MATHEMATICA, 59 (82), N° 1–2, 2017, pp. 15–31

SOME CATEGORICAL ASPECTS IN TOPOLOGICAL FORMAL CONCEPT ANALYSIS

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Abstract. Formal Concept Analysis (FCA) is a prominent field of Applied Mathematics which is grounded on the mathematization of the notion of concept and concept hierarchy, having a wide range of applications in data analysis and knowledge discovery in databases. Topological FCA is investigating issues related to the interplay between Topology and FCA. This paper is devoted to the study of some categorical equivalences in topological FCA. We prove that the category of pseudometric contexts is dually equivalent to a certain category of complete lattices enhanced with a pseudometric, extending by this the Basic Theorem on Concept Lattices to topological FCA.

MSC 2010. 18B35, 54B30, 68T30.

Key words. Formal Concept Analysis, pseudometrics, categorical equivalence.

1. INTRODUCTION

Formal Concept Analysis (FCA) has been introduced by R. Wille as an attempt to restructure lattice theory and to provide order theoretical tools for real-life applications. In his seminal paper [14], Wille states that "restructuring lattice theory is an attempt to reinvigorate connections to our general culture by interpreting the theory as concretely as possible, and in this way to promote better communication between lattice theorists and potential users of lattice theory". Since then, FCA developed as a prominent field of applied mathematics with various applications mainly in Knowledge Discovery and Artificial Intelligence but also with connections to other fields of Mathematics.

There is a vast literature on FCA and its applications. The mathematical foundations are described in [1], while some applications are given in [2]. Synthesizing, FCA has been defined by its founders as a mathematical theory of concepts and concept hierarchies. One can prove that the set of all concepts is a complete lattice, called conceptual hierarchy and used as a basis for further communication and data analysis using algebraic tools.

Gerd Hartung introduced in [4] the notion of a topological context with the aim to represent 0-1-lattices by means of FCA tools. This representation was then completed to a duality, where first only surjective 0-1-lattice homomorphisms were considered. This duality was extended in [5] to arbitrary 0-1-lattice homomorphisms, while the appropriate morphisms in the category of standard topological contexts were defined using multivalued functions (see [7]). Later on, a duality theory for polarity lattices has been developed in [6].

Topological representation theories for algebras of protoconcepts and semiconcepts (as introduced in [13]) have been studied in [9]. While investigating the foundations of Contextual Topology, (pseudo)metric contexts have been introduced in [9] as a tool for approximating objects by their attributes. These contexts generalize the notion of (pseudo)metric spaces. Based on this construction, contextual uniformities have been developed and studied in [10].

This paper is devoted to the study of several categorical aspects related to the construction of (pseudo)metric contexts and their dualities. In particular, the Basic Theorem on Concept Lattices is extended to the pseudometric case.

2. BASIC DEFINITIONS AND RESULTS

We start by recalling some basic facts. For more details we refer to [1].

Definition 2.1. A *formal context* is a triple $\mathbb{K} = (G, M, I)$, where G and M are sets and $I \subseteq G \times M$ is a binary relation. The elements of G are called *objects*, those of M *attributes*, and I is the *incidence relation*.

Given a formal context $\mathbb{K} = (G, M, I)$, one defines for subsets $A \subseteq G$ and $B \subseteq M$ the *concept forming operators* by $A' = \{m \in M \mid gIm \text{ for all } g \in A\}$ and $B' = \{g \in G \mid gIm \text{ for all } m \in B\}$. We denote by $A'' := (A')'$ and by $B'' := (B')'$. The concept forming operators lead to a Galois connection on the power sets of G and M, respectively.

Definition 2.2. A *formal concept* of the formal context $\mathbb{K} = (G, M, I)$ is a pair (A, B) with $A \subseteq G$, $B \subseteq M$ such that $A' = B$ and $B' = A$. The set A is called *extent*, while B is the *intent* of the concept (A, B). The set of all concepts of $\mathbb{K}$ is denoted by $B(\mathbb{K})$.

On the set $B(\mathbb{K})$ of concepts we define the subconcept-superconcept relation by $(A_1, B_1) \leq (A_2, B_2) \Leftrightarrow A_1 \subseteq A_2 (\Leftrightarrow B_1 \supseteq B_2)$.

Theorem 2.3 (The Basic Theorem on Concept Lattices). *Let $\mathbb{K} := (G, M, I)$ be a formal context. The concept lattice $B(\mathbb{K})$ is a complete lattice in which infimum and supremum are given by:*

$$\bigwedge_{t \in T} (A_t, B_t) = \Big(\bigcap_{t \in T} A_t, \big(\bigcup_{t \in T} B_t\big)''\Big),$$
$$\bigvee_{t \in T} (A_t, B_t) = \Big(\big(\bigcup_{t \in T} A_t\big)'', \bigcap_{t \in T} B_t\Big).$$

A complete lattice V is isomorphic to $B(\mathbb{K})$ if and only if there are mappings $\tilde{\gamma} : G \to V$ and $\tilde{\mu} : M \to V$ such that $\tilde{\gamma}(G)$ is supremum-dense in V, $\tilde{\mu}(M)$ is infimum-dense in V, and gIm is equivalent to $\tilde{\gamma}g \leq \tilde{\mu}m$ for all $g \in G$ and all $m \in M$. In particular, $V \simeq B(V, V, \leq)$.

Every object and every attribute can be recovered in the concept lattice of the given context. For an object $g \in G$, we write g' instead of $\{g\}'$ for the object intent $\{m \in M \mid gIm\}$ of the object g. Correspondingly, m' stands for the attribute extent $\{g \in G \mid gIm\}$ of the attribute m. Using the notations from the Basic Theorem, we write γg for the object concept (g'', g') and μm for the attribute concept (m', m'').

DEFINITION 2.4. A formal context (G, M, I) is called *clarified* if for any objects $g, h \in G$ with $g' = h'$ always follows that $g = h$ and, similarly, if for any attributes $m, n \in M$ the equality $m' = n'$ implies $m = n$.

DEFINITION 2.5. A clarified context (G, M, I) is called *row reduced* if every object concept is $\bigvee$-irreducible, and it is called *column reduced*, if every attribute concept is $\bigwedge$-irreducible. A context, which is both row reduced and column reduced, is called *reduced*.

DEFINITION 2.6. A *many-valued context* (G, M, W, I) consists of sets G, M, W, and a ternary relation I between them (i.e., $I \subseteq G \times M \times W$) for which the following holds:

$$(g, m, w) \in I \text{ and } (g, m, v) \in I \quad \text{always imply} \quad w = v.$$

The elements of G are called *objects*, those of M *(many-valued) attributes* and those of W *attribute values* of the many-valued context. Every attribute $m \in M$ can be interpreted as a (partial) map $m \colon G \to W$.

In order to assign concepts to a many-valued context we first have to transform the latter into a formal context, according to some rules, rules which are called *scaling the many-valued context*. The concepts of this *derived* one-valued context are then interpreted as the concepts of the many-valued context. This process is not canonical and depends on the semantics of the attribute values.

DEFINITION 2.7. A *scale* for the attribute m of a many-valued context (G, M, W, I) is a (formal) context $\mathbb{S}_m := (G_m, M_m, I_m)$ with $m(G) \subseteq G_m$. The objects of a scale are called *scale values*, the attributes are called *scale attributes*.

