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SOME INTEGRAL INEQUALITIES OF HERMITE-HADAMARD TYPE FOR GG-CONVEX FUNCTIONS

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Abstract. Some related integral inequalities of Hermite-Hadamard type for GG-convex functions defined on positive intervals are given. Applications for the exponential integral mean are also provided.

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Key words. Convex function, integral inequality, GG-Convex function, Hermite-Hadamard type inequality.

1. INTRODUCTION

We recall first some facts on GG-convex functions and Hermite-Hadamard type inequalities.

The function $f : I \subset (0, \infty) \to (0, \infty)$ is called *GG-convex* on the interval I of real umbers $\mathbb{R}$ if (see [4])

$$f\left(x^{1-\lambda}y^{\lambda}\right) \le [f(x)]^{1-\lambda}[f(y)]^{\lambda} \tag{1}$$

for any $x, y \in I$ and $\lambda \in [0, 1]$. If the inequality is reversed in (1), then the function is called *GG-concave*.

This concept was introduced in 1928 by P. Montel [49], however, the roots of the research in this area can be traced long before him (see [50]).

It is easy to see that (see [50]) the function $f : I \subset (0, \infty) \to (0, \infty)$ is *GG-convex* if and only if the the function $g : \ln I \to \mathbb{R}$, $g = \ln \circ f \circ \exp$ is convex on $\ln I$.

It is known that (see [50]) every real analytic function $f(x) = \sum_{n=0}^{\infty} c_n x^n$, with non-negative coefficients c_n, is a GG-convex function on $(0, r)$, where r is the radius of convergence for f. Therefore, functions like exp, sinh, cosh are GG-convex on $\mathbb{R}$, tan, sec, csc, $\frac{1}{x} - \cot x$ are GG-convex on $\left(0, \frac{\pi}{2}\right)$ and $\frac{1}{1-x}$, $\ln \frac{1}{1-x}$, $\frac{1+x}{1-x}$ are GG-convex on $(0, 1)$. Also, the Γ function is a strictly GG-convex function on $[1, \infty)$.

It is also known that (see [50]), if a function f is GG-convex, then so is $x^{\alpha} f^{\beta}(x)$, for all $\alpha \in \mathbb{R}$ and all $\beta > 0$. If f is continuous and one of the functions $f(x)^x$ and $f(e^{1/\log x})$ is GG-convex, then so is the other.

As pointed out in [50], the *Lobacevski's function*, given by

$$L(x) := -\int_0^x \ln(\cos t)\, dt,$$

is GG-convex on $(0,\pi/2)$ and the *integral sine*, given by

$$\mathrm{Si}\left(x\right):=\int_{0}^{x}\frac{\sin t}{t}\mathrm{d}t,$$

is GG-concave on $(0,\pi/2)$.

We recall the classical Hermite-Hadamard inequality that states that

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_{a}^{b}f\left(t\right)\mathrm{d}t\leq\frac{f\left(a\right)+f\left(b\right)}{2}, \tag{2}$$

for any convex function $f:[a,b]\to\mathbb{R}$.

For related results, see [1–20, 22–53].

We define the *logarithmic mean* $L\left(a,b\right)$ of two positive numbers a and b by

$$L\left(a,b\right):=\begin{cases}\frac{b-a}{\ln b-\ln a}, & \text{if } b\neq a,\\ b, & \text{if } b=a.\end{cases}$$

In 2010, Zhang and Zheng [62] proved the following inequality for a GG-convex function f on $[a,b]$:

$$\frac{1}{\ln b-\ln a}\int_{a}^{b}f\left(t\right)\mathrm{d}t\leq L\left(af\left(a\right),bf\left(b\right)\right). \tag{3}$$

In 2011, Mitroi and Spiridon [48] established, among other results, the following double inequality

$$f\left(I\left(a,b\right)\right)\leq\exp\left(\frac{1}{b-a}\int_{a}^{b}\ln f\left(t\right)\mathrm{d}t\right)\leq\left[f\left(b\right)\right]^{\frac{b-L(a,b)}{b-a}}\left[f\left(a\right)\right]^{\frac{L(a,b)-a}{b-a}}, \tag{4}$$

where $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is GG-convex and $I\left(a,b\right)$ is the *identric mean* of the positive numbers a and b, given by

$$I\left(a,b\right):=\begin{cases}\frac{1}{e}\left(\frac{b^{b}}{a^{a}}\right)^{\frac{1}{b-a}}, & \text{if } b\neq a\\ b, & \text{if } b=a.\end{cases}$$

In 2013, Işcan [41] also proved the following result

$$\begin{aligned}f\left(\sqrt{ab}\right)&\leq\frac{1}{\ln b-\ln a}\int_{a}^{b}\frac{1}{t}\sqrt{f\left(t\right)f\left(\frac{ab}{t}\right)}\mathrm{d}t\\&\leq\frac{1}{\ln b-\ln a}\int_{a}^{b}\frac{f\left(t\right)}{t}\mathrm{d}t\leq L\left(f\left(a\right),f\left(b\right)\right),\end{aligned} \tag{5}$$

provided that $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is GG-convex.

In a recent paper [26], by using some results for GA-convex functions from [24], we proved, among others, the following results.

