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OPTIMAL CONTROL OF A MATHEMATICAL MODEL FOR THE 2014 EBOLA OUTBREAK IN WEST AFRICA

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Abstract. Ebola hemorrhagic fever is a highly infectious and lethal disease that poses serious public health risks in Africa and even countries beyond the African continent. The main goal of this study is to develop a theoretical optimal control treatment of Ebola. The aim of the mathematical model used herein is to make the number of the infectious individuals decrease and the number of recovered individuals increase, while administering an efficient medical treatment (vaccination / medication). Pontryagin's classical control theory is applied to a SEIR mathematical model of Ebola infection characterized by a system of nonlinear differential equations with the following unknown functions: the susceptible individuals, exposed individuals, infectious individuals and recovered individuals. An optimal control strategy is derived for 2014 Ebola outbreaks in Guinea, Sierra Leone, Liberia and Nigeria.

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Key words. Ebola, system of nonlinear differential equations, optimal control, state equations, adjoint equations.

1. INTRODUCTION

Ebola virus is a lethal human pathogen, causing Ebola virus disease (EVD) with an average case fatality rate of 78% [15]. Ebola is transmitted by physical contact with body fluids, secretions, tissues or semen from infected persons. Ebola is not spread through the air, by water, or in general, by food [6]. Individuals exposed to the virus who become infectious do so after an incubation period ranging from 2 to 21 days. Nonspecific symptoms appear, including sudden onset of fever, weakness, vomiting, diarrhea, rash, headache, sore throat, and internal and external bleeding [4].

A complex and unprecedented Ebola epidemic has affected West Africa since December 2013, when the first cases occurred in southern Guinea [3]. This current epidemic was not identified until later in March 2014 ([3]), therefore it facilitated transmission to Sierra Leone, Liberia and a limited outbreak in Nigeria. The World Health Organization declared the Ebola epidemic in West Africa a Public Health Emergency of International Concern in August 2014 [21]. As of April 13, 2016, according to the Centers for Disease Control and Prevention, a total of $3,814$ cases with $2,544$ deaths have been reported in Guinea, $14,124$ cases with $3,956$ deaths in Sierra Leone, $10,678$ cases with $4,810$ deaths in Liberia, and 20 cases with 8 deaths in Nigeria [7].

Mathematical modeling has emerged as an important tool for gaining understanding of the dynamics of the spread of Ebola virus. SIR epidemic models

have been analyzed in [13, 20], SEIR epidemic models in [9, 17, 1, 2], SEIHR epidemic models in [10], SEIHFR epidemic models in [16], and SEEIHDRB epidemic models in [5]. However, there is a scarcity of studies that quantify the effects of control interventions implemented during past Ebola outbreaks [9, 16, 10]. Moreover, with the 2014 Ebola epidemic in West Africa, the development of treatments and vaccines against Ebola is accelerating [10, 11, 12]. For example, an experimental drug with unknown efficacy or safety record in humans has been initiated during the outbreak [12]. Recent experiments in monkeys provide encouraging evidence that this experimental drug could have a significant impact on mortality during Ebola outbreaks [10, 19]. Furthermore, an Ebola vaccine has entered human safety trials in 2014 [10, 14]. Apart from pharmaceutical effects on the infection, we have yet to examine how medication/vaccination changes the overall Ebola virus dynamics. Recently in 2015, Rachah and Torres have considered an optimal control problem for the SIR model to study the effect of vaccination on the spread of Ebola virus in Liberia [20].

In this paper, we apply the classical optimal control theory to the SEIR mathematical model proposed by Althaus [1, 2] for the Guinea, Sierra Leone, Liberia and Nigeria countries. Our goal is to minimize the number of the infectious individuals and maximize the number of the recovered individuals, while administering an efficient medical treatment (medication and vaccination). In Section 2, we present the SEIR mathematical model. In Section 3, we discuss the model with controls and present the objective functional. In Section 4, we state the necessary conditions for the optimal control pair. In Section 5, numerical results are provided. The paper will conclude with Section 6.

2. SEIR MATHEMATICAL MODEL OF EBOLA INFECTION

The objective of this section is to describe the dynamics of the population infected by the Ebola virus mathematically. The SEIR Ebola infection mathematical model examined in [1, 2] is used within this paper. The population analyzed in this model is divided into four important groups: susceptible group, exposed group, infected group and recovered group. Susceptible individuals S in contact with the virus enter the exposed class E at the per-capita rate $\beta I/N$, where β is transmission rate per person per day, N is the total population size, and I/N is the probability that a contact with an infectious individual is made. Exposed individuals move through the incubation period at rate σ before they become infectious individuals I. Infectious individuals I recover or die at rate γ. The expression $1/\sigma$ denotes the average duration of incubation, $1/\gamma$ denotes the average duration of infectiousness, and f denotes the case fatality rate.

