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ESTIMATION OF INEXACT RECIPROCAL-QUINTIC AND RECIPROCAL-SEXTIC FUNCTIONAL EQUATIONS

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Abstract. The purpose of this study is to investigate Ulam stability of reciprocal-quintic and reciprocal-sextic functional equations in non-Archimedean fields pertinent to the results ascertained by Hyers, Rassias, and Găvruta. Illustrative examples are provided to show that the results are not valid for the singular cases.

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1. INTRODUCTION

The theory of stability of functional equations is an emerging field in modern mathematics. The investigation of stability of functional equations has been initiated by the renowned problem of Ulam [25] in 1940. Hyers [12] was the foremost mathematician who vividly provided a partial answer to the question of Ulam. Later on, various generalizations and extension of Hyers' result were ascertained by Bourgin [7], Th. M. Rassias [23], Gruber [11], Aoki [1], J. M. Rassias [17] and Găvruta [10] in different versions. For the past 35 years, the stability of various functional equations were dealt by many mathematicians; one can refer to [2, 3, 8, 13, 16, 18, 26, 27].

In 2010, Ravi and the second author [19] obtained the Ulam-Găvruta-Rassias stability for the Rassias reciprocal functional equation

$$g(x+y) = \frac{g(x)g(y)}{g(x)+g(y)}, \tag{1}$$

where $g : X \longrightarrow \mathbb{R}$ is a map with X being the space of non-zero real numbers. The reciprocal function $g(x) = \frac{c}{x}$ is a solution of the functional equation (1). The functional equation (1) is related with reciprocal formula which appears in an electric circuit consisting of two resistors connected in parallel [21]. Stability results connected with various forms of reciprocal and reciprocal-quadratic, cubic and quartic functional equations can be found in [4], [5], [6], [14], [15], [20], [22] and [24].

In this paper, we introduce the reciprocal-quintic functional equation

$$q(2x+y)+q(2x-y) = \frac{4q(x)q(y)\left[16q(y)+40q(x)^{\frac{2}{5}}q(y)^{\frac{3}{5}}+5q(x)^{\frac{4}{5}}q(y)^{\frac{1}{5}}\right]}{\left[4q(y)^{\frac{2}{5}}-q(x)^{\frac{2}{5}}\right]^{5}} \tag{2}$$

and the reciprocal-sextic functional equation
(3)

$$s(2x+y)+s(2x-y) = \frac{2s(x)s(y)\left[s(x)+60s(x)^{\frac{2}{3}}s(y)^{\frac{1}{3}}+240s(x)^{\frac{1}{3}}+64s(y)\right]}{\left[4s(y)^{\frac{1}{3}}-s(x)^{\frac{1}{3}}\right]^{6}}.$$

It is easy to check that the reciprocal-quintic function $q(x) = \frac{1}{x^5}$ and the reciprocal-sextic function $s(x) = \frac{1}{x^6}$ are solutions of the functional equations (2) and (3), respectively. We prove the generalized Hyers-Ulam stability and stability results controlled by the sum of powers of norms and product of different powers of norms of these functional equations. We also present some suitable examples to show that the results are not stable for the singular cases.

2. PRELIMINARIES

In this section, we sum up the fundamental notions of non-Archimedean field.

Definition 2.1. By a *non-Archimedean field*, we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{K}$ into $[0,\infty)$ such that $|\alpha| = 0$ if and only if $\alpha = 0$, $|\alpha\beta| = |\alpha||\beta|$ and $|\alpha+\beta| \leq \max\{|\alpha|,|\beta|\}$, for all $\alpha,\beta \in \mathbb{K}$.

Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$, for all $n \in \mathbb{N}$. We always assume, in addition, that $|\cdot|$ is non-trivial, i.e., there exists an $a_0 \in \mathbb{K}$ such that $|a_0| \neq 0, 1$. It follows from $|t_n - t_m| \leq \max\{|t_{j+1} - t_j| : m \leq j \leq n-1\}$, for $n > m$, that a sequence $\{t_n\}$ is Cauchy if and only if $\{t_{n+1} - t_n\}$ converges to zero in a non-Archimedean field. By a *complete non-Archimedean field* we mean that every Cauchy sequence is convergent in the field.

