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NECESSARY AND SUFFICIENT CONDITIONS FOR OSCILLATION OF NONLINEAR NEUTRAL FIRST ORDER DIFFERENTIAL EQUATIONS WITH SEVERAL DELAYS

SHYAM SUNDAR SANTRA

Abstract. In this work, necessary and sufficient conditions for oscillations of the solutions of a class of nonlinear first-order neutral differential equations with several delays of the form

$$\big(x(t)+r(t)x(t-\tau)\big)'+\sum_{i=1}^{m}\phi_i(t)H\big(x(t-\sigma_i)\big)=0$$

are established under various ranges of $r(t)$. Finally, two illustrating examples are presented to show the feasibility and the effectiveness of the main results.

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1. INTRODUCTION

Consider a class of first-order nonlinear neutral delay differential equations of the form

$$\big(x(t)+r(t)x(t-\tau)\big)'+\sum_{i=1}^{m}\phi_i(t)H\big(x(t-\sigma_i)\big)=0, \tag{1}$$

where

$$\tau,\sigma_i\in\mathbb{R}_+=(0,+\infty),\ i=1,2...,m,\ r\in C([0,\infty),\mathbb{R}),\ \phi\in C(\mathbb{R}_+,\mathbb{R}_+),$$

and H is nondecreasing with

$$H\in C(\mathbb{R},\mathbb{R})\ \text{ with }\ uH(u)>0\ \text{ for }\ u\neq 0.$$

The purpose of this work is to establish necessary and sufficient conditions for the oscillations of (1) under different ranges of $r(t)$. The motivation of the present paper has come from the work [13]. In [13], Santra has considered

$$\big(x(t)+p(t)x(t-\tau)\big)'+\sum_{i=1}^{m}q_i(t)H\big(x(t-\sigma_i)\big)=f(t), \tag{2}$$

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and

$$(3) \qquad \big(x(t)+p(t)x(t-\tau)\big)' + \sum_{i=1}^{m} q_i(t)H\big(x(t-\sigma_i)\big) = 0.$$

He has established sufficient conditions for the oscillation and the non-oscillation of the solutions of (2) and (3) for any $|p(t)| < +\infty$, when H is linear, sublinear and superlinear. In this direction, we refer to some related works ([1, 2, 3], [11, 12, 13], [15], [17]) and the references therein.

In the last decade, the study of the asymptotic and oscillatory behavior of solutions of neutral differential equations is considered a major area of research. This is because of the development in science and technology and the challenges that the new classes of such equations provide in these applied areas. The delay differential equations play an important role in modeling virtually every physical, technical, or biological process, from celestial motion, to bridge design, to interactions between neurons.

Definition 1.1. By a *solution of the differential equation* (1) we understand a function $x \in C([-\rho,\infty),\mathbb{R})$ such that $x(t)+r(t)x(t-\tau)$ is once continuously differentiable and (1) is satisfied for $t \ge 0$, where $\rho = \max\{\tau,\sigma_i\}$ for $i = 1,\dots,m$, and $\sup\{|x(t)| : t \ge t_0\} > 0$ for every $t_0 \ge 0$. A solution of (1) is said to be *oscillatory* if it has arbitrarily large zeros; otherwise, it is called *non-oscillatory*.

2. NECESSARY AND SUFFICIENT CONDITIONS FOR OSCILLATIONS

In this section, we establish a necessary and sufficient condition for the asymptotic behavior of solutions of a class of first order nonlinear neutral differential equations of the form (1). We need the following lemma.

Lemma 2.1. ([7]) *Let $r, x, z \in C([0,\infty),\mathbb{R})$ be such that $z(t) = x(t) + r(t)x(t-\tau)$, $t \ge \tau > 0$, $x(t) > 0$, $t \ge t_1 > \tau$, $\liminf_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} z(t) = L$ exist. Let $r(t)$ satisfy one of the following conditions:*

i) $0 \le r_1 \le r(t) \le r_2 < 1$,

ii) $1 < r_3 \le r(t) \le r_4 < \infty$,

iii) $-\infty < -r_5 \le r(t) \le 0$,

where $r_i > 0$, $1 \le i \le 5$. Then $L = 0$.