DEFINITION 2.8 ([15]). A *multicontext of signature* $\sigma : P \to I^2$, where I and P are nonempty sets, is a pair (S_I, R_P) consisting of a family $S_I := (S_i)_{i \in I}$ of sets and a family $R_P := (R_p)_{p \in P}$ of binary relations with $R_p \subseteq S_i \times S_j$ if $\sigma p = (i, j)$. A multicontext $\mathbb{K} := (S_I, R_P)$ can be understood as a network of formal contexts $\mathbb{K}_p := (S_i, S_j, R_p)$ with $p \in P$ and $\sigma p = (i, j)$.

3. DUALITY

By the Basic Theorem on Concept Lattices, every complete lattice can be considered as the concept lattice of some formal context and the concept lattice of a formal context is a complete lattice. We will prove that this correspondence is a categorical duality.

DEFINITION 3.1. Consider the class of all reduced formal contexts, and let $\mathbb{K}_1 := (G_1, M_1, I_1)$ and $\mathbb{K}_2 := (G_2, M_2, I_2)$ be objects of this class. A *contextual* *morphism* from $\mathbb{K}_1$ to $\mathbb{K}_2$ is a pair $(\beta^\wedge, \beta^\vee) : \mathbb{K}_1 \to \mathbb{K}_2$, where $\beta^\wedge : M_1 \to M_2$ and $\beta^\vee : G_2 \to G_1$ are mappings satisfying

$$\beta^\vee(g) I_1 m \Leftrightarrow g I_2 \beta^\wedge(m) \text{ for every } g \in G_2 \text{ and } m \in M_1.$$

Since the identity morphism and the composition of two contextual morphisms are obviously satisfying the above condition, the class of formal contexts and the contextual morphisms between them form a category denoted with **FC**.

Consider now the class of all objects of the form (V, G, M), where V is a complete lattice, G a supremum-dense subset of V and M an infimum-dense subset of V. The morphisms between these triples are defined as pairs $(\phi, \psi) : (V_1, G_1, M_1) \to (V_2, G_2, M_2)$, where $\phi : V_1 \to V_2$ and $\psi : V_2 \to V_1$ are mappings having the following properties:

1) ϕ and ψ are monotone;
2) $p \le \psi(\phi(p))$ and $\phi(\psi(q)) \le q$ for every $p \in V_1$ and every $q \in V_2$;
3) $\phi(G_1) \subseteq G_2$ and $\psi(M_2) \subseteq M_1$.

The class of all objects (V, G, M) together with the above defined morphisms define a category denoted by **L**. By the Basic Theorem on Concept Lattices, every object (V, G, M) in **L** generates the formal context $(G, M, \le)$ in **FC**. By this construction, we obtain an object map $T\colon Ob\ \mathbf{L} \to Ob\ \mathbf{FC}$. For morphisms, define $T(\phi, \psi) := (\psi_{|M_2}, \phi_{|G_1})$, where $\psi_{|M_2}$ denotes the restriction of ψ to M_2 and $\phi_{|G_1}$ denotes the restriction of ϕ to G_1. The functor $T\colon \mathbf{L} \to \mathbf{FC}$ is well-defined and contravariant.

For the converse, we define both a correspondence between the objects of **FC** and those of **L**, and a correspondence between the morphisms of these two categories. For this, define first the object map $S : Ob\ \mathbf{FC} \to Ob\ \mathbf{L}$ by $(G, M, I) \mapsto (B(G, M, I), \gamma(G), \mu(M))$. For every morphism $(\beta^\wedge, \beta^\vee) : \mathbb{K}_1 \to \mathbb{K}_2$, define $S(\beta^\wedge, \beta^\vee)\colon (B(\mathbb{K}_2), \gamma(G_2), \mu(M_2)) \to (B(\mathbb{K}_1), \gamma(G_1), \mu(M_1))$ to be the pair of mappings (ϕ, ψ), where $\phi : B(\mathbb{K}_1) \to B(\mathbb{K}_2)$ and $\psi : B(\mathbb{K}_2) \to B(\mathbb{K}_1)$ are defined by

$$\phi(A_1, B_1) := (\beta^\wedge(B_1)', \beta^\wedge(B_1)''),$$

$$\psi(A_2, B_2) = (\beta^\vee(A_2)'', \beta^\vee(A_2)')$$

for all $(A_1, B_1) \in B(\mathbb{K}_1)$ and $(A_2, B_2) \in B(\mathbb{K}_2)$. The functor $S : \mathbf{FC} \to \mathbf{L}$ is well-defined and contravariant. Furthermore, the following theorem is true.

THEOREM 3.2. *The categories* **FC** *and* **L** *are dually equivalent.*

Proof. The following diagram is commutative

$$\begin{array}{ccc} (G_1, M_1, I_1) & \xrightarrow{\iota_1} & TS(G_1, M_1, I_1) \\ {\scriptstyle (\beta,\alpha)}\downarrow & & \downarrow{\scriptstyle TS(\beta,\alpha)} \\ (G_2, M_2, I_2) & \xrightarrow[\iota_2]{} & TS(G_2, M_2, I_2) \end{array}$$

Define $\iota : (G, M, I) \to TS(G, M, I)$ by $\iota := (\mu, \gamma^{-1})$, where γ and μ are the maps from the Basic Theorem on Concept Lattices. Since the contexts are reduced, ι is an isomorphism in **FC**.

On the other hand, for every object (V, G, M) in **L**, define $(j, k) : (V, G, M) \to ST(V, G, M)$. Consider $j : V \to B(G, M, \leq)$ given by $j(p) := (\downarrow p \cap G, \uparrow p \cap M)$ for every $p \in V$, and $k : B(G, M, \leq) \to V$ defined by $k(A, B) := \bigwedge B$ for every concept $(A, B) \in B(G, M, \leq)$. Then (j, k) is an isomorphism in **L** and the following diagram commutes:

$$\begin{array}{ccc} (V_1, G_1, M_1) & \xrightarrow{(j_1,k_1)} & ST(V_1, G_1, M_1) \\ {\scriptstyle(\phi,\psi)}\downarrow & & \downarrow{\scriptstyle ST(\phi,\psi)} \\ (V_2, G_2, M_2) & \xrightarrow[(j_2,k_2)]{} & ST(V_2, G_2, M_2) \end{array}$$

□

4. PSEUDOMETRICS ON CONTEXTS

Contextual Topology has been introduced in [9] with the aim to extend the classical notion of a topological space on a formal context. The need for this generalization was the following remark. If $\mathbb{K} = (G, M, I)$ is a formal context, the incidence relation does not make any qualitative difference between the attributes, all incident attributes are related to the correspondent object. But obviously, in practice some attributes are more relevant than others, respectively the description of an object by these attributes is more accurate, than by others. Hence, we need a measure for the distance between the objects and attributes, i.e., a generalization of the pseudometric on a formal context.

Definition 4.1. Let G and M be two sets. We call *pseudometric between* G **and** M a map $d : G \times M \to \mathbb{R}_+$ satisfying the following rectangle condition:

$$(R) \qquad d(g, m) \leq d(g, n) + d(h, m) + d(h, n), \; g, h \in G, \; m, n \in M,$$

and for every $g \in G$ and $\varepsilon > 0$ there is an attribute $m \in M$ with $d(g, m) < \varepsilon$. Dually, for every $m \in M$ and every $\varepsilon > 0$ there is an object $g \in G$ with $d(g, m) < \varepsilon$.