An example of a non-Archimedean valuation is the map $|\cdot|$ that takes everything but 0 into 1 and $|0| = 0$. This valuation is called trivial. Another famous example of a non-Archimedean valuation on a field $\mathbb{A}$ is the map

$$|\lambda| = \begin{cases} 0, & \text{if } \lambda = 0, \\ \frac{1}{\lambda}, & \text{if } \lambda > 0, \\ -\frac{1}{\lambda}, & \text{if } \lambda < 0, \end{cases}$$

for any $\lambda \in \mathbb{A}$.

Let p be a prime number. For any non-zero rational number $x = p^r \frac{m}{n}$, for which m and n are coprime with the prime number p, consider the p-adic absolute value $|x|_p = p^{-r}$ on $\mathbb{Q}$. It is easy to check that $|\cdot|$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $|\cdot|$, which is denoted by $\mathbb{Q}_p$, is said to be a *p-adic number field.* Note that, if $p > 2$, then $|2^n| = 1$, for all integers n.

Throughout this paper, we consider that $\mathbb{E}$ and $\mathbb{F}$ are a non-Archimedean field and a complete non-Archimedean field, respectively. In the sequel, for a non-Archimdean field $\mathbb{E}$, we put $\mathbb{E}^* = \mathbb{E}\backslash\{0\}$. For simplicity, we define the difference operators D_1q, $D_2s : \mathbb{E}^* \times \mathbb{E}^* \longrightarrow \mathbb{F}$ by

$$\begin{aligned} D_1q(x,y) &= q(2x+y) + q(2x-y) \\ &\quad - \frac{4q(x)q(y)\left[16q(y) + 40q(x)^{\frac{2}{5}}q(y)^{\frac{3}{5}} + 5q(x)^{\frac{4}{5}}q(y)^{\frac{1}{5}}\right]}{\left[4q(y)^{\frac{2}{5}} - q(x)^{\frac{2}{5}}\right]^5} \end{aligned}$$

and

$$\begin{aligned} D_2s(x,y) &= s(2x+y) + s(2x-y) \\ &\quad - \frac{2s(x)s(y)\left[s(x) + 60s(x)^{\frac{2}{3}}s(y)^{\frac{1}{3}} + 240s(x)^{\frac{1}{3}} + 64s(y)\right]}{\left[4s(y)^{\frac{1}{3}} - s(x)^{\frac{1}{3}}\right]^6}, \end{aligned}$$

for all $x, y \in \mathbb{E}^*$.

3. STABILITY OF EQUATIONS (2) AND (3) BOUNDED BY A GENERAL CONTROL FUNCTION

In this section, we investigate the stability of equations (2) and (3) in a non-Archimedean field whose upper bound is controlled by a general function. We also prove the stability results pertaining to Hyers-Ulam stability, Hyers-Ulam-Rassias stability and Ulam-Găvruta-Rassias stability controlled by product-sum of powers of norms and mixed product-sum of powers of norms.

Definition 3.1. A map $q : \mathbb{E}^* \longrightarrow \mathbb{F}$ is called *reciprocal-quintic*, if q satisfies the equation (2). Also, a map $s : \mathbb{E}^* \longrightarrow \mathbb{F}$ is called *reciprocal-sextic*, if s satisfies the equation (3).

Remark 3.2. We note that, according to the discussion after Definition 1 in [4], there is no problem in the definitions of the functional equations (2) and (3).

Theorem 3.3. *Let $k \in \{1, -1\}$ be fixed and let $\psi : \mathbb{E}^* \times \mathbb{E}^* \longrightarrow \mathbb{F}$ be a map such that*

$$\lim_{n\to\infty} \left|\frac{1}{243}\right|^{kn} \psi\left(\frac{x}{3^{kn+\frac{k+1}{2}}}, \frac{y}{3^{kn+\frac{k+1}{2}}}\right) = 0, \tag{4}$$

for all $x, y \in \mathbb{E}^*$. Suppose that $q : \mathbb{E}^* \longrightarrow \mathbb{F}$ is a map satisfying the inequality

$$|D_1q(x,y)| \leq \psi(x,y), \tag{5}$$

for all $x, y \in \mathbb{E}^*$. Then there exists a unique reciprocal-quintic map $Q : \mathbb{E}^* \longrightarrow \mathbb{F}$ such that

(6)

$$|q(x) - Q(x)| \leq \max\left\{\left|\frac{1}{243}\right|^{jk+\frac{k-1}{2}} \psi\left(\frac{x}{3^{jk+\frac{k+1}{2}}}, \frac{x}{3^{jk+\frac{k+1}{2}}}\right) : j \in \mathbb{N}\cup\{0\}\right\},$$

for all $x \in \mathbb{E}^*$.