Remark 2.2. If, in the above lemma, $x(t) < 0$, for all $t \ge \tau > 0$, $\limsup_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} z(t) = L \in \mathbb{R}$ exists, then $L = 0$.

Theorem 2.3. *Let $0 \le r_1 \le r(t) \le r_2 < 1$, $t \in \mathbb{R}_+$. Let H be Lipschitzian on the intervals of the form $[a,b]$, $0 < a < b < \infty$. Then every solution of (1) converges to zero as $t \to \infty$ if and only if*

(A_1) $\int_0^\infty \sum_{i=1}^m \phi_i(t)\mathrm{d}t = \infty$.

Proof. Suppose that (A_1) holds. Let $x(t)$ be a solution of (1) on $[t_x,\infty]$, $t_x \ge 0$. If $x(t)$ is oscillatory, then there is nothing to prove. Suppose the solution satisfies $x(t) > 0$, for $t \geq t_x$. Set

$$z(t) = x(t) + r(t)x(t-\tau),\ t \geq t_0. \tag{4}$$

From (1), it follows that

$$z'(t) = -\sum_{i=1}^{m} \phi_i(t) H\big(x(t-\sigma)\big) < 0, \tag{5}$$

holds and hence $z(t)$ is a decreasing function, for $t \geq t_1 > t_0 + \rho$. Since $z(t) > 0$, for $t \geq t_2$. So, $\lim_{t\to\infty} z(t)$ exists. Consequently, $z(t) > x(t)$ implies that $x(t)$ is bounded. Our objective is to show that $\lim_{t\to\infty} x(t) = 0$. For this, we need to show that $\liminf_{t\to\infty} x(t) = 0$. If $\liminf_{t\to\infty} x(t) \neq 0$, then there exists $t_3 > t_2$ and $\beta > 0$ such that $x(t-\sigma) \geq \beta > 0$ for $t \geq t_3$. Ultimately,

$$\int_{t_3}^{t} \sum_{i=1}^{m} \phi_i(s) H\big(x(s-\sigma)\big) \mathrm{d}s \geq H(\beta) \left[\int_{t_3}^{t} \sum_{i=1}^{m} \phi_i(s) \mathrm{d}s\right] \to +\infty, \text{ as } t \to \infty,$$

due to (A_1). On the other hand, we integrate (5) from t_3 to $t(> t_3)$ to obtain

$$\int_{t_3}^{t} \sum_{i=1}^{m} \phi_i(s) H\big(x(s-\sigma)\big) \mathrm{d}s = -\big[z(s)\big]_{t_3}^{t} < \infty, \text{ as } t \to \infty,$$

which is a contradiction. Therefore, $\liminf_{t\to\infty} x(t) = 0$. Consequently, $\lim_{t\to\infty} z(t) = 0$, due to Lemma 2.1. As a result,

$$0 = \lim_{t\to\infty} z(t) = \limsup_{t\to\infty} \big(x(t) + r(t)x(t-\tau)\big) \geq \limsup_{t\to\infty} x(t)$$

implies that $\limsup_{t\to\infty} x(t) = 0$, that is $\lim_{t\to\infty} x(t) = 0$.

If $x(t) < 0$, for $t \geq t_0$, then we set $y(t) = -x(t)$, for $t \geq t_0$, in (1), we find

$$\big(y(t) + r(t)y(t-\tau)\big)' + \sum_{i=1}^{m} \phi_i(t) H\big(y(t-\sigma_i)\big) = 0,$$

and proceeding as above, we find the same contradiction. This completes the proof of the theorem.