Proposition 4.2. *The pseudometric d between G and M induces the pseudometrics $d^\vee$ and $d^\wedge$ on G and M, respectively.*

Proof. Let us define $d^\vee : G \times G \to \mathbb{R}_+$ by

$$d^\vee(g, h) := \inf_{m \in M}(d(g, m) + d(h, m)), \quad g, h \in G.$$

It follows from the definition of d that $d^\vee(g, g) = 0$ for every $g \in G$ and $d^\vee$ is symmetric. We just have to prove the triangle inequality, i.e., $d^\vee(x, v) +$

$d^\vee(v,u) \geq d^\vee(x,u)$ for all $x,v,u \in G$. Consider $x,v,u \in G$ arbitrary chosen. Using the rectangle inequality, we have

$$\begin{aligned} d^\vee(x,v) + d^\vee(v,u) &= \inf_{y\in M}(d(x,y)+d(v,y)) + \inf_{z\in M}(d(v,z)+d(u,z)) \\ &= \inf_{y,z\in M}(d(x,y)+d(v,y)+d(v,z)+d(u,z)) \\ &\geq \inf_{z\in M}(d(x,z)+d(u,z)) = d^\vee(x,u). \end{aligned}$$

The pseudometric $d^\wedge$ on M is defined in an analogous manner. □

In the following, we shall investigate some categorical properties of the above construction.

Every pseudometric between two sets G and M can be considered as a many-valued context $\mathbb{K} := (G, M, \mathbb{R}_+; d)$. This context will be simply called *the context of the pseudometric d*. The class of all many-valued contexts with fixed value set W is the object class of a category **Ctx**, category in which the morphisms are given by pairs of mappings $(f, f^*)\colon (G_1, M_1, W, e_1) \to (G_2, M_2, W, e_2)$ with $f : G_1 \to G_2$ and $f^* : M_2 \to M_1$ satisfying $e_1(g, f^*(m)) = e_2(f(g), m)$ for every $g \in G_1$ and $m \in M_2$, where $e_i\colon G_i \times M_i \to W$ is a partial map defined by $e_i(g,m) = m(g)$ for every $g \in G_i, m \in M_i, i = 1, 2$. It follows that the class of contexts of pseudometrics constitutes a full subcategory **CtxM** of **Ctx**.

As we have seen in Section 3, there is a categorical duality between the category of formal contexts and that of complete lattices whose morphisms are conveniently chosen. In this section we will discuss the question of extending this duality to pseudometric contexts and pseudometric lattices. We start first with some considerations on pseudometric morphisms and the question of scaling the context of a pseudometric.

LEMMA 4.3. *Let* $(f, f^*) : (G_1, M_1, \mathbb{R}_+, d_1) \to (G_2, M_2, \mathbb{R}_+, d_2)$ *be a morphism between the two contexts of the pseudometrics* d_1 *and* d_2*. We denote by* $d_1^\vee$ *the pseudometric induced by* d_1 *on* G_1 *and by* $d_1^\wedge$ *the pseudometric induced by* d_1 *on* M_1*. Similar* $d_2^\vee$ *and* $d_2^\wedge$ *denote the pseudometrics induced by* d_2 *on* G_2 *and* M_2*, respectively. Then the mappings* f *and* f^**, are expansive with respect to these pseudometrics.*

Proof. Indeed, for every $g, h \in G_1$, we have

$$\begin{aligned} d_2^\vee(f(g), f(h)) &= \inf_{z\in M_2}(d_2(f(g), z) + d_2(f(h), z)) \\ &= \inf_{z\in M_2}(d_1(g, f^*(z)) + d_1(h, f^*(z))) \\ &= \inf_{y\in \mathrm{Im} f^*}(d_1(g,y) + d_1(h,y)) \\ &\geq d_1^\vee(g,h). \end{aligned}$$

□

REMARK 4.4. If f^* is onto then f is an isometry. Dually, we obtain that f^* is expansive too, and if f is onto then f^* is an isometry.

The information stored in a many-valued context can be interpreted by assigning a conceptual structure to the given many-valued context. This asignment is called conceptual scaling. M. Gottsleben showed in [3] that this assignment is a functor called $\mathbf{Skal}_{\mathbb{S}}$ from **Ctx** in **FC**, the latter denoting the category of formal contexts.

DEFINITION 4.5. A formal context $\mathbb{K} := (G, M, I)$ is called a *pseudometric context* if there is a pseudometric $d : G\times M \to \mathbb{R}_+$ between G and M satisfying the following two conditions, called ε-conditions:

$$\forall \varepsilon > 0 \ \forall g \in G \ \exists m \in M : \ gIm \text{ and } d(g, m) < \varepsilon,$$

$$\forall \varepsilon > 0 \ \forall m \in M \ \exists g \in G : \ gIm \text{ and } d(g, m) < \varepsilon.$$

We call a pseudometric context *standard* if for every concept (A, B) of $\mathbb{K}$, we have $d(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\} = 0$. A scale $\mathbb{S} := (X, Y, J)$ with $\mathbb{R} \subseteq X$ is called *pseudometric scale* if the result of the conceptual scaling by $\mathbb{S}$, i.e., $\mathbf{Skal}_{\mathbb{S}}(G, M, \mathbb{R}, d)$ is a pseudometric context.

EXAMPLE 4.6. 1) If $(\mathbb{K}, d)$ is a pseudometric context then the object set and attribute set of $\mathbb{K}$ are pseudometric spaces with the induced pseudometrics $d^{\vee}$ on the object set and $d^{\wedge}$ on the attribute set, respectively. In the following, we start with a formal context $\mathbb{K} := (G, M, I)$ and let $d_1 : G \times G \to \mathbb{R}_+$ and $d_2 := M \times M \to \mathbb{R}_+$ be two pseudometrics on G and M, respectively. Define $d : G\times M \to \mathbb{R}_+$ by $d(g, m) := \min\{d_1(g, m'), d_2(g', m)\}$ for all $(g, m) \in G\times M$. Then d is a pseudometric on the context $\mathbb{K}$.

The ε-condition, i.e., for every $\varepsilon > 0$ and every $g \in G$, there is a $m \in g'$ with $d(g, m) < \varepsilon$, is obviously fulfilled, since for every gIm, we have $d(g, m) = 0$. We only have to prove the rectangle condition. Let $g, h \in G$ and $m, n \in M$ be arbitrary chosen. Then

$$\begin{aligned} d(g, n) + d(h, n) + d(h, m) &= \min\{d_1(g, n'), d_2(g', n)\} \\ &\quad + \min\{d_1(h, n'), d_2(h', n)\} + \min\{d_1(h, m'), d_2(h', m)\} \\ &= \min\{d_1(g, n') + d_1(h, n') + d_1(h, m'), d_2(g', n) + d_2(h', n) + d_2(h', m)\}. \end{aligned}$$

Now

$$\begin{aligned} d_1(g, n') + d_1(h, n') + d_1(h, m') &= \inf_{a\in n'} d_1(g, a) + \inf_{\tilde{a}\in n'} d_1(h, \tilde{a}) + \inf_{b\in m'} d_1(h, b) \\ &= \inf_{a,\tilde{a}\in n'} (d_1(g, a) + d_1(a, \tilde{a}) + d_1(h, a)) + \inf_{b\in m'} d_1(h, b) \\ &\geq \inf_{a,\tilde{a}\in n'} d_1(g, h) + \inf_{b\in m'} d_1(h, b) \\ &= d_1(g, h) + \inf_{b\in m'} d_1(h, b) = \inf_{b\in m'} (d_1(g, h) + d_1(h, b)) \\ &\geq \inf_{b\in m'} d_1(g, b) = d_1(g, m'), \end{aligned}$$

where we have used the triangle inequality for d_1 and the fact that $d_1(g,h) \leq d_1(g,a) + d_1(a,\tilde{a}) + d_1(h,\tilde{a}) \Rightarrow d_1(g,h) \leq \inf_{a\in n'} d_1(g,a) + \inf_{\tilde{a}\in n'} d_1(h,\tilde{a})$). It follows that $d(g,m) \leq d(g,n) + d(h,n) + d(h,m)$.