Proof. Taking (x, y) to be (x, x) in (5), we obtain

$$\left|q(x) - \frac{1}{243^k} q\left(\frac{x}{3^k}\right)\right| \leq |243|^{\frac{|k-1|}{2}} \psi\left(\frac{x}{3^{\frac{k+1}{2}}}, \frac{x}{3^{\frac{k+1}{2}}}\right), \tag{7}$$

for all $x \in \mathbb{X}^*$. Taking x to be $\frac{x}{3^{kn}}$ in (8) and multiplying the resultant by $\left|\frac{1}{243}\right|^{kn}$, we have

$$\begin{aligned}
&\left|\frac{1}{243^{kn}} q\left(\frac{x}{3^{kn}}\right) - \frac{1}{243^{(n+1)k}} q\left(\frac{x}{3^{(n+1)k}}\right)\right| \\
&\leq \left|\frac{1}{243}\right|^{kn+\frac{k-1}{2}} \psi\left(\frac{x}{3^{kn+\frac{k+1}{2}}}, \frac{x}{3^{kn+\frac{k+1}{2}}}\right),
\end{aligned} \tag{8}$$

for all $x \in \mathbb{E}^*$. It follows from relations (4) and (8) that the sequence $\left\{\frac{1}{243^{kn}} q\left(\frac{x}{3^{kn}}\right)\right\}$ is Cauchy. Since $\mathbb{F}$ is complete, this sequence converges to a map $Q : \mathbb{E}^* \longrightarrow \mathbb{F}$, which is defined by

$$Q(x) = \lim_{n\to\infty} \frac{1}{243^{kn}} q\left(\frac{x}{3^{kn}}\right). \tag{9}$$

On the other hand, for each $x \in \mathbb{E}^*$ and non-negative integers n, we have

(10)

$$\begin{aligned}
\left|\frac{1}{243^{kn}} q\left(\frac{x}{3^{kn}}\right) - q(x)\right| &= \left|\sum_{j=0}^{n-1} \left\{\frac{1}{243^{(j+1)k}} q\left(\frac{x}{3^{(j+1)k}}\right) - \frac{1}{243^{jk}} q\left(\frac{x}{3^{jk}}\right)\right\}\right| \\
&\leq \max\left\{\left|\frac{1}{243^{(j+1)k}} q\left(\frac{x}{3^{(j+1)k}}\right) - \frac{1}{243^{jk}} q\left(\frac{x}{3^{jk}}\right)\right| : 0 \leq j < n\right\} \\
&\leq \max\left\{\left|\frac{1}{243}\right|^{jk+\frac{k-1}{2}} \psi\left(\frac{x}{3^{jk+\frac{k+1}{2}}}, \frac{x}{3^{jk+\frac{k+1}{2}}}\right) : 0 \leq j < n\right\}.
\end{aligned}$$

Applying (9) and letting $n \to \infty$ in (10), we find that the inequality (6) holds. Once more, by using (4), (5) and (9), for every $x, y \in \mathbb{E}^*$, we have

$$\begin{aligned}
|D_1 q(x, y)| &= \lim_{n\to\infty} \left|\frac{1}{243}\right|^{kn} \left|D_1 q\left(\frac{x}{3^{kn}}, \frac{y}{3^{kn}}\right)\right| \\
&\leq \lim_{n\to\infty} \left|\frac{1}{243}\right|^{kn} \psi\left(\frac{x}{3^{kn}}, \frac{y}{3^{kn}}\right) = 0.
\end{aligned}$$

Thus, the map Q satisfies (2) and hence it is a reciprocal-quintic map. In order to prove the uniqueness of Q, let us consider $Q' : \mathbb{E}^* \longrightarrow \mathbb{F}$ as another reciprocal-quintic map satisfying (6). Then