THEOREM 1.1. *Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be a GG-convex function on $[a,b]$. Then, for any $\lambda\in[0,1]$, we have*

$$
\begin{aligned}
f\left(\sqrt{ab}\right) &\le \left[f\left(a^{\frac{1-\lambda}{2}}b^{\frac{\lambda+1}{2}}\right)\right]^{1-\lambda}\left[f\left(a^{\frac{2-\lambda}{2}}b^{\frac{\lambda}{2}}\right)\right]^{\lambda}\\
&\le \exp\left(\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(t)}{t}\mathrm{d}t\right)\\
&\le \sqrt{f\left(a^{1-\lambda}b^{\lambda}\right)[f(b)]^{1-\lambda}[f(a)]^{\lambda}}\le\sqrt{f(a)f(b)}.
\end{aligned}\tag{6}
$$

THEOREM 1.2. *Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be a GG-convex function on $[a,b]$. Then we have the following inequalities*

$$
1\le\frac{\sqrt{f(a)f(b)}}{\exp\left(\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(s)}{s}ds\right)}\le\left(\frac{b}{a}\right)^{\frac{1}{8}\left(\frac{f'_-(b)b}{f(b)}-\frac{f'_+(a)a}{f(a)}\right)}\tag{7}
$$

and

$$
1\le\frac{\exp\left(\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(s)}{s}ds\right)}{f\left(\sqrt{ab}\right)}\le\left(\frac{b}{a}\right)^{\frac{1}{8}\left(\frac{f'_-(b)b}{f(b)}-\frac{f'_+(a)a}{f(a)}\right)}.\tag{8}
$$

Motivated by the above results, we establish in this paper some new inequalities of Hermite-Hadamard type for GG-convex functions. Applications for the exponential integral mean are also provided.

2. NEW RESULTS

We start with the following inequality for powers of GG-convex functions.

THEOREM 2.1. *Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be a GG-convex function on $[a,b]$.*

(i) *If $p\in\left(0,\frac{1}{2}\right]$, then we have*

$$
\begin{aligned}
f\left(\sqrt{ab}\right) &\le \exp\left[\frac{1}{\ln b-\ln a}\int_a^b\frac{\ln f(t)}{t}\mathrm{d}t\right]\\
&\le\left(\frac{1}{\ln b-\ln a}\int_a^b\frac{1}{t}f^p(t)\,f^p\left(\frac{\sqrt{ab}}{t}\right)\mathrm{d}t\right)^{\frac{1}{2p}}\\
&\le\left(\frac{1}{\ln b-\ln a}\int_a^b\frac{1}{t}f^{2p}(t)\,\mathrm{d}t\right)^{\frac{1}{2p}}\le\frac{1}{\ln b-\ln a}\int_a^b\frac{f(t)}{t}\mathrm{d}t\\
&\le L\left(f(a),f(b)\right).
\end{aligned}\tag{9}
$$

(ii) *For $p>0$, but $p\neq\frac{1}{2}$, we also have*

$$
\begin{aligned}
f\left(\sqrt{ab}\right) &\leq \exp\left[\frac{1}{\ln b-\ln a}\int_a^b \frac{\ln f(t)}{t}\mathrm{d}t\right] \\
&\leq \left(\frac{1}{\ln b-\ln a}\int_a^b \frac{1}{t} f^p(t) f^p\left(\frac{\sqrt{ab}}{t}\right)\mathrm{d}t\right)^{\frac{1}{2p}} \\
&\leq \left(\frac{1}{\ln b-\ln a}\int_a^b \frac{1}{t} f^{2p}(t)\,\mathrm{d}t\right)^{\frac{1}{2p}} \\
&\leq \left[L_{2p-1}\left(f(a),f(b)\right)\right]^{1-\frac{1}{2p}}\left[L\left(f(a),f(b)\right)\right]^{\frac{1}{2p}}.
\end{aligned}
\tag{10}
$$

If we take $p=\frac{1}{4}$ in (9), then we get

$$
\begin{aligned}
f\left(\sqrt{ab}\right) &\leq \exp\left[\frac{1}{\ln b-\ln a}\int_a^b \frac{\ln f(t)}{t}\mathrm{d}t\right] \\
&\leq \left(\frac{1}{\ln b-\ln a}\int_a^b \frac{1}{t}\sqrt[4]{f(t) f\left(\frac{\sqrt{ab}}{t}\right)}\mathrm{d}t\right)^{2} \\
&\leq \left(\frac{1}{\ln b-\ln a}\int_a^b \frac{1}{t}\sqrt{f(t)}\mathrm{d}t\right)^{2} \\
&\leq \frac{1}{\ln b-\ln a}\int_a^b \frac{f(t)}{t}\mathrm{d}t \leq L\left(f(a),f(b)\right).
\end{aligned}
\tag{11}
$$

The case $p=\frac{1}{2}$ produces

$$
\begin{aligned}
f\left(\sqrt{ab}\right) &\leq \exp\left[\frac{1}{\ln b-\ln a}\int_a^b \frac{\ln f(t)}{t}\mathrm{d}t\right] \\
&\leq \frac{1}{\ln b-\ln a}\int_a^b \frac{1}{t}\sqrt{f(t) f\left(\frac{\sqrt{ab}}{t}\right)}\mathrm{d}t \leq \frac{1}{\ln b-\ln a}\int_a^b \frac{f(t)}{t}\mathrm{d}t \\
&\leq L\left(f(a),f(b)\right).
\end{aligned}
\tag{12}
$$

Also, if we take $p=1$ in (10), then we get

$$
\begin{aligned}
f\left(\sqrt{ab}\right) &\leq \exp\left[\frac{1}{\ln b-\ln a}\int_a^b \frac{\ln f(t)}{t}\mathrm{d}t\right] \\
&\leq \sqrt{\frac{1}{\ln b-\ln a}\int_a^b \frac{1}{t} f(t) f\left(\frac{\sqrt{ab}}{t}\right)\mathrm{d}t}
\end{aligned}
\tag{13}
$$

$$\leq \sqrt{\frac{1}{\ln b - \ln a} \int_a^b \frac{1}{t} f^2(t)\, \mathrm{d}t}$$
$$\leq \sqrt{A(f(a), f(b))} \sqrt{L(f(a), f(b))}.$$

We also have the following result.