Therefore, the system of four ordinary differential equations describing the Ebola dynamics over the time horizon $t_0 \leq t \leq t_f$ is given by

$$\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= -\beta SI/N, S\left(t_0\right) = S_0 \\
\frac{\mathrm{d}E}{\mathrm{d}t} &= \beta SI/N - \sigma E, E\left(t_0\right) = E_0 \\
\frac{\mathrm{d}I}{\mathrm{d}t} &= \sigma E - \gamma I, I\left(t_0\right) = I_0 \\
\frac{\mathrm{d}R}{\mathrm{d}t} &= \left(1-f\right)\gamma I, R\left(t_0\right) = R_0.
\end{aligned}$$

All of the parameter values in the above equations are assumed to be positive.

For the purpose of this paper, we will normalize the equations (by substituting $S/N \to S$, $E/N \to E$, $I/N \to I$, $R/N \to R$) and we will use the normalized equations where the variables are dimensionless

$$\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= -\beta SI, S\left(t_0\right) = S_0 \in [0,1] \\
\frac{\mathrm{d}E}{\mathrm{d}t} &= \beta SI - \sigma E, E\left(t_0\right) = E_0 \in [0,1] \\
\frac{\mathrm{d}I}{\mathrm{d}t} &= \sigma E - \gamma I, I\left(t_0\right) = I_0 \in [0,1] \\
\frac{\mathrm{d}R}{\mathrm{d}t} &= \left(1-f\right)\gamma I, R\left(t_0\right) = R_0 \in [0,1].
\end{aligned}$$

One key parameter describing the spread of an infection is the basic reproduction number, denoted by $\mathcal{R}_0$, understood as the number of cases one infected case generates on average over the course of its infectious period, in an otherwise uninfected population. This metric is useful because it helps determine whether or not an infectious disease can spread through a population. When $\mathcal{R}_0 < 1$, the infection will die out in the long run, but if $\mathcal{R}_0 > 1$, the infection will be able to spread in a population. Generally, the larger the value of $\mathcal{R}_0$ the harder it is to control the epidemic.

Several Ebola studies have fitted mathematical models to data from previous oubreaks in order to provide estimates for the reproduction number [9, 17, 16]. Estimates of the reproduction number during the 2014 outbreak in Guinea, Sierra Leone, Liberia and Nigeria have been provided by Althaus in [1, 2]. Lower, but greater than 1, $\mathcal{R}_0$ values have been estimated for Guinea and Liberia, a larger $\mathcal{R}_0$ value has been determined for Sierra Leone, while a very large $\mathcal{R}_0$ value has been predicted for Nigeria. Therefore, it is expected to be able to control the Ebola outbreak in Guinea and Liberia easier compared to the Ebola outbreak in Nigeria.

3. OPTIMAL CONTROL OF EBOLA INFECTION

Controlling infectious diseases has been an increasingly challenging and important issue in biomedicine. According to the latest experiments on EVD, there is proof and hope that the Ebola infection can be controlled by vaccination and medication [10, 11, 12, 14, 19]. An optimal control problem of Ebola infection that takes into account these controlling venues is proposed herein. The system of equations that describe the SEIR model with vaccination and medication could be formulated as follows

$$
\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= -\beta SI - u_1(t)S, S\left(t_0\right) = S_0 \\
\frac{\mathrm{d}E}{\mathrm{d}t} &= \beta SI - \sigma E, E\left(t_0\right) = E_0 \\
\frac{\mathrm{d}I}{\mathrm{d}t} &= \sigma E - \gamma I, I\left(t_0\right) = I_0 \\
\frac{\mathrm{d}R}{\mathrm{d}t} &= \left(1 - f\left(1 - u_2(t)\right)\right)\gamma I + u_1(t)S, R\left(t_0\right) = R_0,
\end{aligned} \tag{1}
$$

where

$$0 \leq a_1 \leq u_1\left(t\right) \leq b_1 \leq 1,$$

u_1 denotes the "effectiveness" of vaccination,

$$0 \leq a_2 \leq u_2\left(t\right) \leq b_2 \leq 1$$

and u_2 denotes the "effectiveness" of medication.