Next, we suppose that

$$\int_0^\infty \sum_{i=1}^{m} \phi_i(t) \mathrm{d}t < \infty \tag{6}$$

and we need to show that the equation (1) admits a non-oscillatory solution which does not tend to zero as $t \to \infty$, when the limit exists. Suppose there exists $t_1 > 0$ such that

$$\int_{t_1}^\infty \sum_{i=1}^{m} \phi_i(s) \mathrm{d}s < \frac{1 - r_2}{10K},$$

where $K = \max\{K_1, H(1)\}$ and K_1 is the Lipschitz constant of H on $\left[\frac{2(1-r_2)}{5}, 1\right]$. For $t_2 > t_1$, we set $Y = BC([t_2, \infty), \mathbb{R})$, the space of real-valued bounded continuous functions on $[t_2, \infty)$. Clearly, Y is a Banach space with respect to the sup norm defined by

$$||y|| = \sup\{|y(t)| : t \geq t_2\}.$$

Define

$$S = \left\{ u \in Y : \ \frac{2(1-r_2)}{5} \leq u(t) \leq 1, \ t \geq t_2 \right\}.$$

Clearly, S is a closed and convex subspace of Y. Let $T : S \to S$ be defined by

$$Tx(t) = \begin{cases} Tx(t_2 + \rho), & t \in [t_2, t_2 + \rho] \\ -r(t)x(t-\tau) + \frac{2+3r_2}{5} + \int_t^\infty \sum_{i=1}^m \phi_i(s) H\big(x(s-\sigma)\big)\mathrm{d}s, & t \geq t_2 + \rho. \end{cases}$$

For every $x \in S$,

$$\begin{aligned} Tx(t) &\leq \frac{2+3r_2}{5} + H(1)\left[\int_t^\infty \sum_{i=1}^m \phi_i(s)\mathrm{d}s\right] \\ &< \frac{2+3r_2}{5} + \frac{1-r_2}{10} = \frac{1+r_2}{2} < 1 \end{aligned}$$

and

$$\begin{aligned} Tx(t) &\geq -r(t)x(t-\tau) + \frac{2+3r_2}{5} \\ &\geq -r_2 + \frac{2+3r_2}{5} = \frac{2(1-r_2)}{5} \end{aligned}$$

imply that $Tx \in S$. Now, for $y_1, y_2 \in S$,

$$\begin{aligned} |Ty_1(t) - Ty_2(t)| \leq {}& |r(t)||y_1(t-\tau) - y_2(t-\tau)| \\ &+ K_1 \int_t^\infty \sum_{i=1}^m \phi_i(s)|y_1(s-\sigma) - y_2(s-\sigma)|\mathrm{d}s, \end{aligned}$$

that is

$$\begin{aligned} |Ty_1(t) - Ty_2(t)| &\leq r_2||y_1 - y_2|| + K_1||y_1 - y_2|| \left[\int_t^\infty \sum_{i=1}^m \phi_i(s)\mathrm{d}s\right] \\ &< \left(r_2 + \frac{1-r_2}{10}\right)||y_1 - y_2||, \end{aligned}$$

which implies that

$$||Ty_1 - Ty_2|| \leq \mu||y_1 - y_2||,$$

and thus T is a contraction mapping, where $\mu = r_2 + \frac{1-r_2}{10} = \frac{1+9r_2}{10} < 1$. Since S is complete and T is a contraction on S, by Banach's fixed point theorem, T has a unique fixed point x in $\left[\frac{2(1-r_2)}{5}, 1\right]$. Hence $Tx = x$ and

$$x(t) = \begin{cases} x(t_2+\rho), & t \in [t_2, t_2+\rho] \\ -r(t)x(t-\tau) + \frac{2+3r_2}{5} + \int_t^\infty \sum_{i=1}^m \phi_i(s) H\big(x(s-\sigma)\big)\mathrm{d}s, & t \geq t_2+\rho \end{cases}$$

is a non-oscillatory solution of (1) on $\left[\frac{2(1-r_2)}{5}, 1\right]$ such that $\lim_{t\to\infty} x(t) \neq 0$. Therefore, (A_1) is necessary. This completes the proof of the theorem. □