Theorem 2.2. *Let $f : [a,b] \subset (0,\infty) \to (0,\infty)$ be a GG-convex function on $[a,b]$. Then, for every $s \in [a,b]$, we have the inequality*

$$(14)\quad \begin{aligned}&(\ln b - \ln s) f(b) + (\ln s - \ln a) f(a) - \int_a^b \frac{f(t)}{t} \mathrm{d}t \\ &\geq \int_a^b \frac{f(t) \ln f(t)}{t} \mathrm{d}t - \ln f(s) \int_a^b \frac{f(t)}{t} \mathrm{d}t.\end{aligned}$$

If we take $s = G(a,b) = \sqrt{ab}$ in (14), then we get

$$(15)\quad \begin{aligned}&(\ln b - \ln a) \frac{f(b) + f(a)}{2} - \int_a^b \frac{f(t)}{t} \mathrm{d}t \\ &\geq \int_a^b \frac{f(t) \ln f(t)}{t} \mathrm{d}t - \ln f(G(a,b)) \int_a^b \frac{f(t)}{t} \mathrm{d}t.\end{aligned}$$

Also, if we take $s = I(a,b)$, then we get from (14) that

$$(16)\quad \begin{aligned}&(\ln b - \ln I(a,b)) f(b) + (\ln I(a,b) - \ln a) f(a) - \int_a^b \frac{f(t)}{t} \mathrm{d}t \\ &\geq \int_a^b \frac{f(t) \ln f(t)}{t} \mathrm{d}t - \ln f(I(a,b)) \int_a^b \frac{f(t)}{t} \mathrm{d}t.\end{aligned}$$

Since simple calculations show that

$$\ln b - \ln I(a,b) = \frac{L(a,b) - a}{L(a,b)}$$

and

$$\ln I(a,b) - \ln a = \frac{b - L(a,b)}{L(a,b)},$$

the inequality (16) can be written as

$$(17)\quad \begin{aligned}&\frac{L(a,b) - a}{L(a,b)} f(b) + \frac{b - L(a,b)}{L(a,b)} f(a) - \int_a^b \frac{f(t)}{t} \mathrm{d}t \\ &\geq \int_a^b \frac{f(t) \ln f(t)}{t} \mathrm{d}y - \ln f(I(a,b)) \int_a^b \frac{f(t)}{t} \mathrm{d}t.\end{aligned}$$

Moreover, if we take the integral mean in (14), then we get

$$
\begin{aligned}
&(\ln b-\ln I(a,b)) f(b)+(\ln I(a,b)-\ln a) f(a)-\int_a^b \frac{f(t)}{t}\mathrm{d}t \\
&\geq \int_a^b \frac{f(t)\ln f(t)}{t}\mathrm{d}t-\frac{1}{b-a}\int_a^b \ln f(s)\,ds \int_a^b \frac{f(t)}{t}\mathrm{d}t.
\end{aligned} \tag{18}
$$

This can be also written as

$$
\begin{aligned}
&\frac{L(a,b)-a}{L(a,b)} f(b)+\frac{b-L(a,b)}{L(a,b)} f(a)-\int_a^b \frac{f(t)}{t}\mathrm{d}t \\
&\geq \int_a^b \frac{f(t)\ln f(t)}{t}\mathrm{d}t-\frac{1}{b-a}\int_a^b \ln f(s)\,ds \int_a^b \frac{f(t)}{t}\mathrm{d}t.
\end{aligned} \tag{19}
$$

From a different perspective we have the following result.

Theorem 2.3. *Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be a GG-convex function on $[a,b]$. Then we have the inequality*

$$
\begin{aligned}
&\frac{f(b)+f(a)}{2}-\frac{1}{\ln b-\ln a}\int_a^b \frac{f(t)}{t}\mathrm{d}t \\
&\geq \int_a^b \frac{f(t)}{t}\ln f(t)\,\mathrm{d}t-\int_a^b \frac{f(t)}{t}\mathrm{d}t\frac{1}{\ln b-\ln a}\int_a^b \frac{\ln f(t)}{t}\mathrm{d}t \geq 0.
\end{aligned} \tag{20}
$$

Also, we can state the following result as well.

Theorem 2.4. *Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be a GG-convex function on $[a,b]$. Then we have the inequality*

$$
\begin{aligned}
&f(b)\left(\ln b-\frac{\int_a^b \frac{f(t)\ln t}{t}\mathrm{d}t}{\int_a^b \frac{f(t)}{t}\mathrm{d}t}\right)+f(a)\left(\frac{\int_a^b \frac{f(t)\ln t}{t}\mathrm{d}t}{\int_a^b \frac{f(t)}{t}\mathrm{d}t}-\ln a\right)-\int_a^b \frac{f(t)}{t}\mathrm{d}t \\
&\geq \int_a^b \frac{f(t)}{t}\ln f(t)\,\mathrm{d}t-\int_a^b \frac{f(t)}{t}\ln f\left(\exp\left(\frac{\int_a^b \frac{f(t)\ln t}{t}\mathrm{d}t}{\int_a^b \frac{f(t)}{t}\mathrm{d}t}\right)\right) \geq 0.
\end{aligned} \tag{21}
$$

Finally, we have the following theorem.