$$
\begin{aligned}
\left|Q(x) - Q'(x)\right| &= \lim_{m\to\infty} \left|\frac{1}{243}\right|^{km} \left|Q\left(\frac{x}{3^{km}}x\right) - Q'\left(\frac{x}{3^{km}}\right)\right| \\
&\leq \lim_{m\to\infty} \left|\frac{1}{243}\right|^{km} \max\left\{\left|Q\left(\frac{x}{3^{km}}\right) - q\left(\frac{x}{3^{km}}\right)\right|, \left|q\left(\frac{x}{3^{km}}\right) - Q'\left(\frac{x}{3^{km}}\right)\right|\right\} \\
&\leq \lim_{m\to\infty}\lim_{n\to\infty} \max\Big\{\max\Big\{ \left|\frac{1}{243}\right|^{(j+m)l+\frac{k-1}{2}} \psi\left(\frac{x}{3^{(j+m)l+\frac{k+1}{2}}}, \frac{x}{3^{(j+m)l+\frac{k+1}{2}}}\right) : \\
&\qquad m \leq j \leq n+m\Big\}\Big\} = 0,
\end{aligned}
$$

for all $x \in \mathbb{E}^*$, which shows that Q is unique. This completes the proof. □

Here and subsequently, we assume that $|2| < 1$. The following corollaries are direct consequences of Theorem 3.3 associated with the stability of equation (2).

COROLLARY 3.4. *Let $\theta > 0$ be a constant. If $q : \mathbb{E}^* \longrightarrow \mathbb{F}$ satisfies $|D_1 q(x,y)| \leq \theta$, for all $x, y \in \mathbb{E}^*$, then there exists a unique reciprocal-quintic map $Q : \mathbb{E}^* \longrightarrow \mathbb{F}$ satisfying (2) and $|q(x) - Q(x)| \leq \theta$, for all $x \in \mathbb{E}^*$.*

Proof. Letting $\psi(x,y) = \theta$, for the case $k = -1$ in Theorem 3.3, we arrive at the required result. □

COROLLARY 3.5. *Let $\theta \geq 0$ and $p \neq -5$ be fixed constants. If $q : \mathbb{E}^* \longrightarrow \mathbb{F}$ satisfies $|D_1 q(x,y)| \leq \theta\left(|x|^p + |y|^p\right)$, for all $x, y \in \mathbb{E}^*$, then there exists a unique reciprocal-quintic map $Q : \mathbb{E}^* \longrightarrow \mathbb{F}$ satisfying (2) and*

$$
|q(x) - Q(x)| \leq \begin{cases} \frac{|2|\theta}{|3|^p} |x|^p, & p > -5, \\ |2|\theta|3|^5 |x|^p, & p < -5, \end{cases}
$$

for all $x \in \mathbb{E}^*$.

Proof. Considering $\psi(x,y) = \theta\left(|x|^p + |y|^p\right)$ in Theorem 3.3, the desired result follows directly. □

COROLLARY 3.6. *Let $q : \mathbb{E}^* \longrightarrow \mathbb{F}$ be a map and let the real numbers $r, s, p = r + s \neq -5$ and $\theta \geq 0$ be such that $|D_1 q(x,y)| \leq \theta |x|^r |y|^s$, for all x, $y \in \mathbb{E}^*$. Then there exists a unique reciprocal-quintic map $Q : \mathbb{E}^* \longrightarrow \mathbb{F}$ satisfying (2) and*

$$
|q(x) - Q(x)| \leq \begin{cases} \frac{\theta}{|3|^p} |x|^p, & p > -5, \\ \theta|3|^5| |x|^p, & p < -5, \end{cases}
$$

for all $x \in \mathbb{E}^*$.

Proof. Assigning $\psi(x,y) = \theta |x|^r |y|^s$, for all $x, y \in \mathbb{E}^*$, in Theorem 3.3, we finish the proof. □

Corollary 3.7. *Let $\theta \geq 0$ and $p \neq -5$ be real numbers and $q : \mathbb{E}^* \longrightarrow \mathbb{F}$ be a map satisfying the functional inequality*

$$|D_1 q(x,y)| \leq \theta \left(|x|^{\frac{p}{2}} |y|^{\frac{p}{2}} + (|x|^p + |y|^p) \right),$$

for all x, $y \in \mathbb{E}^*$. Then there exists a unique reciprocal-quintic map $Q : \mathbb{E}^* \longrightarrow \mathbb{F}$ satisfying (2) and

$$|q(x) - Q(x)| \leq \begin{cases} \frac{|3|\theta}{|3|^p} |x|^p, & p > -5, \\ |3|\theta |3|^5 |x|^p, & p < -5, \end{cases}$$

for all $x \in \mathbb{E}^*$.