The same construction works for the maximum metric, i.e., $D : G\times M \to \mathbb{R}$, $D(g,m) := \max\{d_1(g,m'), d_2(g',m)\}$ for every $(g,m) \in G \times M$.

There are examples of metrics of a context $\mathbb{K}$ which do not arise from this construction. Take for example the euclidian metric on the set of rationals $\mathbb{Q}$. If $r < s$ then $d(r,s) \neq 0$ but for every $g \in \mathbb{Q}$ and $m \in \mathbb{Q}$, we have $d(g,g') = d(m',m) = 0$.

2) While working with a pseudometric space, we are often thinking in terms of distance less then an appropriate ε. This conceptualization is in fact a pseudometric scale. Let d be a pseudometric between G and M and $\varepsilon \geq 0$. We consider the scale $\mathbb{S}_\varepsilon := (\mathbb{R}, \varepsilon, \leq)$. The context $\mathbb{S}_\varepsilon$ is a metric scale. Indeed, $\mathbf{Skal}_{\mathbb{S}_\varepsilon}(G,M,\mathbb{R},d) = (G,M,P_\varepsilon)$ where $(g,m) \in P_\varepsilon$ if and only if $d(g,m) \leq \varepsilon$. More, if (C,D) is a concept of $\mathbb{K}_\varepsilon := (G,M,P_\varepsilon)$ then $C = \{g \in G \mid d(g,m) \leq \varepsilon \text{ for every } m \in D\}$ and dually, $D = \{m \in M \mid d(g,m) \leq \varepsilon \text{ for every } g \in C\}$. Similar to the case of a pseudometric space, we denote for $g \in G$ and $m \in M$,

$$B^{\mathbb{K}_\varepsilon}(g,\varepsilon) := \{m \in M \mid d(g,m) \leq \varepsilon\} = g^\varepsilon,$$

$$B^{\mathbb{K}_\varepsilon}(m,\varepsilon) := \{g \in G \mid d(g,m) \leq \varepsilon\} = m^\varepsilon,$$

and for $X \subseteq G$ and $Y \subseteq M$,

$$B^{\mathbb{K}_\varepsilon}(X,\varepsilon) := \{m \in M \mid d(x,m) \leq \varepsilon \text{ for all } x \in X\} = \bigcap_{g\in X} g^\varepsilon,$$

$$B^{\mathbb{K}_\varepsilon}(Y,\varepsilon) := \{g \in G \mid d(g,y) \leq \varepsilon \text{ for all } y \in Y\} = \bigcap_{m\in Y} g^\varepsilon.$$

For an arbitrary chosen concept (C,D), we obtain, by the Basic Theorem on Concept Lattices that $(C,D) = (B^{\mathbb{K}_\varepsilon}(D,\varepsilon), B^{\mathbb{K}_\varepsilon}(C,\varepsilon))$.

On the other hand, $d(C,D) = \inf\{d(g,m) \mid g \in C, m \in D\}$. Since d is a pseudometric between G and M, then for every $g \in G$ and $m \in M$, we have $d(g,M) = d(G,m) = 0$, hence for every $\eta \geq 0$, there is an $m_\eta \in M$ with $d(g,m_\eta) \leq \eta$, and a $g_\eta \in M$ with $d(g_\eta,m) \leq \eta$. For every $g \in G$, we have $d(g,g^\varepsilon) = \inf_{m\in g^\varepsilon} d(g,m)$ and using the above statement, we conclude that $d(g,g^\varepsilon) = 0$ since $m_\eta \in D$ and $d(g,m_\eta) \leq \eta$ for every $\eta \leq \varepsilon$. Analogous arguments shows that $d(m^\varepsilon,m) = 0$, i.e., $\mathbb{S}_\varepsilon$ is a pseudometric scale.

Definition 4.7. A map $d : G \times M \to \mathbb{R}_+$ is called *metric* between the sets G and M if and only if d is a pseudometric between G and M and

$$\forall n \in M : d(g,n) = d(h,n) \Rightarrow g = h,$$

$$\forall g \in G : d(g,m) = d(g,n) \Rightarrow m = n.$$

An analogous definition yields the notion of a *metric* on the formal context $\mathbb{K}$.

Remark 4.8. If d is a metric between G and M, then the induced pseudometrics on G and M are metrics too.

Let $g, h \in G$ with $d^{\vee}(g,h) = 0$. Then $\inf_{m\in M}(d(g,m) + d(h,m)) = 0$. From the rectangle condition follows that $d(h,n) \leq d(g,m) + d(h,m) + d(g,n)$ for every $m, n \in M$, i.e., $d(h,n) \leq d(g,n)$. By symmetry, we have that $d(g,n) = d(h,n)$ for every $n \in M$ and since d was a metric on $\mathbb{K}$, we have that $g = h$, i.e., $d^{\vee}$ is a metric on G. Analogous arguments proves that $d^{\wedge}$ is a metric on M.

Remark 4.9. The last condition in the definition of a metric on a context $\mathbb{K}$ is the analogon of the "clarifying" a formal context. Hence metric contexts are, in some sense, clarified pseudometric contexts

Proposition 4.10. *Every pseudometric context $(\mathbb{K}, d)$ can be factorized to a metric context. The concept lattice of the factor context is a surjective image of the concept lattice of $\mathbb{K}$, but these two complete lattices are generally not isomorphic.*

Proof. Let $(\mathbb{K}, d) := (G, M, I; d)$ be a pseudometric context. Consider the following equivalence relations on G and M, respectively:

$$R_d \subseteq G \times G \text{ with } (g,h) \in R_d :\Leftrightarrow \forall n \in M : d(g,n) = d(h,n),$$

$$S_d \subseteq M \times M \text{ with } (m,n) \in S_d :\Leftrightarrow \forall g \in G : d(g,m) = d(g,n),$$

satisfying the following compatibility condition with the incidence of $\mathbb{K}$:

$$gIm, gRg_1, mSm_1 \Rightarrow g_1Im_1.$$

We define the context $\tilde{\mathbb{K}} := (G/R_d, M/S_d, J)$ where $[g]J[m] :\Leftrightarrow gIm$. The claim is that $\tilde{\mathbb{K}}$ is a metric context where the metric on $\tilde{\mathbb{K}}$ is defined appropriately. The incidence relation J of $\tilde{\mathbb{K}}$ is well defined. For every pair $([g],[m]) \in J$ and every representant $h \in [g]$ and $n \in [m]$, we have gRh and mSn and, by the compatibility condition, hIn, i.e., $([h],[n]) \in J$.