THEOREM 2.5. *Let $f : [a,b] \subset (0,\infty) \to (0,\infty)$ be a GG-convex function on $[a,b]$. Then we have the inequality*

$$\begin{aligned}
&\left(\frac{1}{\ln b-\ln a}\int_a^b \frac{f^{2p}(t)}{t}\mathrm{d}t\right)^{\frac{1}{2p}} \\
&\geq \left(\frac{1}{\ln b-\ln a}\int_a^b \left[f^{2p}(t)\right]^{1-\alpha}\left[f^{2p}\left(\frac{ab}{t}\right)\right]^{\alpha}\mathrm{d}t\right)^{\frac{1}{2p}} \\
&\geq \left(\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{a^{1-\alpha}b^{\alpha}}^{a^{\alpha}b^{1-\alpha}} \frac{f^{2p}(t)}{t}\mathrm{d}t\right)^{\frac{1}{2p}} \qquad (22)\\
&\geq \left(\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{a^{1-\alpha}b^{\alpha}}^{a^{\alpha}b^{1-\alpha}} \frac{f^{p}(t)f^{p}\left(\frac{ab}{t}\right)}{t}\mathrm{d}t\right)^{\frac{1}{2p}} \\
&\geq \exp\left[\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{a^{1-\alpha}b^{\alpha}}^{a^{\alpha}b^{1-\alpha}} \frac{\ln f(t)}{t}\mathrm{d}t\right] \geq f\left(\sqrt{ab}\right),
\end{aligned}$$

for every $\alpha \in [0,1] \setminus \left\{\frac{1}{2}\right\}$ and $p > 0$.

For $p = \frac{1}{4}$ in (22), we get

$$\begin{aligned}
&\left(\frac{1}{\ln b-\ln a}\int_a^b \frac{\sqrt{f(t)}}{t}\mathrm{d}t\right)^{2} \\
&\geq \left(\frac{1}{\ln b-\ln a}\int_a^b \sqrt{f^{1-\alpha}(t)}\left[\sqrt{f^{\alpha}\left(\frac{ab}{t}\right)}\right]\mathrm{d}t\right)^{2} \\
&\geq \left(\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{a^{1-\alpha}b^{\alpha}}^{a^{\alpha}b^{1-\alpha}} \frac{\sqrt{f(t)}}{t}\mathrm{d}t\right)^{2} \qquad (23)\\
&\geq \left(\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{a^{1-\alpha}b^{\alpha}}^{a^{\alpha}b^{1-\alpha}} \frac{\sqrt[4]{f(t)f\left(\frac{ab}{t}\right)}}{t}\mathrm{d}t\right)^{2} \\
&\geq \exp\left[\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{a^{1-\alpha}b^{\alpha}}^{a^{\alpha}b^{1-\alpha}} \frac{\ln f(t)}{t}\mathrm{d}t\right] \geq f\left(\sqrt{ab}\right).
\end{aligned}$$

If we take $p = \frac{1}{2}$ in (22), then we get

$$\begin{aligned}
&\frac{1}{\ln b-\ln a}\int_a^b \frac{f(t)}{t}\mathrm{d}t \qquad (24)\\
&\geq \frac{1}{\ln b-\ln a}\int_a^b [f(t)]^{1-\alpha}\left[f\left(\frac{ab}{t}\right)\right]^{\alpha}\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{a^{1-\alpha}b^{\alpha}}^{a^{\alpha}b^{1-\alpha}}\frac{f(t)}{t}\mathrm{d}t\\
&\geq \frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{a^{1-\alpha}b^{\alpha}}^{a^{\alpha}b^{1-\alpha}}\frac{\sqrt{f(t)\,f\left(\frac{ab}{t}\right)}}{t}\mathrm{d}t\\
&\geq \exp\left[\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{a^{1-\alpha}b^{\alpha}}^{a^{\alpha}b^{1-\alpha}}\frac{\ln f(t)}{t}\mathrm{d}t\right]\geq f\left(\sqrt{ab}\right).
\end{aligned}
$$

Finally, by taking $p=1$ in (22), we get

$$
\begin{aligned}
&\sqrt{\frac{1}{\ln b-\ln a}\int_{a}^{b}\frac{f^{2}(t)}{t}\mathrm{d}t}\\
&\geq \sqrt{\frac{1}{\ln b-\ln a}\int_{a}^{b}\left[f^{2}(t)\right]^{1-\alpha}\left[f^{2}\left(\frac{ab}{t}\right)\right]^{\alpha}\mathrm{d}t}\\
&\geq \sqrt{\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{a^{1-\alpha}b^{\alpha}}^{a^{\alpha}b^{1-\alpha}}\frac{f^{2}(t)}{t}\mathrm{d}t}\\
&\geq \sqrt{\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{a^{1-\alpha}b^{\alpha}}^{a^{\alpha}b^{1-\alpha}}\frac{f(t)\,f\left(\frac{ab}{t}\right)}{t}\mathrm{d}t}\\
&\geq \exp\left[\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{a^{1-\alpha}b^{\alpha}}^{a^{\alpha}b^{1-\alpha}}\frac{\ln f(t)}{t}\mathrm{d}t\right]\geq f\left(\sqrt{ab}\right).
\end{aligned}
\tag{25}
$$

3. PROOFS

In a recent paper [25], we established the following inequalities for a log-convex function $g:[c,d]\to(0,\infty)$.