Proof. One can obtain the result by choosing

$$\psi(x,y) = \theta\Big(\big|x\big|^{\frac{p}{2}} \big|y\big|^{\frac{p}{2}} + \big(\big|x\big|^p + \big|y\big|^p \big) \Big)$$

in Theorem 3.3. □

In analogy with Theorem 3.3, we state the following result for the stability of the functional equation (3). We include some parts of the proof for the sake of completeness.

Theorem 3.8. *Let $k \in \{1, -1\}$ be fixed and let $\chi : \mathbb{E}^* \times \mathbb{E}^* \longrightarrow \mathbb{F}$ be a map such that*

$$\lim_{n\to\infty} \left| \frac{1}{729} \right|^{kn} \chi \left(\frac{x}{3^{kn+\frac{k+1}{2}}}, \frac{y}{3^{kn+\frac{k+1}{2}}} \right) = 0, \tag{11}$$

for all $x, y \in \mathbb{E}^*$. Suppose that $s : \mathbb{E}^* \longrightarrow \mathbb{F}$ is a map satisfying the inequality

$$|D_2 s(x,y)| \leq \chi(x,y), \tag{12}$$

for all $x, y \in \mathbb{E}^*$. Then there exists a unique reciprocal-sextic map $S : \mathbb{E}^* \longrightarrow \mathbb{F}$ such that

$$|s(x) - S(x)| \leq \max \left\{ \left| \frac{1}{729} \right|^{jk+\frac{k-1}{2}} \chi \left(\frac{x}{3^{jk+\frac{k+1}{2}}}, \frac{x}{3^{jk+\frac{k+1}{2}}} \right) : j \in \mathbb{N} \cup \{0\} \right\}, \tag{13}$$

for all $x \in \mathbb{E}^*$.

Proof. Choosing (x, y) to be (x, x) in (12), we get

$$\left| s(x) - \frac{1}{729^k} s\left(\frac{x}{3^k}\right) \right| \leq |729|^{\frac{|k-1|}{2}} \chi \left(\frac{x}{3^{\frac{k+1}{2}}}, \frac{x}{3^{\frac{k+1}{2}}} \right), \tag{14}$$

for all $x \in \mathbb{E}^*$. Taking x to be $\frac{x}{3^{kn}}$ in (14) and then multiplying by $\left|\frac{1}{729}\right|^{kn}$, we arrive at

$$\begin{aligned} &\left|\frac{1}{729^{kn}} s\left(\frac{x}{3^{kn}}\right) - \frac{1}{729^{(n+1)k}} s\left(\frac{x}{3^{(n+1)k}}\right)\right| \\ &\leq \left|\frac{1}{729}\right|^{kn+\frac{k-1}{2}} \chi\left(\frac{x}{3^{kn+\frac{k+1}{2}}}, \frac{x}{3^{kn+\frac{k+1}{2}}}\right), \end{aligned} \tag{15}$$

for all $x \in \mathbb{E}^*$. Relations (11) and (15) imply that $\left\{\frac{1}{729^{kn}} s\left(\frac{x}{3^{kn}}\right)\right\}$ is a Cauchy sequence. Since $\mathbb{F}$ is complete, there exists a map $S : \mathbb{E}^* \longrightarrow \mathbb{F}$ such that

$$S(x) = \lim_{n\to\infty} \frac{1}{729^{kn}} s\left(\frac{x}{3^{kn}}\right), \tag{16}$$

for all $x \in \mathbb{E}^*$. The remaining part of the proof is similar to the corresponding one given in the proof of Theorem 3.3. □

The following corollaries point out some immediate effects of the stability of the functional equation (3), which is used in Theorem 3.8.

Corollary 3.9. *Let $\eta > 0$. Suppose that $s : \mathbb{E}^* \longrightarrow \mathbb{F}$ satisfies $|D_2 s(x, y)| \leq \eta$, for all $x, y \in \mathbb{E}^*$. Then there exists a unique reciprocal-sextic map $S : \mathbb{E}^* \longrightarrow \mathbb{F}$ satisfying (3) and $|s(x) - S(x)| \leq \eta$ for all $x \in \mathbb{E}^*$.*

Proof. It is sufficient to set $\chi(x, y) = \eta$ in Theorem 3.8, when $k = -1$. □

Corollary 3.10. *Let $\eta \geq 0$ and $\beta \neq -6$ be fixed. If $s : \mathbb{E}^* \longrightarrow \mathbb{F}$ satisfies $|D_2 s(x, y)| \leq \eta\left(|x|^\beta + |y|^\beta\right)$, for all $x, y \in \mathbb{E}^*$, then there exists a unique reciprocal-sextic map $S : \mathbb{E}^* \longrightarrow \mathbb{F}$ satisfying (3) and*

$$|s(x) - S(x)| \leq \begin{cases} \frac{|2|\eta}{|3|^\beta} |x|^\beta, & \beta > -6, \\ |2|\eta|3|^6 |x|^\beta, & \beta < -6, \end{cases}$$

for all $x \in \mathbb{E}^*$.