Due to vaccination, the susceptible population is reduced by $u_1(t)S$ and the successfully vaccinated individuals become recovered. Due to medication, the fatality rate is reduced by $\left(1 - u_2\left(t\right)\right)$. It is assumed that medication does not change the average duration of infectiousness.

We formulate an optimal control problem to minimize the infectious population and to maximize the recovered population, but also to minimize the cost of treatment

$$J(u_1, u_2) = \int_{t_0}^{t_f} \left(C_{11}I(t) - C_{12}R(t) + \frac{C_{21}}{2}{u_1}^2(t) + \frac{C_{22}}{2}{u_2}^2(t) \right) \mathrm{d}t, \tag{2}$$

subject to the system of differential equations (1) and to the range restrictions on the continuous controls

$$(u_1, u_2) : [t_0, t_f] \times [t_0, t_f] \to [a_1, b_1] \times [a_2, b_2].$$

The coefficients C_{11}, C_{12}, C_{21} and C_{22} are strictly positive weight constants chosen to balance the values of the infectious and recovered populations and the control functions, respectively, and/or chosen to emphasize the most important term(s) / aspect(s) in the control problem. The negative sign of the recovered population comes from the need to maximize this population. The square of the control functions and the fractions with denominator 2 are taken for the sake of mathematical manipulations that will follow next. In addition,

since it is expected that the effects of the controls to be non-linear, the quadratic cost terms $u_i^2(t), i = 1, 2$ will reflect these effects. The weight constants on the controls, C_{21}, C_{22}, include a measure of the cost associated with vaccination and medication. The higher the weights, the greater the expense. The lower and upper bounds for u_1, u_2 correspond to minimum/maximum control.

Therefore, the optimal control problem can be written as follows

$$\min_{u_1,u_2} J(u_1, u_2) \tag{3}$$

subject to

$$\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= -\beta SI - u_1(t)S, S(t_0) = S_0 \\
\frac{\mathrm{d}E}{\mathrm{d}t} &= \beta SI - \sigma E, E(t_0) = E_0 \\
\frac{\mathrm{d}I}{\mathrm{d}t} &= \sigma E - \gamma I, I(t_0) = I_0 \\
\frac{\mathrm{d}R}{\mathrm{d}t} &= (1 - f(1 - u_2(t)))\gamma I + u_1(t)S, R(t_0) = R_0.
\end{aligned} \tag{4}$$

4. OPTIMALITY CONDITIONS

The optimal control problem is solved using the Pontryagin's Minimum Principle [8, 18]. The optimal control pair (u_1^*, u_2^*) which gives the optimal treatment will be derived.

THEOREM 4.1. *If (u_1^*, u_2^*) is an optimal control pair of the optimal control problem (3), S^*, E^*I^*, R^* are the corresponding state variables of the state system (4), then there exist adjoint (co-state) variables $\Psi_S, \Psi_E, \Psi_I, \Psi_R$ which satisfy*

$$\begin{aligned}
\frac{\mathrm{d}\Psi_S}{\mathrm{d}t} &= \beta I^*(\Psi_S - \Psi_E) + u_1(\Psi_S - \Psi_R) \\
\frac{\mathrm{d}\Psi_E}{\mathrm{d}t} &= \sigma(\Psi_E - \Psi_I) \\
\frac{\mathrm{d}\Psi_I}{\mathrm{d}t} &= -C_{11} + \beta S^*(\Psi_S - \Psi_E) + \gamma\Psi_I - (1 - f(1 - u_2))\gamma\Psi_R \\
\frac{\mathrm{d}\Psi_R}{\mathrm{d}t} &= C_{12}
\end{aligned} \tag{5}$$

and transversality conditions (terminal conditions)

$$\Psi_S(t_f) = 0, \Psi_E(t_f) = 0, \Psi_I(t_f) = 0, \Psi_R(t_f) = 0. \tag{6}$$

Furthermore,

$$u_1{}^* = \min\left\{\max\left\{a_1, \frac{S^*(\Psi_S - \Psi_R)}{C_{21}}\right\}, b_1\right\}, C_{21} > 0$$

$$u_2{}^* = \min\left\{\max\left\{a_2, -\frac{f\gamma I^*\Psi_R}{C_{22}}\right\}, b_2\right\}, C_{22} > 0. \tag{7}$$