If $p\in\left(0,\frac{1}{2}\right]$, then we have

$$
\begin{aligned}
g\left(\frac{c+d}{2}\right)&\leq \exp\left[\frac{1}{d-c}\int_{c}^{d}\ln g(x)\,\mathrm{d}x\right]\\
&\leq \left(\frac{1}{d-c}\int_{c}^{d}g^{p}(x)\,g^{p}(c+d-x)\,\mathrm{d}x\right)^{\frac{1}{2p}}\\
&\leq \left(\frac{1}{d-c}\int_{c}^{d}g^{2p}(x)\,\mathrm{d}x\right)^{\frac{1}{2p}}\leq \frac{1}{d-c}\int_{c}^{d}g(x)\,\mathrm{d}x\\
&\leq L\left(g(c),g(d)\right).
\end{aligned}
\tag{26}
$$

For $p>0$, but $p\neq \frac{1}{2}$, we also have

$$
\begin{aligned}
g\left(\frac{c+d}{2}\right) &\leq \exp\left[\frac{1}{d-c}\int_c^d \ln g\left(x\right)\mathrm{d}x\right] \\
&\leq \left(\frac{1}{d-c}\int_c^d g^p\left(x\right)g^p\left(c+d-x\right)\mathrm{d}x\right)^{\frac{1}{2p}} \\
&\leq \left(\frac{1}{d-c}\int_c^d g^{2p}\left(x\right)\mathrm{d}x\right)^{\frac{1}{2p}} \\
&\leq \left[L_{2p-1}\left(g\left(c\right),g\left(d\right)\right)\right]^{1-\frac{1}{2p}}\left[L\left(g\left(c\right),g\left(d\right)\right)\right]^{\frac{1}{2p}}.
\end{aligned}
\tag{27}
$$

If $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is a GG-convex function on $[a,b]$, then, by taking $g:=f\circ\exp$, $c=\ln a$ and $d=\ln b$, we have that g is log-convex on $[c,d]$ and, by (26) and (27), we thus get

$$
\begin{aligned}
&f\circ\exp\left(\frac{\ln a+\ln b}{2}\right) \\
&\leq \exp\left[\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b}\ln\left[f\circ\exp\left(x\right)\right]\mathrm{d}x\right] \\
&\leq \left(\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b}\left[f\circ\exp\left(x\right)\right]^p\left[f\circ\exp\left(\ln a+\ln b-x\right)\right]^p\mathrm{d}x\right)^{\frac{1}{2p}} \\
&\leq \left(\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b}\left[f\circ\exp\left(x\right)\right]^{2p}\mathrm{d}x\right)^{\frac{1}{2p}} \\
&\leq \frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b}f\circ\exp\left(x\right)\mathrm{d}x\leq L\left(f\circ\exp\left(\ln a\right),f\circ\exp\left(d\right)\right),
\end{aligned}
\tag{28}
$$

for $p\in\left(0,\frac{1}{2}\right]$, and

$$
\begin{aligned}
&f\circ\exp\left(\frac{\ln a+\ln b}{2}\right) \\
&\leq \exp\left[\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b}\ln\left[f\circ\exp\left(x\right)\right]\mathrm{d}x\right] \\
&\leq \left(\frac{1}{\ln b-\ln a}\left[f\circ\exp\left(x\right)\right]^p\left[f\circ\exp\left(\ln a+\ln b-x\right)\right]^p\mathrm{d}x\right)^{\frac{1}{2p}} \\
&\leq \left(\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b}\left[f\circ\exp\left(x\right)\right]^{2p}\mathrm{d}x\right)^{\frac{1}{2p}} \\
&\leq \left[L_{2p-1}\left(f\circ\exp\left(\ln a\right),f\circ\exp\left(\ln b\right)\right)\right]^{1-\frac{1}{2p}} \\
&\times\left[L\left(f\circ\exp\left(\ln a\right),f\circ\exp\left(\ln b\right)\right)\right]^{\frac{1}{2p}},
\end{aligned}
\tag{29}
$$

for $p>0$, but $p\neq\frac{1}{2}$.

The inequalities (28) and (29) can be written equivalently as

$$
\begin{aligned}
f\left(\sqrt{ab}\right) &\leq \exp\left[\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b}\ln\left[f\circ\exp\left(x\right)\right]\mathrm{d}x\right]\\
&\leq \left(\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b}\left[f\circ\exp\left(x\right)\right]^{p}\left[f\left(\frac{\sqrt{ab}}{\exp\left(x\right)}\right)\right]^{p}\mathrm{d}x\right)^{\frac{1}{2p}}\\
&\leq \left(\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b}\left[f\circ\exp\left(x\right)\right]^{2p}\mathrm{d}x\right)^{\frac{1}{2p}}\\
&\leq \frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b}f\circ\exp\left(x\right)\mathrm{d}x\leq L\left(f\left(a\right),f\left(b\right)\right)
\end{aligned}
\tag{30}
$$

and

$$
\begin{aligned}
f\left(\sqrt{ab}\right) &\leq \exp\left[\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b}\ln\left[f\circ\exp\left(x\right)\right]\mathrm{d}x\right]\\
&\leq \left(\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b}\left[f\circ\exp\left(x\right)\right]^{p}\left[f\left(\frac{\sqrt{ab}}{\exp\left(x\right)}\right)\right]^{p}\mathrm{d}x\right)^{\frac{1}{2p}}\\
&\leq \left(\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b}\left[f\circ\exp\left(x\right)\right]^{2p}\mathrm{d}x\right)^{\frac{1}{2p}}\\
&\leq \left[L_{2p-1}\left(f\left(a\right),f\left(b\right)\right)\right]^{1-\frac{1}{2p}}\left[L\left(f\left(a\right),f\left(b\right)\right)\right]^{\frac{1}{2p}}.
\end{aligned}
\tag{31}
$$

Now, by making the change of variable $\exp\left(x\right)=t$ in the integrals in (30) and (31), we obtain the desired results (9) and (10).