Proof. Allowing $\chi(x, y) = \eta\left(|x|^\beta + |y|^\beta\right)$, for all $x, y \in \mathbb{E}^*$, in Theorem 3.8, we arrive at the desired result. □

Corollary 3.11. *Let $s : \mathbb{E}^* \longrightarrow \mathbb{F}$ be a map and assume that there exist real numbers $p, q, \beta = p + q \neq -6$ and $\eta \geq 0$ such that*

$$|D_2 s(x, y)| \leq \eta\, |x|^p\, |y|^q,$$

for all $x, y \in \mathbb{E}^*$. Then there exists a unique reciprocal-sextic map $S : \mathbb{E}^* \longrightarrow \mathbb{F}$ satisfying (3) and

$$|s(x) - S(x)| \leq \begin{cases} \frac{\eta}{|3|^\beta} |x|^\beta, & \beta > -6, \\ \eta|3|^6 |x|^\beta, & \beta < -6, \end{cases}$$

for all $x \in \mathbb{E}^*$.

Proof. The result follows by taking $\chi(x,y) = \eta\,|x|^p\,|y|^q$, for all $x, y \in \mathbb{E}^*$, in Theorem 3.8. $\square$

Corollary 3.12. *Let $\eta \geq 0$ and $\beta \neq -6$ be real numbers and $s : \mathbb{E}^* \longrightarrow \mathbb{F}$ be a map satisfying the functional inequality*

$$|D_2 s(x,y)| \leq \eta \left(|x|^{\frac{\beta}{2}}\,|y|^{\frac{\beta}{2}} + \left(|x|^{\beta} + |y|^{\beta} \right) \right),$$

for all $x, y \in \mathbb{E}^*$. Then there exists a unique reciprocal-sextic map $S : \mathbb{E}^* \longrightarrow \mathbb{F}$ satisfying (3) *and*

$$|s(x) - S(x)| \leq \begin{cases} \frac{|3|\eta}{|3|^{\beta}}\,|x|^{\beta}, & \beta > -6, \\ |3|\eta|3|^6\,|x|^{\beta}, & \beta < -6, \end{cases}$$

for all $x \in \mathbb{E}^*$.

Proof. It is easy to obtain the result by taking

$$\chi(x,y) = \eta \left(|x|^{\frac{\beta}{2}}\,|y|^{\frac{\beta}{2}} + \left(|x|^{\beta} + |y|^{\beta} \right) \right)$$

in Theorem 3.8. $\square$

We close the paper with two exmples. In fact, by using the idea of the familiar counter-example presented by Gajda [9], we obtain related examples, for $p = -5$ in Corollary 3.5 and for $\beta = -6$ in Corollary 3.10, on $\mathbb{R}$ with the usual $|\cdot|$. Note that $(\mathbb{R}, |\cdot|)$ is an Archimedean field.

Example 3.13. Consider the function

$$\zeta(x) = \begin{cases} \frac{\sigma}{x^5}, & \text{for } x \in (1,\infty), \\ \sigma, & \text{otherwise}, \end{cases} \tag{17}$$

where $\zeta : \mathbb{R}^* \longrightarrow \mathbb{R}$. Let $f : \mathbb{R}^* \longrightarrow \mathbb{R}$ be defined by

$$f(x) = \sum_{n=0}^{\infty} 243^{-n} \zeta(3^{-n} x), \tag{18}$$

for all $x \in \mathbb{R}$. Let the function $f : \mathbb{R}^* \longrightarrow \mathbb{R}$ defined in (18) satisfy the functional inequality

$$|D_1 f(x,y)| \leq \frac{365\sigma}{121} \left(|x|^{-5} + |y|^{-5} \right), \tag{19}$$

for all $x, y \in \mathbb{R}^*$. We shall show that there do not exist a reciprocal-quintic map $q : \mathbb{R}^* \longrightarrow \mathbb{R}$ and a constant $\rho > 0$ such that

$$|f(x) - q(x)| \leq \rho\,|x|^{-5}, \tag{20}$$

for all $x \in \mathbb{R}^*$. For this, let us first prove that f satisfies (19). By some computations, we have

$$|f(x)| = \left| \sum_{n=0}^{\infty} 243^{-n} \zeta(3^{-n} x) \right| \leq \sum_{n=0}^{\infty} \frac{\sigma}{243^n} = \frac{243\sigma}{242}.$$