Define $\tilde{d} : G/R_d \times M/S_d \to \mathbb{R}$ by $\tilde{d}([g],[m]) := d(g,m)$. The map $\tilde{d}$ is well defined. Indeed, for gRh and mSn, we have $d(g,p) = d(h,p)$ and $d(q,m) = d(q,n)$ for every $p \in M$ and every $q \in G$. Choose $p = m$, then $d(g,m) = d(h,m)$ by the definition of R_d. By choosing $q = h$, we obtain $d(g,m) = d(h,m) = d(h,n)$ by the definition of S_d, i.e., $\tilde{d}$ is well defined. One can easy check that $\tilde{d}$ is a metric on $\tilde{\mathbb{K}}$.

Let us denote by $B(\mathbb{K})$ and $B(\tilde{\mathbb{K}})$ the concept lattices of the contexts $\mathbb{K}$ and $\tilde{\mathbb{K}}$. Define $\phi : B(\mathbb{K}) \to B(\tilde{\mathbb{K}})$ by $\phi(A,B) := ([A],[B])$ where $[A]$ denotes the set $[A] := \{[g] \mid g \in A\}$. For every concept $(A,B) \in B(\mathbb{K})$, we have

$$[A]^J := \{[n] \in M/S_d \mid \forall [g] \in [A], [g]J[n]\} = \{[n] \in M/S_d \mid \forall g \in A, gIn\} = [B].$$

Analogous arguments show that $[B]^J = [A]$, i.e., the map ϕ is well-defined.

Moreover, ϕ is clearly a complete lattice homomorphism and it is onto but not necessary one-to-one. For every $(C,D) \in B(\tilde{\mathbb{K}})$, define $A := \{g \in G \mid [g] \in C\}$ and $B := \{m \in M \mid [m] \in D\}$. Now $A^I = \{n \in M \mid \forall g \in A, gIn\} = \{n \in M \mid \forall g \in A, [g]J[n]\} = \{n \in M \mid \forall x \in C, xJ[n]\} = B$, i.e., $(A,B) \in B(\mathbb{K})$, which proves that ϕ is onto. Since the context $\tilde{\mathbb{K}}$ was obtained by factorization, it is not difficult to see that ϕ is generally not one-to-one. □

In the following, we will describe the context of a pseudometric as a multicontext. The incidence relations of this multicontext play a certain role in characterizing the topological properties of a pseudometric context [9].

Let $d : G\times M \to \mathbb{R}$ be a pseudometric between G and M. The pseudometric d generates a multicontext $\mathbb{K}$ where $I := \{1, 2\}, P := [0,\infty]$, $S_1 := G$, and $S_2 := M$. The relations R_ε with $\varepsilon \geq 0$ are defined by $R_\varepsilon := \{(g,m) \in G \times M \mid d(g,m) \leq \varepsilon\}$ and are satisfying the following properties:

(M) $\quad R_\varepsilon(g,m) \Rightarrow R_\delta(g,m),\ \delta \geq \varepsilon$

$$R_\varepsilon(g,m) \wedge R_\delta(g,h) \wedge R_\eta(h,m) \Rightarrow R_{\varepsilon+\delta+\eta}(h,n)$$

(M_0) $\quad \forall g\ \forall\varepsilon \in \mathbb{R}_+\ \exists m\ R_\varepsilon(g,m)$

(M_∞) $\quad \forall\delta \in \mathbb{R}_+\ \delta > \varepsilon,\ R_\delta(x,y) \Rightarrow R_\varepsilon(x,y)$

The class of multicontexts $(G,M,P_\varepsilon)_{\varepsilon\geq 0}$ where G and M are sets and $(P_\varepsilon)_{\varepsilon\geq 0}$ is a family of binary relations between G and M satisfying $(M), (M_0)$, and (M_∞) yields a category denoted with **RelM** in which the morphisms $(f, f^*) : (G_1,M_1,P_\varepsilon)_{\varepsilon\geq 0} \to (G_2,M_2,Q_\varepsilon)_{\varepsilon\geq 0}$, are pairs of mappings $f : G_1 \to G_2$ and $f^* : M_2 \to M_1$ satisfying $P_\varepsilon(g, f^*(m)) \Leftrightarrow Q_\varepsilon(f(g),m)$ for $\varepsilon \geq 0, g \in G_1$, and $m \in M_1$. The composition of $(f, f^*)\colon (G_1,M_1,P_\varepsilon)_{\varepsilon\geq 0} \to (G_2,M_2,Q_\varepsilon)_{\varepsilon\geq 0}$ and $(g, g^*)\colon (G_2,M_2,Q_\varepsilon)_{\varepsilon\geq 0} \to (G_3,M_3,R_\varepsilon)_{\varepsilon\geq 0}$ defined by $(g, g^*) \circ (f, f^*) = (g \circ f, f^* \circ g^*)$ is also a morphism. It satisfies the compatibility condition with the given family of relations, since

$$P_\varepsilon(h, f^*(g^*(m)) \Leftrightarrow Q_\varepsilon(f(h), g^*(m)) \Leftrightarrow R_\varepsilon(g(f(h)),m).$$

Proposition 4.11. *The category* **CtxM** *of contexts of pseudometrics is equivalent to* **RelM**.

Proof. Let us consider the functor F : **CtxM** $\to$ **RelM** defined on objects by $F(G,M,\mathbb{R}, d) := (G,M,P_\varepsilon)_{\varepsilon\geq 0}$ and on morphisms by $F(f, f^*) := (f, f^*)$.

Let $(f, f^*)\colon F(G_1,M_1,\mathbb{R}_+, d_1) \to F(G_2,M_2,\mathbb{R}_+, d_2)$ be a morphism in the category **RelM**, $(f, f^*)\colon (G_1,M_1,P_\varepsilon)_{\varepsilon\geq 0} \to (G_2,M_2,Q_\varepsilon)_{\varepsilon\geq 0}$, where $(P_\varepsilon)_{\varepsilon\geq 0}$ and $(Q_\varepsilon)_{\varepsilon\geq 0}$ are the relations induced by d_1 on $G_1\times M_1$ and by d_2 on $G_2\times M_2$, respectively.

Since $f : G_1 \to G_2$ and $f^* : M_2 \to M_1$, we will prove that $(f, f^*) : (G_1,M_1,\mathbb{R}, d_1) \to (G_2,M_2,\mathbb{R}, d_2)$ is a morphism in **CtxM**. Choosing $\varepsilon := d_1(g, f^*(m))$ we conclude that $P_\varepsilon(g, f^*(m))$ is equivalent to $Q_\varepsilon(f(g),m)$ if and only if $d_2(f(g),m) \leq d_1(g, f^*(m))$.

Let $d_2(f(g),m) =: \delta$ then $(f(g),m) \in Q_\delta$ is equivalent to $(g, f^*(m)) \in P_\delta$, which holds true if and only if $d_1(g, f^*(m)) \leq d_2(f(g),m)$, concluding that F is faithful.

For the equivalence, we only have to prove that for every object $(G, M, P_\varepsilon)_{\varepsilon \geq 0}$ in the category **RelM**, there is a an object $(G, M, \mathbb{R}, d)$ in **CtxM** with

$$F(G, M, \mathbb{R}, d) \cong (G, M, P_\varepsilon)_{\varepsilon \geq 0},$$

which express the fact that the family $(P_\varepsilon)_{\varepsilon \geq 0}$ induces a convenient pseudometric between G and M. For any multicontext $(G, M, P_\varepsilon)_{\varepsilon \geq 0}$ in **RelM** we define the pseudometric $d : G \times M \to \mathbb{R}$ induced by $(P_\varepsilon)_{\varepsilon \geq 0}$ between G and M by $d(g, m) := \inf\{\delta \geq 0 \mid P_\delta(g, m)\}$. Axioms $(M), (M_0)$ and (M_∞) guarantees that d is well defined and a pseudometric between G and M, which finishes the proof. □

As we have seen, the category of formal contexts is dual equivalent to that of complete lattices. To extend this duality to the metric case, it is naturally to ask whether the given pseudometric can be extended on the correspondent concept lattice.