Proof. Denote the Hamiltonian H of the optimal control problem by

$$\begin{aligned} &H\left(S, E, I, R, \Psi_S, \Psi_E, \Psi_I, \Psi_R, w_{11}, w_{12}, w_{21}, w_{22}, u_1, u_2\right) \\ &\quad = C_{11}I - C_{12}R + \frac{C_{21}}{2}u_1^2 + \frac{C_{22}}{2}u_2^2 + \Psi_S\left(-\beta SI - u_1 S\right) \\ &+ \Psi_E\left(\beta SI - \sigma E\right) + \Psi_I\left(\sigma E - \gamma I\right) + \Psi_R\left(\left(1 - f\left(1 - u_2\right)\right)\gamma I + u_1 S\right) \\ &\quad + w_{11}\left(b_1 - u_1\right) + w_{12}\left(u_1 - a_1\right) + w_{21}\left(b_2 - u_2\right) + w_{22}\left(u_2 - a_2\right), \end{aligned}$$

where, $w_{11}(t), w_{12}(t), w_{21}(t), w_{22}(t) \geq 0$ are penalty multipliers satisfying

$$\begin{aligned} w_{11}(b_1 - u_1) = 0, w_{12}(u_1 - a_1) = 0, \quad at \quad u_1 = u_1^*\\ w_{21}(b_2 - u_2) = 0, w_{22}(u_2 - a_2) = 0, \quad at \quad u_2 = u_2^*. \end{aligned}$$

Using Pontryagin's Minimum Principle [18]

$$\frac{\mathrm{d}\Psi_S}{\mathrm{d}t} = -\frac{\partial H}{\partial S}, \frac{\mathrm{d}\Psi_E}{\mathrm{d}t} = -\frac{\partial H}{\partial E}, \frac{\mathrm{d}\Psi_I}{\mathrm{d}t} = -\frac{\partial H}{\partial I}, \frac{\mathrm{d}\Psi_R}{\mathrm{d}t} = -\frac{\partial H}{\partial R},$$

we derive the following equations

$$\begin{aligned} \frac{\mathrm{d}\Psi_S}{\mathrm{d}t} &= \beta I^*\left(\Psi_S - \Psi_E\right) + u_1\left(\Psi_S - \Psi_R\right) \\ \frac{\mathrm{d}\Psi_E}{\mathrm{d}t} &= \sigma(\Psi_E - \Psi_I) \\ \frac{\mathrm{d}\Psi_I}{\mathrm{d}t} &= -C_{11} + \beta S^*\left(\Psi_S - \Psi_E\right) + \gamma\Psi_I - \left(1 - f\left(1 - u_2\right)\right)\gamma\Psi_R \\ \frac{\mathrm{d}\Psi_R}{\mathrm{d}t} &= C_{12}. \end{aligned}$$

The transversality conditions have the expression provided in (6).

The Hamiltonian is minimized with respect to the controls at the optimal control pair, so we differentiate H with respect to u_1 and u_2, respectively, and impose equality to zero

$$\begin{aligned} \frac{\partial H}{\partial u_1} &= C_{21}u_1 + S^*\left(\Psi_R - \Psi_S\right) - w_{11} + w_{12} = 0, \text{ at } u_1 = u_1{}^* \\ \frac{\partial H}{\partial u_2} &= C_{22}u_2 + f\gamma I^*\Psi_R - w_{21} + w_{22} = 0, \text{ at } u_2 = u_2{}^*. \end{aligned}$$

Solving for u_1^* and u_2^* on the interior sets we obtain

$$\begin{aligned} u_1{}^* &= \frac{S^*\left(\Psi_S - \Psi_R\right) + w_{11} - w_{12}}{C_{21}} \\ u_2{}^* &= \frac{-f\gamma I^*\Psi_R + w_{21} - w_{22}}{C_{22}}. \end{aligned}$$

By standard control arguments involving the bounds on the controls, we conclude

$$u_1^* = \begin{cases} \frac{S^*(\Psi_S - \Psi_R)}{C_{21}}, & \text{if } a_1 < \frac{S^*(\Psi_S - \Psi_R)}{C_{21}} < b_1 \\ a_1, & \text{if } \frac{S^*(\Psi_S - \Psi_R)}{C_{21}} \leq a_1 \\ b_1, & \text{if } \frac{S^*(\Psi_S - \Psi_R)}{C_{21}} \geq b_1. \end{cases}$$

By the same argument,

$$u_2^* = \begin{cases} -\frac{f\gamma I^* \Psi_R}{C_{22}}, & \text{if } a_2 < -\frac{f\gamma I^* \Psi_R}{C_{22}} < b_2 \\ a_2, & \text{if } -\frac{f\gamma I^* \Psi_R}{C_{22}} \leq a_2 \\ b_2, & \text{if } -\frac{f\gamma I^* \Psi_R}{C_{22}} \geq b_2. \end{cases}$$