So, we have that f is bounded by $\frac{243\sigma}{242}$ on $\mathbb{R}$. If $|x|^{-5}+|y|^{-5}\geq 1$, then the left hand side of (19) is less than $\frac{365\sigma}{121}$. Now, suppose that $0<|x|^{-5}+|y|^{-5}<1$. Hence, there exists a positive integer k such that

$$\frac{1}{243^{k+1}}\leq |x|^{-5}+|y|^{-5}<\frac{1}{243^k}. \tag{21}$$

Thus, the relation (21) implies $243^k\left(|x|^{-5}+|y|^{-5}\right)<1$, or, equivalently, $243^k x^{-5}<1, 243^k y^{-5}<1$. So $\frac{x^5}{243^k}>1$, $\frac{y^5}{243^k}>1$. The last inequalities imply that $\frac{x^5}{243^{k-1}}>243>1$, $\frac{y^5}{243^{k-1}}>243>1$ and consequently

$$\frac{1}{3^{k-1}}(x)>1, \frac{1}{3^{k-1}}(y)>1, \frac{1}{3^{k-1}}(2x+y)>1, \frac{1}{3^{k-1}}(2x-y)>1.$$

Therefore, for each value $n=0,1,2,\ldots,k-1$, we obtain

$$\frac{1}{3^n}(x)>1, \frac{1}{3^n}(y)>1, \frac{1}{3^n}(2x+y)>1, \frac{1}{3^n}(2x-y)>1$$

and thus $D_1\zeta(3^{-n}x,3^{-n}y)=0$, for $n=0,1,2,\ldots,k-1$. Using (17) and the definition of f, we obtain

$$\begin{aligned}|D_1f(x,y)| &\leq \sum_{n=k}^{\infty}\frac{\sigma}{243^n}+\sum_{n=k}^{\infty}\frac{\sigma}{243^n}+\frac{244}{243}\sum_{n=k}^{\infty}\frac{\sigma}{243^n}\\ &\leq \frac{730\sigma}{243}\frac{1}{243^k}\left(1-\frac{1}{243}\right)^{-1}\leq \frac{730\sigma}{242}\frac{1}{243^k}\leq\frac{730\sigma}{242}\frac{1}{243^{k+1}}\\ &\leq \frac{365\sigma}{121}\left(|x|^{-5}+|y|^{-5}\right),\end{aligned}$$

for all $x,y\in\mathbb{R}^*$. This means that inequality (19) holds. We claim that the reciprocal-quintic functional equation (2) is not stable for $p=-5$ in Corollary 3.5. Assume that there exists a reciprocal-quintic map $q:\mathbb{R}^*\longrightarrow\mathbb{R}$ satisfying (20). Then we have

$$|f(x)|\leq(\rho+1)|x|^{-5}. \tag{22}$$

However, we can pick a positive integer m with $m\sigma>\rho+1$. If $x\in\left(1,3^{m-1}\right)$, then $3^{-n}x\in(1,\infty)$, for all $n=0,1,2,\ldots,m-1$, and thus

$$|f(x)|=\sum_{n=0}^{\infty}\frac{\zeta(3^{-n}x)}{243^n}\geq\sum_{n=0}^{m-1}\frac{\frac{243^n\sigma}{x^5}}{243^n}=\frac{m\sigma}{x^5}>(\rho+1)x^{-5},$$

which contradicts (22). Therefore, the reciprocal-quintic functional equation (2) is not stable for $p=-5$ in Corollary 3.5.