LEMMA 4.12. *Let* $\mathbb{K} := (G, M, I; \rho)$ *be a pseudometric context. The map* $d : B(G, M, I) \times B(G, M, I) \to \mathbb{R}$*, defined by*

$$d((A, B), (C, D)) := \max\{\rho(A, D), \rho(C, B)\},$$

is a pseudometric on $B(G, M, I)$*, the concept lattice of* $\mathbb{K}$.

Proof. Let $(A, B), (C, D)$, and (E, F) be some formal concepts of $\mathbb{K}$. Then $d((A, B), (A, B)) = \rho(A, B) = 0$ because of the properties of ρ. The map d is symmetric by definition, so we only have to prove the triangle inequality.

$$\begin{aligned} d((A, B), (C, D)) &+ d((C, D), (E, F)) \\ &= \max\{\rho(A, D), \rho(C, B)\} + \max\{\rho((C, F), \rho(E, D)\} \\ &= \max\{\inf\{\rho(a, d) \mid a \in A, d \in D\}, \inf\{\rho(c, b) \mid c \in C, b \in B\}\} \\ &\quad + \max\{\inf\{\rho(c, f) \mid c \in C, f \in F\}, \inf\{\rho(e, d) \mid e \in E, d \in D\}\}. \end{aligned}$$

Now,

$$\begin{aligned} d((A, B), (E, F)) &= \max\{\rho(A, F), \rho(E, B)\} \\ &= \max\{\inf\{\rho(a, f) \mid a \in A, f \in F\}, \inf\{\rho(e, b) \mid e \in E, b \in B\}\} \\ &\leq \max\{\rho(a, f), \rho(e, b)\} \\ &\leq \max\{\rho(a, d) + \rho(e, d) + \rho(e, f), \rho(e, f) + \rho(c, f) + \rho(c, b)\} \end{aligned}$$

for every $a \in A, b \in B, c \in C, d \in D, e \in E$ and $f \in F$.

It follows that

$$\begin{aligned}
d((A,B),(C,D)) \leq & \max\{\inf\{\rho(a,d) \mid a\in A, d\in D\} \\
& + \inf\{\rho(e,f) \mid e\in E, f\in F\} \\
& + \inf\{\rho(e,d) \mid e\in E, d\in D\},\ \inf\{\rho(e,f) \mid e\in E, f\in F\} \\
& + \inf\{\rho(c,f) \mid c\in C, f\in F\} + \inf\{\rho(c,b) \mid c\in C, b\in B\}\} \\
\leq & \max\{\inf\rho(a,d), \inf\rho(c,b)\} + \max\{\inf\rho(c,f), \inf\rho(e,d)\} \\
= & d((A,B),(C,D)) + d((C,D),(E,F)),
\end{aligned}$$

since $\inf\{\rho(e,f) \mid e\in E, f\in F\} = \rho(E,F) = 0$. □

We denote by **MC** the category of pseudometric contexts. The morphisms are exactly those context morphisms $(f,f^*) : (G_1,M_1,I_1;\rho_1) \to (G_2,M_2,I_2;\rho_2)$ in **FC** satisfying $\rho_2(f^{-1}(g_1)f^*(m_1)) \leq \rho_1(g,m)$ for all $g\in G_1$ and $m\in M_1$, and $\rho_1(f(g_2), f^{*-1}(m_2)) \leq \rho_2(g_2,m_2)$ for all $g\in G_2$ and $m\in M_2$.

Remark 4.13. 1) $gI_1f^*(m) \Leftrightarrow f(g)I_2m$ is equivalent to $g\in f^*(m)' \Leftrightarrow f(g)\in m'$, which implies $f(f^*(m)')\subseteq m'$. Dually, we obtain that $f^*(f(g)')\subseteq g'$.

2) The class **SMC** of standard pseudometric contexts is a full subcategory of **MC**, the category of pseudometric contexts.

3) A more natural description of a morphism in **MC** would be that inherited from **Ctx**: Since to every pseudometric we can associate the context of that pseudometric, the morphisms in **MC** could be viewed as those inherited from **Ctx**. But this would lead to a very restrictive description of the morphisms between the correspondent concept latices as isometrics.

Every pseudometric d on a context $\mathbb{K}$ induces a pseudometric $\rho : B(\mathbb{K})\times B(\mathbb{K})\to\mathbb{R}$ by $\rho((A,B),(C,D)) = \max\{d(A,D), d(C,B)\}$ on $B(\mathbb{K})$, the complete lattice of formal concepts of $\mathbb{K}$. Its restriction to $\gamma G\times\mu M$, i.e., $T\circ S(\mathbb{K}) = (\gamma G, \mu M, \leq; \rho)$ is obviously a pseudometric between γG and μM and $(\gamma G,\mu M,\leq;\rho)$ is a metric context. To see this, take an arbitrary $g\in G$. Now

$$\begin{aligned}
\rho(\gamma g, \gamma g^{\leq}) &= \rho(\gamma g, \{m\in M \mid \gamma g\leq \mu m\}) \\
&= \inf\{\rho(\gamma g, \mu m) \mid m\in g'\} \\
&= \inf\{\max\{d(g'',m''), d(m',g')\} \mid m\in g'\}.
\end{aligned}$$

Since $\{m\}\subseteq g'$, it follows that $g''\subseteq m'$ and $m''\subseteq g'$. The distance between g and g' is zero, which implies that for every $\varepsilon>0$, there is an $m_0\in g'$ with $d(g,m_0)<\varepsilon$, hence $d(g'',m_0'')<\varepsilon$.

On the other hand, $d(m_0',g') = \inf\{d(h,n) \mid h\in m_0', n\in g'\}$. By the rectangle inequality, we have

$$\begin{aligned}
d(h,n) &\leq d(g,m_0) + d(g,n) + d(h,m_0) \\
&\leq \varepsilon + d(g,n) + d(h,m_0),
\end{aligned}$$

hence

$$\begin{aligned}\inf\{d(h,n) \mid h \in m_0',\, n \in g'\} &\le \varepsilon + \inf\{d(g,n) + d(h,m_0) \mid h \in m_0', n \in g'\}\\ &= \varepsilon + \inf\{d(g,n) \mid h \in g'\} + \inf\{d(h,m_0) \mid h \in m_0'\}\\ &= \varepsilon + d(g,g') + d(m_0',m_0)\\ &= \varepsilon.\end{aligned}$$

We conclude that $\rho(\gamma g, \gamma g^{\le}) = 0$ and also $\rho(\mu m^{\le}, \mu m) = 0$.

Remark 4.14. We have proved even more. For all objects $g \in G$ and for all $\varepsilon > 0$, there is an attribute $m \in M$ with $\rho(\gamma g, \mu m) < \varepsilon$ and symmetrically for the attributes, that means every object concept can be approximated by some attribute concept and, dually, every attribute concept can be approximated by some object concept.