In compact notation,

$$\begin{aligned} {u_1}^* &= \min\left\{\max\left\{a_1, \frac{S^*\left(\Psi_S - \Psi_R\right)}{C_{21}}\right\}, b_1\right\}, C_{21} > 0 \\ {u_2}^* &= \min\left\{\max\left\{a_2, -\frac{f\gamma I^* \Psi_R}{C_{22}}\right\}, b_2\right\}, C_{22} > 0. \end{aligned}$$

□

We have derived the optimality system for the optimal control pair (u_1^*, u_2^*) for the state system (4) with given initial conditions and the adjoint system (5) with transversality conditions (6).

5. NUMERICAL RESULTS

Analytical solution for optimal control is difficult to obtain since the state system is non-linear. In addition, we have initial conditions for the state variables and terminal conditions for the adjoint variables. Therefore, the optimality system has been solved numerically, using an Euler integration scheme, based on the following algorithm

(1) Choose an initial guess of control (u_1, u_2);
(2) Solve the state system forward in time;
(3) Solve the adjoint system backward in time;
(4) Update the control using the optimality condition (7);
(5) Repeat the iterations #2, #3 and #4, until convergence of controls is achieved.

We will illustrate the numerical algorithm on the Ebola outbreaks in Guinea, Sierra Leone, Liberia and Nigeria and the estimates of the parameters determined by Althaus in [1, 2] (see Table 1).

We will discuss the optimal control problem for the outbreaks in Guinea, Sierra Leone, Liberia and Nigeria for the same control objective (same values of the weights $C_{11}, C_{12}, C_{21}, C_{22}$), as well as the optimal control for the Nigeria outbreak for different control objectives (different values of C_{11}, C_{12}, C_{21}, C_{22}).

Parameter	Guinea	Sierra Leone	Liberia	Nigeria
β	0.27	0.45	0.28	1.22
$\frac{1}{\sigma}$	5.3	5.3	5.3	9.31
$\frac{1}{\gamma}$	5.61	5.61	5.61	7.41
f	0.74	0.48	0.71	0.39
$\mathcal{R}_0$	1.51	2.53	1.59	9.01

Table 1 – Coefficients estimated by Althaus et. al., 2014, 2015

We choose to study the outbreak in Nigeria, as it exhibits the largest reproduction number, and it is more critical to be addressed compared to outbreaks with lower reproduction numbers.

Same Control Objective for all Outbreaks

The weight values in the objective functional $J(u_1, u_2)$ are chosen to balance the magnitude of the infectious and recovered populations and treatment functions, therefore we consider the case described by

$$C_{11} = 100, C_{12} = 1, C_{21} = 1, C_{22} = 0.1.$$

We have chosen $C_{11} \gg C_{12}$ to balance the magnitude of I (that is aimed to approach 0) and R (that is aimed to approach 1). Along the same lines, $C_{21} > C_{22}$, as the cost of population vaccination is usually greater than the cost of administering medication. The bounds imposed on control are

$$a_1 = 0, a_2 = 0, b_1 = 0.8, b_2 = 0.8$$

(80% maximum efficacy of the treatment) and the treatment strategy is determined for a period of 90 days.

The basic reproduction number $\mathcal{R}_0$ is greater than 1 for all four cases, indicating that Ebola virus has been spreading within the healthy population.

Figures 5.1, 5.3, 5.5, 5.7 describe the behavior of I, R without and with control for all four outbreaks. Figures 5.2, 5.4, 5.6, and 5.8 illustrate the control functions u_1 (vaccination), u_2 (medication) for the same outbreaks, respectively.

One can remark that the infectious population I is minimized during the treatment period for all outbreaks, due to sustained vaccination. The recovered population R approaches 100% with control, due to concomitant administration of vaccination to susceptible individuals and medication to infected individuals.

It is observed that for outbreaks with lower reproduction number, the infected population without control peaks later within the control window, therefore, vaccination / medication should be administered for a longer period of time (see Figures 5.2 and 5.6). As the reproduction number increases, the infected population without control peaks higher and earlier within the control

window, therefore, it is enough to administer vaccination /medication for a shorter period of time (see Figures 5.4 and 5.8).

The price to control outbreaks with lower reproduction number is greater that the price corresponding to higher reproduction numbers. As the reproduction number increases, though, it is imperative to apply vaccination early, to reduce the infected population as soon as possible.