Example 3.14. Define the function $\xi:\mathbb{R}^*\longrightarrow\mathbb{R}$ via

$$\xi(x)=\begin{cases}\frac{\omega}{x^6}, & \text{for } x\in(1,\infty),\\ \omega, & \text{otherwise.}\end{cases} \tag{23}$$

Let $g : \mathbb{R}^* \longrightarrow \mathbb{R}$ be defined by

$$g(x) = \sum_{n=0}^{\infty} 729^{-n}\xi(3^{-n}x), \tag{24}$$

for all $x \in \mathbb{R}$. Assume that the function g satisfies the functional inequality

$$|D_2 g(x,y)| \leq \frac{547\omega}{182}\left(|x|^{-6} + |y|^{-6}\right), \tag{25}$$

for all $x, y \in \mathbb{R}^*$. Then there do not exist a reciprocal-sextic map $s : \mathbb{R}^* \longrightarrow \mathbb{R}$ and a constant $\gamma > 0$ such that

$$|g(x) - s(x)| \leq \gamma\, |x|^{-6}, \tag{26}$$

for all $x \in \mathbb{R}^*$. To prove this, we note that

$$|g(x)| = \left|\sum_{n=0}^{\infty} 729^{-n}\xi(3^{-n}x)\right| \leq \sum_{n=0}^{\infty} \frac{\omega}{729^n} = \frac{729\omega}{728}.$$

Hence, we see that g is bounded by $\frac{729\omega}{728}$ on $\mathbb{R}$. If $|x|^{-6} + |y|^{-6} \geq 1$, then the left hand side of (25) is less than $\frac{547\omega}{182}$. Now suppose that $0 < |x|^{-6} + |y|^{-6} < 1$. Then there exists a positive integer m such that

$$\frac{1}{729^{m+1}} \leq |x|^{-6} + |y|^{-6} < \frac{1}{729^m}. \tag{27}$$

By arguments similar to those used in Example 3.13, the relation $|x|^{-6} + |y|^{-6} < \frac{1}{729^m}$ implies

$$\frac{1}{3^{m-1}}(x) > 1,\ \frac{1}{3^{m-1}}(y) > 1,\ \frac{1}{3^{m-1}}(2x+y) > 1,\ \frac{1}{3^{m-1}}(2x-y) > 1.$$

Thus, for every $n = 0, 1, 2, \dots, m-1$, we obtain

$$\frac{1}{3^n}(x) > 1,\ \frac{1}{3^n}(y) > 1,\ \frac{1}{3^n}(2x+y) > 1,\ \frac{1}{3^n}(2x-y) > 1$$

and thus $D_2\xi(3^{-n}x, 3^{-n}y) = 0$, for $n = 0, 1, 2, \dots, m-1$. Applying (23) and the definition of g, we find

$$\begin{aligned}
|D_2 g(x,y)| &\leq \sum_{n=m}^{\infty} \frac{\omega}{729^n} + \sum_{n=m}^{\infty} \frac{\omega}{729^n} + \frac{730}{729}\sum_{n=k}^{\infty} \frac{\omega}{729^n} \\
&\leq 2\omega \sum_{n=m}^{\infty} \frac{1}{729^n} + \frac{730\omega}{729}\sum_{n=m}^{\infty} \frac{1}{729^n} \\
&\leq \frac{2188\omega}{729}\frac{1}{729^m}\left(1 - \frac{1}{729}\right)^{-1} \\
&\leq \frac{2188\omega}{728}\frac{1}{729^m} \leq \frac{2188\omega}{728}\frac{1}{729^{m+1}} \\
&\leq \frac{547\omega}{182}\left(|x|^{-6} + |y|^{-6}\right),
\end{aligned}$$

for all $x, y \in \mathbb{R}^*$. So the inequality (25) holds. Assume that there exists a reciprocal-sextic map $q : \mathbb{R}^* \longrightarrow \mathbb{R}$ satisfying (26). Then

$$|g(x)| \leq (\gamma + 1)|x|^{-6}. \tag{28}$$

In other words, we can chose a positive integer k with $k\omega > \gamma + 1$. If $x \in \big(1, 3^{k-1}\big)$, then $3^{-n}x \in (1, \infty)$, for all $n = 0, 1, 2, \ldots, k-1$, and so

$$|g(x)| = \sum_{n=0}^{\infty} \frac{\xi(3^{-n}x)}{729^n} \geq \sum_{n=0}^{k-1} \frac{\frac{729^n\omega}{x^6}}{729^n} = \frac{k\omega}{x^6} > (\gamma + 1)x^{-6},$$

which contradicts (28). Therefore, the reciprocal-sextic functional equation (3) is not stable for $\beta = -6$ in Corollary 3.10.

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