THEOREM 2.4. *Let $1 < r_3 \leq r(t) \leq r_4 < \infty$, $t \in \mathbb{R}_+$ and $r_3^2 > r_4$. Suppose that H is Lipschitzian on the intervals of the form $[a, b]$, $0 < a < b < \infty$. Then every solution of* (1) *converges to zero as $t \to \infty$ if and only if (A_1) holds.*

Proof. The sufficient part is the same as in the proof of Theorem 2.3. For the necessary part, we suppose that (6) holds. It is possible to find $t_1 > 0$ such that

$$\int_{t_1}^\infty \sum_{i=1}^m \phi_i(t)\mathrm{d}t < \frac{r_3 - 1}{2K},$$

where $K = \max\{K_1, K_2\}$, K_1 is the Lipschitz constant of H on $[a, b]$ and $K_2 = G(b)$ such that

$$a = \frac{2\lambda({r_3}^2 - r_4) - r_4(r_3 - 1)}{2{r_3}^2 r_4}$$

$$b = \frac{r_3 - 1 + 2\lambda}{2r_3}, \qquad \lambda > \frac{r_4(r_3-1)}{2({r_3}^2 - r_4)} > 0.$$

Let $Y = BC([t_2, \infty), \mathbb{R})$ be the space of real-valued bounded continuous functions on $[t_2, \infty)$. Clearly, Y is a Banach space with respect to the sup norm defined by

$$||y|| = \sup\{|y(t)| : t \geq t_2\}.$$

Define

$$S = \{u \in Y : \ a \leq u(t) \leq b, \ t \geq t_2\}.$$

It is easy to verify that S is a closed convex subspace of Y. Let $T : S \to S$ be such that

$$Tx(t) =$$

$$\begin{cases} Tx(t_2+\rho), & t \in [t_2, t_2+\rho] \\ -\frac{x(t+\tau)}{r(t+\tau)} + \frac{\lambda}{r(t+\tau)} + \frac{1}{r(t+\tau)}\left[\int_{t+\tau}^\infty \sum_{i=1}^m \phi_i(s) H\big(x(s-\sigma)\big)\mathrm{d}s\right], & t \geq t_2+\rho. \end{cases}$$

For every $x \in S$,

$$\begin{aligned} Tx(t) &\leq \frac{H(b)}{r(t+\tau)} \left[\int_{t+\tau}^{\infty} \sum_{i=1}^{m} \phi_i(s) \mathrm{d}s \right] + \frac{\lambda}{r(t+\tau)} \\ &\leq \frac{1}{r_3} \left[\frac{r_3 - 1}{2} + \lambda \right] = b \end{aligned}$$

and

$$\begin{aligned} Tx(t) &\geq -\frac{x(t+\tau)}{r(t+\tau)} + \frac{\lambda}{r(t+\tau)} \\ &> -\frac{b}{r_3} + \frac{\lambda}{r_4} \\ &= -\frac{r_3 - 1 + 2\lambda}{2r_3^2} + \frac{\lambda}{r_4} \\ &= \frac{2\lambda({r_3}^2 - r_4) - r_4(r_3 - 1)}{2{r_3}^2 r_4} = a \end{aligned}$$

imply that $Tx \in S$. For $y_1, y_2 \in S$

$$\begin{aligned} |Ty_1y(t) - Ty_2(t)| \leq{}& \frac{1}{|r(t+\tau)|} |y_1(t+\tau) - y_2(t+\tau)| \\ &+ \frac{K}{|r(t+\tau)|} \left[\int_{t+\tau}^{\infty} \sum_{i=1}^{m} \phi_i(s) |y_1(s-\sigma) - y_2(s-\sigma)| \mathrm{d}s \right], \end{aligned}$$

that is

$$\begin{aligned} |Ty_1(t) - Ty_2(t)| &\leq \frac{1}{r_3} ||y_1 - y_2|| + \frac{K}{r_3} ||y_1 - y_2|| \left[\int_{t+\tau}^{\infty} \sum_{i=1}^{m} \phi_i(s) \mathrm{d}s \right] \\ &< \left(\frac{1}{r_3} + \frac{r_3 - 1}{2r_3} \right) ||y_1 - y_2||, \end{aligned}$$