In [25] we also obtained the following result.

Let $g:[c,d]\to(0,\infty)$ be a log-convex function on $[c,d]$. Then, for any $x\in[c,d]$, we have

$$
\begin{aligned}
&g\left(d\right)\left(d-x\right)+g\left(c\right)\left(x-c\right)-\int_{c}^{d}g\left(y\right)\mathrm{d}y\\
&\geq \int_{c}^{d}g\left(y\right)\ln g\left(y\right)\mathrm{d}y-\ln g\left(x\right)\int_{c}^{d}g\left(y\right)\mathrm{d}y.
\end{aligned}
\tag{32}
$$

A simple proof of this fact is as follows. Since the function $\ln g$ is convex on $[c,d]$, then we have by the gradient inequality

$$
\ln g\left(x\right)-\ln g\left(y\right)\geq\frac{g'_{+}\left(y\right)}{g\left(y\right)}\left(x-y\right),
\tag{33}
$$

for any $x \in [c,d]$ and $y \in (c,d)$. If we multiply (33) by $g(y) > 0$ and integrate it on $[c,d]$ with respect to y, we get

$$\begin{aligned}\ln g(x)\int_c^d g(y)\,\mathrm{d}y - \int_c^d g(y)\ln g(y)\,\mathrm{d}y &\\ \geq \int_c^d g'_+(y)(x-y)\,\mathrm{d}y &\\ = g(y)(x-y)\big|_c^d + \int_c^d g(y)\,\mathrm{d}y &\\ = g(d)(x-d) + g(c)(c-x) + \int_c^d g(y)\,\mathrm{d}y, &\end{aligned}$$

which is equivalent to (32).

If $f : [a,b] \subset (0,\infty) \to (0,\infty)$ is a GG-convex function on $[a,b]$, then, by taking $g := f \circ \exp$; $c = \ln a$, $d = \ln b$ and $x = \ln s$, $s \in [a,b]$, we have that g is log-convex on $[c,d]$ and, by (32), we get

$$\begin{aligned}&(\ln b - \ln s)\, f\circ\exp(\ln b) + (\ln s - \ln a)\, f\circ\exp(\ln a) - \int_{\ln a}^{\ln b} f\circ\exp(y)\,\mathrm{d}y\\ &\geq \int_{\ln a}^{\ln b} f\circ\exp(y)\ln(f\circ\exp(y))\,\mathrm{d}y - \ln(f\circ\exp(\ln s))\int_{\ln a}^{\ln b} f\circ\exp(y)\,\mathrm{d}y,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&(\ln b - \ln s)\, f(b) + (\ln s - \ln a)\, f(a) - \int_{\ln a}^{\ln b} f\circ\exp(y)\,\mathrm{d}y\\ &\qquad\geq \int_{\ln a}^{\ln b} f\circ\exp(y)\ln(f\circ\exp(y))\,\mathrm{d}y - \ln f(s)\int_{\ln a}^{\ln b} f\circ\exp(y)\,\mathrm{d}y,\end{aligned}$$

which holds for each $s \in [a,b]$. Now, if we make the change of variable $\exp(y) = t$ in this last inequality, then we obtain the desired result (14).

The following lemma (see also [25]) on log-convex functions improves the trapezoid inequality for convex functions $h : [c,d] \to \mathbb{R}$

$$\frac{h(d)+h(c)}{2} - \frac{1}{d-c}\int_c^d h(y)\,\mathrm{d}y \geq 0.$$

Lemma 3.1. *Let $g : [c,d] \to (0,\infty)$ be a log-convex function on $[c,d]$. Then*

$$\begin{aligned}&\frac{g(d)+g(c)}{2} - \frac{1}{d-c}\int_c^d g(y)\,\mathrm{d}y\\ &\geq \int_c^d g(y)\ln g(y)\,\mathrm{d}y - \int_c^d g(y)\,\mathrm{d}y\frac{1}{d-c}\int_c^d \ln g(y)\,\mathrm{d}y \geq 0.\end{aligned}\tag{34}$$

Proof. If we take the integral mean over x in (32), then we get

$$\frac{1}{d-c}\int_c^d [g(d)(d-x)+g(c)(x-c)]\,\mathrm{d}x-\int_c^d g(y)\,\mathrm{d}y \geq \int_c^d g(y)\ln g(y)\,\mathrm{d}y-\int_c^d g(y)\,\mathrm{d}y\frac{1}{d-c}\int_c^d \ln g(x)\,\mathrm{d}x,$$

and, since

$$\frac{1}{d-c}\int_c^d [g(d)(d-x)+g(c)(x-c)]\,\mathrm{d}x=\frac{g(d)+g(c)}{2}-\frac{1}{d-c}\int_c^d g(y)\,\mathrm{d}y,$$

the first inequality in (34) is proved.