Different Control Objective for Nigeria Outbreak

Different control objectives will be chosen to emphasize different possible outcomes on the EVD optimal control. The weights that describe these different objectives are given in the following table

Case	C_{11}	C_{12}	C_{21}	C_{22}
Different Control for Nigeria Outbreak: Case 1	100	100	1	1
Different Control for Nigeria Outbreak: Case 2	1	0.1	10	0.1
Different Control for Nigeria Outbreak: Case 3	1	0.1	1	0.001
Previous Control for Nigeria Outbreak	100	1	1	0.1

Table 2 – Different Weights of Objective Functional $J(u_1, u_2)$ for Nigeria Outbreak

Figures 5.9, 5.10, 5.11, 5.12, 5.13, 5.14 show the state variables without and with control and the optimal controls for Case 1, Case 2 and Case 3, respectively. We will compare these results to the Nigeria case discussed previously at "Same Control Objective for all Outbreaks".

For Case 1, since $C_{11} = C_{12}$, greater emphasis is given to the recovered population R compared to the infected population I. In addition, the vaccination / medication are equally as expensive, since $C_{21} = C_{22}$. The optimal control in this case suggests sustained vaccination and longer administration of medication (see Figures 5.8 and 5.10).

Case 2 is considered with the purpose of reducing the cost of the vaccination u_1, since $C_{21} \gg C_{22}$. Comparing Figures 5.8 and 5.12, one can see that the vaccination timeframe is greatly reduced, however, the administration of medication is increased and postponed until after the vaccination is over.

The main goal of Case 3 is to allow more control compared to Case 2, while still trying to reduce the cost of vaccination. From Figures 5.12 and 5.14, it can be seen that the vaccination / medication are administered longer, as expected. In addition, since the medication is cheap in this case (C_{22} is very low), medication is administered right from the beginning and for a longer period of time.

In other words, the vaccination / medication strategies differ from one objective to another, to "adjust" to the goals determined by the coefficients C_{11}, C_{12}, C_{21}, C_{22} of the objective functional $J(u_1, u_2)$.

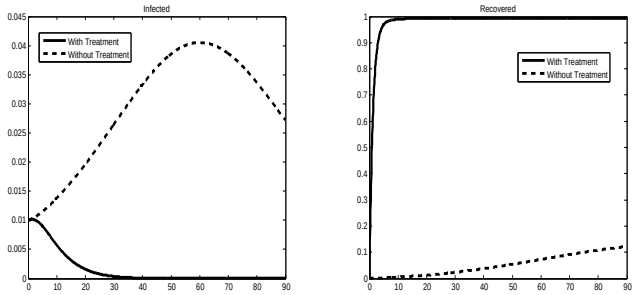

Fig. 5.1 – I and R Populations with/without Control in Guinea.

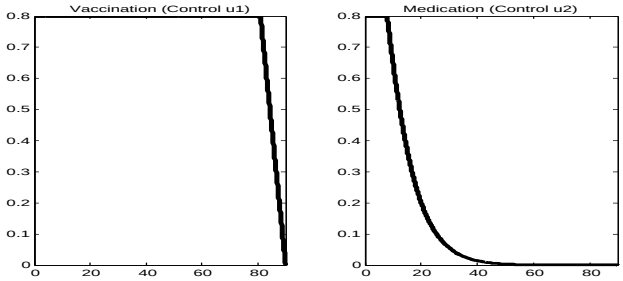

Fig. 5.2 – Controls $u_1(t), u_2(t)$ in Guinea.

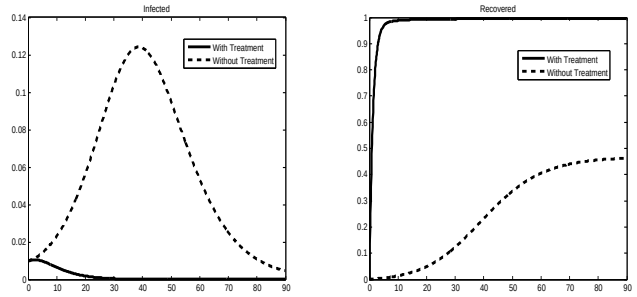

Fig. 5.3 – I and R Populations with/without Control in Sierra Leone.