which implies that

$$||Ty_1 - Ty_2|| \leq \mu ||y_1 - y_2||,$$

and thus T is a contraction, where $\mu = \left(\frac{1}{r_3} + \frac{r_3 - 1}{2r_3} \right) < 1$. Hence by Banach's fixed point theorem, T has a unique fixed point which is a non-oscillatory solution of (1) on $[a, b]$. Thus the proof of the theorem is complete. □

THEOREM 2.5. *Let* $-1 < -r_5 \leq r(t) \leq 0$, $t \in \mathbb{R}_+$, $r_5 > 0$. *Then every solution of* (1) *converges to zero as* $t \to \infty$ *if and only if* (A_1) *holds.*

Proof. Proceeding as in the proof of Theorem 2.3, we obtain (5). Hence, $z(t)$ is monotonic on $[t_2, \infty)$, $t_2 > t_1$. Let $z(t) > 0$ for $t \geq t_2$. So, $\lim_{t \to \infty} z(t)$ exists. Let $z(t) < 0$, for $t \geq t_2$. We claim that $x(t)$ is bounded. If not, there exists $\{\eta_n\}$ such that $\eta_n \to \infty$ as $n \to \infty$, $x(\eta_n) \to \infty$ as $n \to \infty$ and

$$x(\eta_n) = \max\{x(s) : t_2 \leq s \leq \eta_n\}.$$

Therefore,

$$\begin{aligned}z(\eta_n) &= x(\eta_n) + r(\eta_n)x(\eta_n - \tau)\\ &\geq (1 - r_5)x(\eta_n)\\ &\to +\infty, \ \ as \ \ n \to \infty,\end{aligned}$$

which is in contradiction with $z(t) > 0$. So, our claim holds. Consequently, $z(t) \leq x(t)$ implies that $\lim_{t\to\infty} z(t)$ exists. Hence, for any $z(t)$, $x(t)$ is bounded. Using the same type of argument as in the proof of Theorem 2.3, it is easy to show that $\liminf_{t\to\infty} x(t) = 0$ and by Lemma 2.1, $\lim_{t\to\infty} z(t) = 0$. Indeed,

$$\begin{aligned}0 = \lim_{t\to\infty} z(t) &= \limsup_{t\to\infty}\big(x(t) + r(t)x(t-\tau)\big)\\ &\geq \limsup_{t\to\infty} x(t) + \liminf_{t\to\infty}\big(-r_5 x(t-\tau)\big)\\ &= (1 - r_5)\limsup_{t\to\infty} x(t)\end{aligned}$$

implies that $\limsup_{t\to\infty} x(t) = 0$. The rest of the proof follows from Theorem 2.3.

Next, we suppose that (6) holds. Then there exist $t_1 > 0$ such that

$$\int_{t_1}^{\infty} \sum_{i=1}^{m} \phi_i(t)\mathrm{d}t < \frac{1-r_5}{5H(1)}, \ \ t \geq t_1.$$

For $t_2 > t_1$, let $Y = BC([t_2,\infty),\mathbb{R})$ be the space of all real-valued bounded continuous functions defined on $[t_2,\infty)$. Clearly, Y is a Banach space with respect to the sup norm defined by

$$||y|| = \sup\{|y(t)| : t \geq t_2\}.$$

Let $K = \{y \in Y : y(t) \geq 0, \ t \geq t_2\}$. Then, Y is a partially ordered Banach space (see p.30 in [7]). For $u, v \in Y$, we define $u \leq v$, if $u - v \in K$. Let

$$S = \left\{X \in Y : \ \frac{1-r_5}{5} \leq x(t) \leq 1, \ t \geq t_2\right\}.$$

