(y))\\ &= (f\circ p)(x)\odot(g\circ q)(y)\\ &= (f\circ p)\odot(g\circ q)\\ &= (f\odot g)\circ(p\times q)(x,y),\end{aligned}\tag{5}$$

and the desired property follows. □

COROLLARY 3.2. *If M, N are two manifolds, then the following inequality holds* $\gamma_{S^1}(M\times N)\le\gamma_{S^1}(M)\gamma_{S^1}(N)$. *Moreover, if* $\chi(M),\chi(N)\neq 0$, *then* $\gamma_{S^1}(M\times N)\ge 1$.

Proof. Let $f:M\to S^1$, $g:N\to S^1$ be Morse functions such that $\mu(f)=\gamma_{S^1}(M)$ and $\mu(g)=\gamma_{S^1}(N)$. According to Proposition 3.1, the product $f\odot g$ is also a Morse function and $\mu(f\odot g)\le \mathrm{card}(C(f)\times C(g))$, as $C(f\odot g)\subseteq C(f)\times C(g)$ and due to Proposition 2.1. Thus,

$$\begin{aligned}\gamma_{S^1}(M\times N) &\le \mu(f\odot g)\le \mathrm{card}(C(f)\times C(g))\\ &= \mathrm{card}(C(f))\mathrm{card}(C(g)) = \gamma_{S^1}(M)\gamma_{S^1}(N).\end{aligned}$$

Finally, if $\chi(M),\chi(N)\neq 0$, then $\varphi_{S^1}(M\times N)\ge 1$, due to Proposition 2.1. Thus, $\gamma_{S^1}(M\times N)\ge\varphi_{S^1}(M\times N)\ge 1$. □

EXAMPLE 3.3. Let (G,g) be a n-dimensional Lie group endowed with the Riemannian metric g and the sphere bundle defined as

$$SG=\{(g,v)\in TG:\|v\|=1\}\cong G\times S^{n-1}.$$

If $n\ge 3$, then $\varphi_{S^1}(SG)\le 2\varphi_{S^1}(G)$.

EXAMPLE 3.4. If $n\ge 2$, then the following inequalities hold

(1) $\gamma_{S^1}(M\times S^n)\le 2\gamma_{S^1}(M)$. If $\chi(M)\neq 0$, then $\varphi_{S^1}(M\times S^n)\ge 1$.
(2) $\gamma_{S^1}(M\times SO(n))\le 2^{n-1}\gamma_{S^1}(M)$.
(3) $\gamma_{S^1}(M\times Spin(n))\le 2^{n}\gamma_{S^1}(M)$.
(4) If $n\ge 3$ and $1\le k\le n-1$, then $\gamma_{S^1}(G_{k,n}\times M)\le\binom{n+k}{k}\gamma_{S^1}(M)$.

THEOREM 3.5 ([3]). *For the circular Morse-Smale characteristic of a closed surface* $\Sigma\neq\mathbb{RP}^2$, *the following relation holds* $\gamma_{S^1}(\Sigma)=|\chi(\Sigma)|$, *where* $\chi(\Sigma)$ *is the Euler-Poincaré characteristic of* Σ.

EXAMPLE 3.6. The following inequalities hold:

(1) $1 \leq \gamma_{S^1}\left(\Sigma_g \times \Sigma_{g'}\right) \leq \left|\chi\left(\Sigma_g \times \Sigma_{g'}\right)\right| = (2g-2)(2g'-2)$ for surfaces of genus $g, g' \geq 2$. In particular, $1 \leq \gamma_{S^1}(\Sigma_2 \times \Sigma_2) \leq 4$;

(2) $\gamma_{S^1}(M \times \Sigma_g) \leq (2g-2)\gamma_{S^1}(M)$.

PROPOSITION 3.7 ([2]). *Let $U(n)$ denote the unitary group and let $SU(n)$ be the special unitary group. Then the following inequalities hold*

(1) $n \leq \varphi(U(n)) \leq \gamma(U(n)) \leq 2^n$;

(2) $n-1 \leq \varphi_{S^1}(SU(n)) = \varphi(SU(n)) \leq \gamma(SU(n)) = \gamma_{S^1}(SU(n)) \leq 2^{n-1}$.

REMARK 3.8. The unitary group is diffeomorphic to the product $SU(n) \times S^1$. Therefore, $0 = \varphi_{S^1}(U(n)) < n \leq \varphi(U(n))$.

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