Let us have a closer look to $(B(\mathbb{K}), \gamma G, \mu M; \rho)$. Since $\mathbb{K}$ was a standard pseudometric context, then $d(A,B) = 0$ for all concepts $(A,B) \in B(\mathbb{K})$. That means, for example, that for all $\varepsilon > 0$, there is an $m \in B$ with $d(A,m) < \varepsilon$, hence $d(A, m'') < \varepsilon$. Since $m \in B$ implies $A \subseteq m'$, we have $d(m', B) \le d(A,B) = 0$. We conclude that $B(\mathbb{K})$ satisfies the following density property:

(Δ) For every $(A,B) \in B(\mathbb{K})$, there is a $g \in A$ and an $m \in B$ with $\rho((A,B), \gamma g) < \varepsilon$ and $\rho((A,B), \mu m) < \varepsilon$.

Let us consider the category **Lm** whose object class consists of triples (V,G,M), where V is a complete lattice, G is a supremum dense subset of V, M is an infimum dense subset of V and $d : V \times V \to \mathbb{R}$ is a pseudometric on V, satisfying

$$(D) \forall p \in V\ \forall \varepsilon > 0\ \exists g \in G : g \le p, \exists m \in M : p \le m, d(g,p) < \varepsilon, d(p,m) < \varepsilon.$$

The morphisms in this category are given by the morphisms of **L**, i.e., pairs of mappings $(f, f^*) : (V_1, G_1, M_1; d_1) \to (V_2, G_2, M_2; d_2)$, where f and f^* are contractions, satisfying the following condition:

(d) $d_1(f^{-1}(g_2), f^*(m_2)) \le d_2(g_2, m_2)$ for every $g \in G_2$ and $m \in M_2$
$d_2(f(g_1), f^{*-1}(m_1)) \le d_1(g_1, m_1)$ for every $g_1 \in G_1$ and $m_1 \in M_1$.

Theorem 4.15. *The categories* **SMC** *and* **Lm** *are dually equivalent.*

Proof. Since the imposed conditions where likely to extend the given duality between the category of formal contexts **FC** and that of complete lattices **L**, we have to check whether the restrictions to **SMC** and **Lm** of the functors T and S are well-defined. Let us consider the functor $T : \mathbf{L} \to \mathbf{FC}$ and denote its restriction to **Lm** also by T. We will prove that the restriction of T still acts on objects and morphisms within the category **SMC**.

For $(V,G,M;d)$ in **Lm**, we have $T(V,G,M;d) = (G,M,\le;d)$ where we have denoted by d its restriction $d := d_{|G\times M}$ to $G \times M$. The context $(G, M, \le; d)$

is a pseudometric context since density implies $d(g, g^{\leq}) = 0 = d(m^{\leq}, m)$ for $g \in G$ and $m \in M$.

Let $(A, B) \in B(G, M, \leq)$. Then there is a $p \in V$ with $(A, B) = (\downarrow p \cap G, \uparrow p \cap M)$. Density implies that for all $\varepsilon > 0$, there is a $g \in \downarrow p \cap G$ and an $m \in \uparrow p \cap M$ with $d(g, p) < \varepsilon$ and $d(m, p) < \varepsilon$, i.e., $d(g, m) < 2\varepsilon$, which implies that $d(A, B) = 0$. We conclude that the functor T is well defined on objects. For the well-definedness on morphisms, let us consider $(f, f^*) : (V_1, G_1, M_1; \rho_1) \to (V_2, G_2, M_2; \rho_2)$ a morphism in **Lm**. Then $T(f, f^*) = (f^*_{|M_2}, f_{|G_1}) : (G_2, M_2, \leq; \rho_2) \to (G_1, M_1, \leq; \rho_1)$. By the definition, it is obvious that condition (d) for contexts is satisfied by $T(f, f^*)$. We conclude that the restriction of T to the pseudometric case, $T : \mathbf{Lm} \to \mathbf{SMC}$ is well-defined.

Let us now consider the functor $S : \mathbf{FC} \to \mathbf{L}$. Applying S to a standard pseudometric context, we obtain $S(G, M, I; d) = (B(G, M, I), \gamma G, \mu M; \rho)$. The restriction of S to the pseudometric case is well-defined on objects, since $d(A, B) = 0$ for every $(A, B) \in B(G, M, I)$ implies the existence of elements $g \in A$ and $m \in B$ with $d(A, m) < \varepsilon$ and $d(g, B) < \varepsilon$ for some $\varepsilon > 0$. By the definition of ρ, we have $\rho((A, B), \mu m) = \max\{d(A, m''), d(m', B)\}$. Since $d(A, m) < \varepsilon$, it follows that $d(A, m'') < \varepsilon$, and $m \in B$ leads to $A \subseteq m'$, which implies $d(m', B) = 0$ by $d(A, B) = 0$. Hence $\rho((A, B), \mu m) < \varepsilon$, and $\rho((A, B), \gamma g) < \varepsilon$ by analogous arguments. We just have proved that the density condition holds in $(B(G, M, I), \gamma G, \mu M; \rho)$.

The functor S is also well-defined on morphisms in **SMC**. To see this, let $\mathbb{K}_1 := (G_1, M_1, I_1; \rho_1)$ and $\mathbb{K}_2 := (G_2, M_2, I_2; \rho_2)$ be two standard pseudometric contexts in **SMC** and $(\alpha, \beta) : \mathbb{K}_1 \to \mathbb{K}_2$ a morphism between $\mathbb{K}_1$ and $\mathbb{K}_2$. Then $S(\alpha, \beta) = (\psi, \phi) : (B(\mathbb{K}_2), \gamma G_2, \mu M_2; d_2) \to (B(\mathbb{K}_1), \gamma G_1, \mu M_1; d_1)$, where $\phi : \underline{\mathfrak{B}}(\mathbb{K}_1) \to B(\mathbb{K}_2)$ and $\psi : B(\mathbb{K}_2) \to B(\mathbb{K}_1)$ are given by

$$\phi(A_1, B_1) = (\beta(B_1)', \beta(B_1)'') \text{ for every } (A_1, B_1) \in B(\mathbb{K}_1), \text{ and}$$

$$\psi(A_2, B_2) = (\alpha(A_2)'', \alpha(A_2)') \text{ for every } (A_2, B_2) \in B(\mathbb{K}_2).$$

Now we have

$$\begin{aligned} d_2(\phi(A_1, B_1), \phi(C_1, D_1)) &= d_2((\beta(B_1)', \beta(B_1)''), (\beta(D_1)', \beta(D_1)'')) \\ &= \max\{\rho_2(\beta(B_1)', \beta(D_1)''), \rho_2(\beta(D_1)', \beta(B_1)'')\}. \end{aligned}$$

The following is true

$$\begin{aligned} \beta(B_1)' &= \{g_2 \in G_2 \mid g_2 I_2 n_2 \text{ for all } n_2 \in \beta(B_1)\} \\ &= \{g_2 \in G_2 \mid g_2 I_2 b(m_1) \text{ for all } m_1 \in B_1\} \\ &= \{g_2 \in G_2 \mid \alpha(g_2) I_1 m_1 \text{ for all } m_1 \in B_1\} \\ &= \{g_2 \in G_2 \mid \alpha(g_2) \in B_1' (= A_1)\} = \alpha^{-1}(A_1). \end{aligned}$$

We can analogously prove that $\beta(D_1)' = \alpha^{-1}(C_1)$.