Since ln is an increasing function on $(0,\infty)$, we have

$$(g(x)-g(y))(\ln g(x)-\ln g(y))\geq 0,$$

for any $x,y\in[c,d]$, showing that the functions g and $\ln g$ are synchronous on $[c,d]$.

By making use of the Čebyšev integral inequality for the synchronous functions $g,h:[c,d]\to\mathbb{R}$, namely

$$\frac{1}{d-c}\int_c^d g(x)h(x)\,\mathrm{d}x\geq\frac{1}{d-c}\int_c^d g(x)\,\mathrm{d}x\frac{1}{d-c}\int_c^d h(x)\,\mathrm{d}x,$$

we have

$$\frac{1}{d-c}\int_c^d g(x)\ln g(x)\,\mathrm{d}x\geq\frac{1}{d-c}\int_c^d g(x)\,\mathrm{d}x\frac{1}{d-c}\int_c^d \ln g(x)\,\mathrm{d}x,$$

which proves the last part of (34). □

If $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is a GG-convex function on $[a,b]$, then, by taking $g:=f\circ\exp$, $c=\ln a$, and $d=\ln b$, we have that g is log-convex on $[c,d]$ and, by (34), we get

$$\begin{aligned}&\frac{f(b)+f(a)}{2}-\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b} f\circ\exp(y)\,\mathrm{d}y\\ &\geq\int_{\ln a}^{\ln b} f\circ\exp(y)\ln(f\circ\exp(y))\,\mathrm{d}y\\ &-\int_{\ln a}^{\ln b} f\circ\exp(y)\,\mathrm{d}y\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b}\ln(f\circ\exp(y))\,\mathrm{d}y\geq 0.\end{aligned}\tag{35}$$

By changing the variable $\exp(y)=t$ in (35), we deduce the desired inequality (20).

We state the following lemma from [25].

LEMMA 3.2. *Let $g : [c, d] \to (0, \infty)$ be a log-convex function on $[c, d]$. Then*

$$g(d)\left(d - \frac{\int_c^d yg(y)\,\mathrm{d}y}{\int_c^d g(y)\,\mathrm{d}y}\right) + g(c)\left(\frac{\int_c^d yg(y)\,\mathrm{d}y}{\int_c^d g(y)\,\mathrm{d}y} - c\right) - \int_c^d g(y)\,\mathrm{d}y$$
$$\geq \int_c^d g(y)\ln g(y)\,\mathrm{d}y - \int_c^d g(y)\,\mathrm{d}y \ln g\left(\frac{\int_c^d yg(y)\,\mathrm{d}y}{\int_c^d g(y)\,\mathrm{d}y}\right) \geq 0. \tag{36}$$

Proof. The first inequality follows from (32), by taking

$$x = \frac{\int_c^d yg(y)\,\mathrm{d}y}{\int_c^d g(y)\,\mathrm{d}y} \in [c, d],$$

since $g(y) > 0$, for any $y \in [c, d]$.

By Jensen's inequality, for the convex function $\ln g$ and the positive weight g, we have

$$\frac{\int_c^d g(y)\ln g(y)\,\mathrm{d}y}{\int_c^d g(y)\,\mathrm{d}y} \geq g\left(\frac{\int_c^d g(y)\,y\mathrm{d}y}{\int_c^d g(y)\,\mathrm{d}y}\right),$$

which proves the second inequality in (36). □

If $f : [a, b] \subset (0, \infty) \to (0, \infty)$ is a GG-convex function on $[a, b]$, then, by taking $g := f \circ \exp$, $c = \ln a$, and $d = \ln b$, we have that g is log-convex on $[c, d]$ and, by (36), we have

$$f(b)\left(\ln b - \frac{\int_{\ln a}^{\ln b} yf \circ \exp(y)\,\mathrm{d}y}{\int_{\ln a}^{\ln b} f \circ \exp(y)\,\mathrm{d}y}\right)$$
$$+ f(c)\left(\frac{\int_{\ln a}^{\ln b} yf \circ \exp(y)\,\mathrm{d}y}{\int_{\ln a}^{\ln b} f \circ \exp(y)\,\mathrm{d}y} - \ln a\right)$$
$$- \int_{\ln a}^{\ln b} f \circ \exp(y)\,\mathrm{d}y \tag{37}$$
$$\geq \int_{\ln a}^{\ln b} f \circ \exp(y)\ln(f \circ \exp(y))\,\mathrm{d}y$$
$$- \int_{\ln a}^{\ln b} f \circ \exp(y)\,\mathrm{d}y \times \ln(f \circ \exp)\left(\frac{\int_{\ln a}^{\ln b} y(f \circ \exp(y))\,\mathrm{d}y}{\int_{\ln a}^{\ln b} f \circ \exp(y)\,\mathrm{d}y}\right) \geq 0.$$

By changing the variable $\exp(y) = t$ in (37), we get (21).

In [25] we also proved the following result.