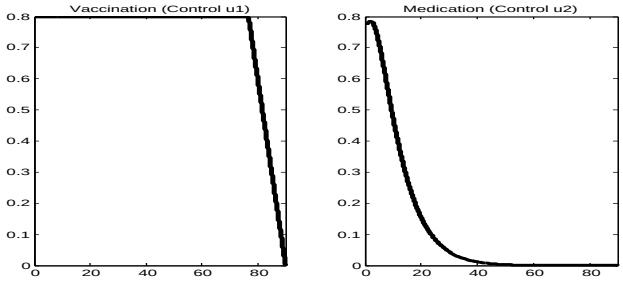

Fig. 5.4 – Controls $u_1(t), u_2(t)$ in Sierra Leone.

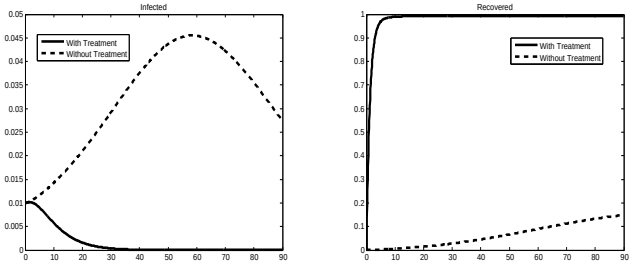

Fig. 5.5 – I and R Populations with/without Control in Liberia.

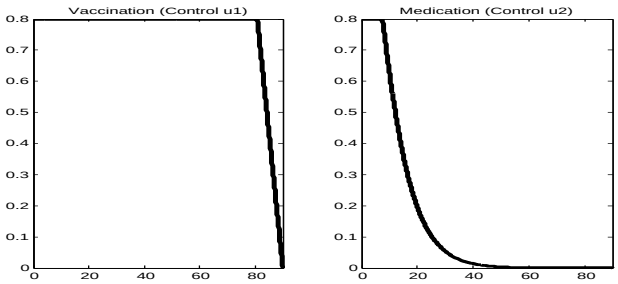

Fig. 5.6 – Controls $u_1(t), u_2(t)$ in Liberia.

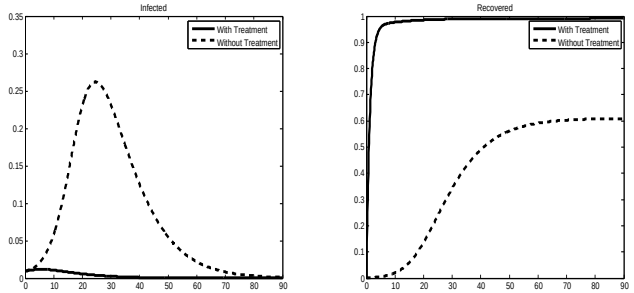

Fig. 5.7 – I and R Populations with/without Control in Nigeria.

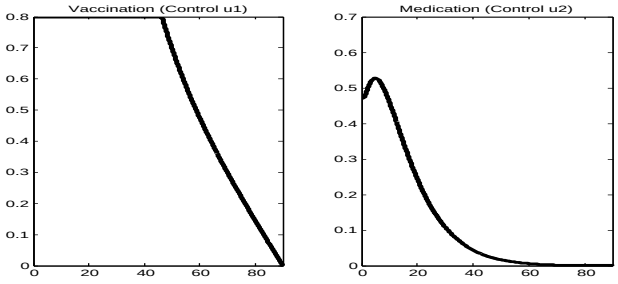

Fig. 5.8 – Controls $u_1(t), u_2(t)$ in Nigeria.

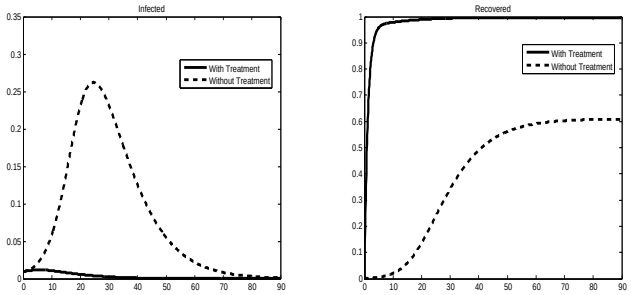

Fig. 5.9 – Case 1: I and R Populations with/without Control in Nigeria.