Since $(C_1, D_1) \in B(\mathbb{K}_1)$, we have $C_1 = \bigcap_{m_1 \in D_1} m_1'$. Now

$$\begin{aligned}\alpha^{-1}(C_1) &= \bigcap_{m_1 \in D_1} \alpha^{-1}(m_1') = \bigcap_{m_1 \in D_1} \beta(m_1'')' \\ &= \bigcap_{m_1 \in D_1} \beta(m_1)' = (\bigcup_{m_1 \in D_1} \beta(m_1))',\end{aligned}$$

which implies $\beta(D_1)'' = \bigcup_{m_1 \in D_1} \beta(m_1)$. Using (d), we obtain

$$\begin{aligned}\rho_2(\beta(B_1)', \beta(D_1'')) &= \rho_2(\bigcup_{g_1 \in A_1} \alpha^{-1}(g_1), \bigcup_{m_1 \in D_1} \beta(m_1)) \\ &\leq \rho_2(\alpha^{-1}(g_1), \beta(m_1)) \\ &\leq \rho_1(g_1, m_1)\end{aligned}$$

for every $g_1 \in A_1$ and $m_1 \in D_1$. It follows that $\rho_2(\beta(B_1)', \beta(D_1)'') \leq \rho_1(A_1, D_1)$, and similarly $\rho_2(\beta(D_1)', \beta(B_1)'') \leq \rho_1(C_1, B_1)$, i.e., the restriction of S to the metric case is well-defined.

Let $(V, G, M; d)$ be an object in **Lm**. Then (V, G, M) is isomorphic to $(B(G, M, \leq), \gamma G, \mu M)$. We prove that this isomorphism is an isometry. For (A, B) and $(C, D) \in B(G, M, \leq)$, there exists elements p and q in V, with $(A, B) = (\downarrow p \cap G, \uparrow q \cap M)$ and $(C, D) = (\downarrow q \cap G, \uparrow q \cap M)$.

By definition, $\rho((A, B), (C, D)) = \max\{d(A, D), d(C, B)\}$. Using the density condition we obtain that for all $\varepsilon > 0$, there are some $a \in A$ and some $\delta \in D$ with $d(p, a) < \varepsilon$ and $d(q, \delta) < \varepsilon$. Now $d(a, \delta) \leq d(a, p) + d(p, q) + d(q, \delta)$, and so $|d(a, \delta) - d(p, q)| < 2\varepsilon$; hence $d(A, D) = d(p, q)$. Similarly, $d(C, B) = d(p, q)$ concluding that the given isomorphism in **L** is an isometry, i.e., an isomorphism in **Lm**.

If $\mathbb{K} := (G, M, I; d)$ is a standard pseudometric context, then (G, M, I) is isomorphic to $(\gamma G, \mu M, \leq)$ in **FC**. We only have to prove that this isomorphism is also an an isomorphism in **SMC**.

By definition, $\rho(\gamma g, \mu m) = \max\{d(g'', m''), d(m', g')\}$ for every $g \in G$ and $m \in M$; hence for every $\varepsilon > 0$, there are $n \in g'$ and $h \in m'$ with $d(g, n) < \varepsilon$ and $d(h, b) < \varepsilon$, which by leads to $d(g, m) \leq d(g, n) + d(h, n) + d(h, b) < 2\varepsilon + d(h, n)$ by the rectangle inequality, implying $d(g, m) \leq d(m', g')$. Since $g \in g''$ and $m \in m''$, we obtain $d(g'', m'') \leq d(g, m)$.

But $\mathbb{K}$ was a standard pseudometric context, i.e., $d(g, g') = d(m', m) = 0$ for every $g \in G$ and $m \in M$, that means that for every $\varepsilon > 0$, there are $n \in g'$ and $h \in m'$ with $d(g, n) < \varepsilon$ and $d(h, m) < \varepsilon$ wherefrom follows that $d(h, n) < 2\varepsilon + d(g, m)$, hence $d(m', g') \leq d(g, m)$.

In conclusion, we just have proved the commutativity of the following two diagrams in **SMC** and **Lm**, respectively

$$\begin{array}{ccc}(G_1, M_1, I_1; d_1) & \xrightarrow{\iota_1} & TS(G_1, M_1, I_1; d_1) \\ {\scriptstyle(\beta,\alpha)}\downarrow & & \downarrow{\scriptstyle TS(\beta,\alpha)} \\ (G_2, M_2, I_2; d_2) & \xrightarrow[\iota_2]{} & TS(G_2, M_2, I_2; d_2)\end{array}$$

$$\begin{array}{ccc}(V_1, G_1, M_1; \rho_1) & \xrightarrow{j_1} & (\underline{\mathfrak{B}}(G_1, M_1, \leq), \gamma G_1, \mu M_1; \rho_1) \\ {\scriptstyle(\phi,\psi)}\downarrow & & \downarrow{\scriptstyle(\psi_{|M_2}, \phi_{|G_1})} \\ (V_2, G_2, M_2; \rho_2) & \xrightarrow[j_2]{} & (\underline{\mathfrak{B}}(G_2, M_2, \leq), \gamma G_2, \mu M_2; \rho_2)\end{array}$$

which concludes the proof. □

We have seen that every pseudometric on a context $\mathbb{K}$ leads to a family of relations $(P_\varepsilon)_{\varepsilon\geq 0}$ allowing a representation of pseudometric contexts as multicontexts; moreover, this representation is a categorical equivalence. Since this representation depends only on the properties of the pseudometric d on the context $\mathbb{K}$, we conclude that the same construction can be made for every pseudometric lattice in **Lm**. Consider the category $\mathbf{L}'$ of complete lattices (V, G, M) enhanced with a family of relations $(P_\varepsilon)_{\varepsilon\geq 0}$ which are satisfying the following axioms

(M) $$P_\varepsilon(x, y) \to P_\delta(x, y), \delta \geq \varepsilon,$$

$$P_\varepsilon(x, y) \wedge P_\delta(y, z) \to P_{\varepsilon+\delta}(x, z);$$

(M_∞) $$\forall \delta > \varepsilon : P_\delta(x, y) \to P_\varepsilon(x, y), \varepsilon \geq 0;$$

(D) $$\forall x\ \exists g\ \exists m\ \forall \varepsilon :\ P_\varepsilon(x, g) \wedge P_\varepsilon(x, m).$$

The morphisms of this category are those from **L**, i.e., pair of mappings (f, f^*) where $f : V_1 \to V_2$ and $f^* : V_2 \to V_1$ respect the relations $(P_\varepsilon)_{\varepsilon\geq 0}$, i.e., for $(f, f^*) : (V_1, G_1, M_1, (P_\varepsilon)) \to (V_2, G_2, M_2, (Q_\varepsilon))$ the following holds

$$\forall \varepsilon > 0\ \forall g \in V_1\ \forall m \in V_2 :\ P_\varepsilon(g, f^*(m)) \Leftrightarrow Q_\varepsilon(f(g), m).$$

By all the above results, the following holds true.

Theorem 4.16. *The following categories are equivalent*

$$\begin{array}{ccc}\mathbf{Lm} & \xrightarrow{\cong} & \mathbf{L}' \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ \mathbf{SMC}^{op} & \xrightarrow[\cong]{} & \mathbf{C}'^{op}\end{array}$$

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Received July 28, 2017
Accepted September 1, 2017

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