Let $g : [c, d] \to (0, \infty)$ be a log-convex function. Then, for every $\alpha \in [0, 1] \setminus \left\{\frac{1}{2}\right\}$, we have, for $p > 0$, that

$$\begin{aligned}
&\left(\frac{1}{d-c}\int_c^d g^{2p}(x)\,\mathrm{d}x\right)^{\frac{1}{2p}} \\
&\geq \left(\frac{1}{d-c}\int_c^d \left[g^{2p}(x)\right]^{1-\alpha}\left[g^{2p}(c+d-x)\right]^{\alpha}\mathrm{d}x\right)^{\frac{1}{2p}} \\
&\geq \left(\frac{1}{(1-2\alpha)(d-c)}\int_{(1-\alpha)c+\alpha d}^{\alpha c+(1-\alpha)d} g^{2p}(u)\,du\right)^{\frac{1}{2p}} \qquad (38)\\
&\geq \left(\frac{1}{(1-2\alpha)(d-c)}\int_{(1-\alpha)c+\alpha d}^{\alpha c+(1-\alpha)d} g^{p}(u)\,g^{p}(c+d-u)\,\mathrm{d}x\right)^{\frac{1}{2p}} \\
&\geq \exp\left[\frac{1}{(1-2\alpha)(d-c)}\int_{(1-\alpha)c+\alpha d}^{\alpha c+(1-\alpha)d} \ln g(u)\,du\right] \geq g\left(\frac{c+d}{2}\right).
\end{aligned}$$

If $f : [a, b] \subset (0, \infty) \to (0, \infty)$ is a GG-convex function on $[a, b]$, then, by taking $g := f \circ \exp$, $c = \ln a$, and $d = \ln b$, we have that g is log-convex on $[c, d]$ and, by (38), we have

$$\begin{aligned}
&\left(\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b} f^{2p}\circ\exp(x)\,\mathrm{d}x\right)^{\frac{1}{2p}} \\
&\geq \left(\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b} \left[f^{2p}\circ\exp(x)\right]^{1-\alpha}\left[f^{2p}\left(\frac{ab}{\exp(x)}\right)\right]^{\alpha}\mathrm{d}x\right)^{\frac{1}{2p}} \\
&\geq \left(\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{\ln(a^{1-\alpha}b^{\alpha})}^{\ln\left(a^{\alpha}b^{1-\alpha}\right)} f^{2p}\circ\exp(x)\,\mathrm{d}x\right)^{\frac{1}{2p}} \\
&\geq \left(\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{\ln(a^{1-\alpha}b^{\alpha})}^{\ln\left(a^{\alpha}b^{1-\alpha}\right)} f^{p}\circ\exp(x)\,f^{p}\left(\frac{ab}{\exp(x)}\right)\mathrm{d}x\right)^{\frac{1}{2p}} \\
&\geq \exp\left[\frac{1}{(1-2\alpha)(\ln b-\ln a)}\int_{\ln(a^{1-\alpha}b^{\alpha})}^{\ln\left(a^{\alpha}b^{1-\alpha}\right)} \ln f\circ\exp(x)\,\mathrm{d}x\right] \geq f\left(\sqrt{ab}\right),
\end{aligned}$$

which, by changing the variable $t = \exp x$, is equivalent to (22).

4. APPLICATIONS FOR EXPONENTIAL INTEGRAL MEAN

First, we consider the *exponential integral* $\mathrm{Ei} : \mathbb{R} \to \mathbb{R}$ defined by

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t}\mathrm{d}t,\ x \in \mathbb{R},$$

and, for $b > a > 0$, define the *exponential integral mean* by

$$\mathrm{Ei}_{\mu}(a,b) := \frac{\mathrm{Ei}(b) - \mathrm{Ei}(a)}{\ln b - \ln a} = \frac{\int_a^b \frac{e^t}{t}\mathrm{d}t}{\ln b - \ln a}.$$

If we use inequality (11) in the form

$$\exp\left[\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(t)}{t}\mathrm{d}t\right] \le \frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{t}\mathrm{d}t \le L(f(a), f(b)), \tag{39}$$

for the GG-convex function $f(t) = \exp t$, then we get the basic inequality

$$\exp(L(a,b)) \le \mathrm{Ei}_{\mu}(a,b) \le L(\exp(a), \exp(b)) = E(a,b), \tag{40}$$

where

$$E(a,b) := \frac{\exp b - \exp a}{b-a},\ \ b > a > 0.$$

From (20) for the GG-convex function $f(t) = \exp t$, we have

$$A(\exp(a), \exp(b)) - \mathrm{Ei}_{\mu}(a,b) \ge (b-a)[E(a,b) - \mathrm{Ei}_{\mu}(a,b)] \ge 0. \tag{41}$$

If we use inequality (24) in the form

$$\frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{t}\mathrm{d}t \ge \frac{1}{(1-2\alpha)(\ln b - \ln a)}\int_{a^{1-\alpha}b^{\alpha}}^{a^{\alpha}b^{1-\alpha}} \frac{f(t)}{t}\mathrm{d}t$$
$$\ge \exp\left[\frac{1}{(1-2\alpha)(\ln b - \ln a)}\int_{a^{1-\alpha}b^{\alpha}}^{a^{\alpha}b^{1-\alpha}} \frac{\ln f(t)}{t}\mathrm{d}t\right] \ge f\left(\sqrt{ab}\right),$$

for the GG-convex function $f(t) = \exp t$, then we also have

$$\begin{aligned} \mathrm{Ei}_{\mu}(a,b) &\ge \mathrm{Ei}_{\mu}\left(a^{1-\alpha}b^{\alpha}, a^{\alpha}b^{1-\alpha}\right) \\ &\ge \exp\left(L\left(a^{1-\alpha}b^{\alpha}, a^{\alpha}b^{1-\alpha}\right)\right) \ge \exp(G(a,b)), \end{aligned} \tag{42}$$

for every $\alpha \in [0,1] \setminus \left\{\frac{1}{2}\right\}$.

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