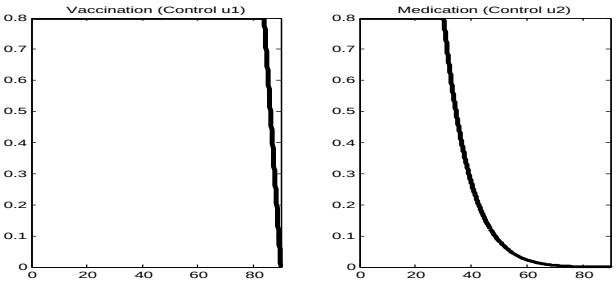

Fig. 5.10 – Case 1: Controls $u_1(t), u_2(t)$ in Nigeria.

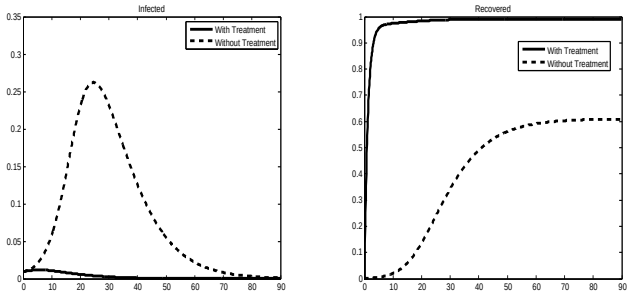

Fig. 5.11 – Case 2: I and R Populations with/without Control in Nigeria.

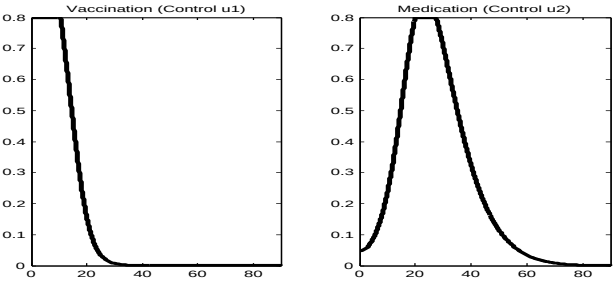

Fig. 5.12 – Case 2: Controls $u_1(t), u_2(t)$ in Nigeria.

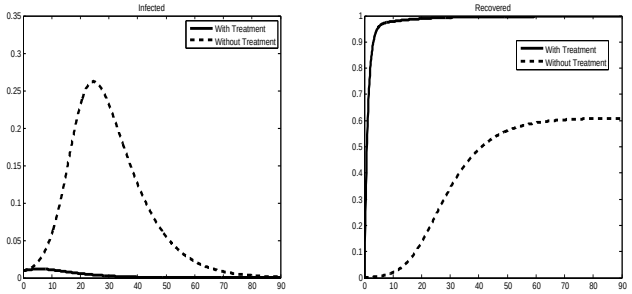

Fig. 5.13 – Case 3: I and R Populations with/without Control in Nigeria.

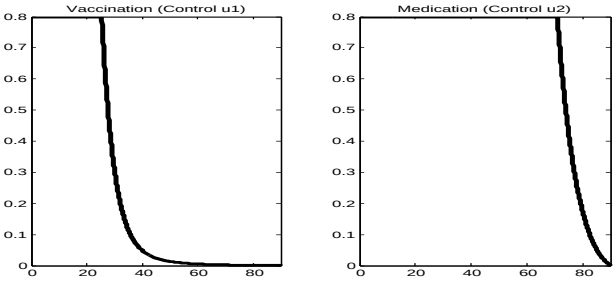

Fig. 5.14 – Case 3: Controls $u_1(t), u_2(t)$ in Nigeria.

6. CONCLUSION

In this paper we formulated a control problem for Ebola virus that aims the minimization of infected population and maximization of recovered population. Using Pontryagin's Minimum Principle, we proved the existence of the optimal control pair (u_1^*, u_2^*) for the vaccination/medication. The optimal controls were computed using numerical methods. Our solutions indicate that optimal control treatment differs from one outbreak to another and from one optimal goal to another.

For Ebola diseases with low values of the basic reproduction number, such as Guinea and Liberia' cases, vaccination and medication should be administered longer. As the reproduction number increases, such as Sierra Leone and Nigeria's cases, the administration of vaccination /medication is reduced in time. This is due to the fact that the virus is more potent and intervention is needed only early within the control window until the danger has passed.

In addition the optimal control protocols are highly dependent on the emphasis formulated in the objective functional, i.e. the values of the weight coefficients. Depending on the cost of vaccination/medication, custom control strategies could be determined to minimize the number of infected individuals and maximize the number of recovered individuals.

Finally, since continuous vaccination and medication is usually not feasible and the dynamics of Ebola infection is far more complicated than it was captured in this optimal control problem, the vaccination and medication strategies computed numerically herein can only be viewed as possible recommendations for practical usage.

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