If $x_0(t) = \frac{1-r_5}{5}$, then $x_0 \in S$ and $x_0 =$ g.l.b. of S. Further, if $\phi \subset S^* \subset S$, then

$$S^* = \left\{x \in Y : \ l_1 \leq x(t) \leq l_2, \ \frac{1-r_5}{5} \leq l_1, \ l_2 \leq 1\right\}.$$

Let $v_0(t) = l_2'$, $t \geq t_3$, where $l_2' = \sup\{l_2 : \frac{1-r_5}{5} \leq l_2 \leq 1\}$. Then $v_0 \in S$ and $v_0 =$ l.u.b. of S^*. For $t_3 = t_2 + \rho$, define $T : S \to S$ by

$$Tx(t) = \begin{cases} Tx(t_3), & t \in [t_2, t_3]\\ -r(t)x(t-\tau) + \frac{1-r_5}{5} + \int_t^{\infty}\sum_{i=1}^{m}\phi_i(s)H\big(x(s-\sigma)\big)\mathrm{d}s, t \geq t_3. \end{cases}$$

For every $x \in S$, $Tx(t) \geq \frac{1-r_5}{5}$ and

$$\begin{aligned} Tx(t) &\leq r_5 + \frac{1-r_5}{5} + H(1)\left[\int_t^\infty \sum_{i=1}^m \phi_i(s)\mathrm{d}s\right] \\ &< \frac{2+3r_5}{5} < 1 \end{aligned}$$

imply that $Tx \in S$. Now, for $x_1, x_2 \in S$, it is easy to verify that $x_1 \leq x_2$ implies that $Tx_1 \leq Tx_2$. Hence by the Knaster-Tarski fixed point theorem (see Theorem 1.7.3 in [7]), T has a unique fixed point such that $\lim_{t\to\infty} x(t) \neq 0$. This completes the proof of the theorem. $\square$

THEOREM 2.6. *Let $-\infty < -r_6 \leq r(t) \leq -r_7 < -1$, $t \in \mathbb{R}_+$ and $r_6, r_7 > 0$. Let H be Lipschitzian on the intervals of the form $[a,b]$, $0 < a < b < \infty$. Then every bounded solution of* (1) *converges to zero as $t \to \infty$ if and only if* (A_1) *holds.*

Proof. The proof of the theorem follows from the proof of the Theorem 2.4. For the necessary part, we need to mention the following inequality

$$\int_{t_1}^\infty \sum_{i=1}^m \phi_i(t)\mathrm{d}t < \frac{r_7-1}{2K},$$

where $K = \max\{K_1, K_2\}$, K_1 is the Lipschitz constant of H on $[a,b]$, $K_2 = H(b)$ such that

$$a = \frac{2\lambda r_7 - r_6(r_7-1)}{2r_6r_7}, \quad b = \frac{\lambda}{r_7-1}$$

for

$$\lambda > \frac{r_6(r_7-1)}{2r_7} > 0,$$

and

$$Tx(t) = \begin{cases} Tx(t_2+\rho), & t \in [t_2, t_2+\rho] \\ -\frac{x(t+\tau)}{r(t+\tau)} - \frac{\lambda}{r(t+\tau)} + \frac{1}{r(t+\tau)}\left[\int_{t+\tau}^\infty \sum_{i=1}^m \phi_i(s)H\big(x(s-\sigma)\big)\mathrm{d}s\right], \\ t \geq t_2+\rho. \end{cases}$$

This completes the proof of the theorem. $\square$

REMARK 2.7. In the above theorems, H could be linear, sublinear or superlinear.

REMARK 2.8. Lemma 2.1 does not include $r(t) \equiv 1$, for all t (see for e.g [7]). The present analysis does not allow the case $r(t) \equiv -1$, for all t. Hence, in our discussion, a necessary and sufficient condition is established, excluding $r(t) = \pm 1$, for all t. It seems that a different approach is necessary to study the case $r(t) = \pm 1$. However, in the following, the author succeeded to establish necessary and sufficient conditions for the case $r(t) = -1$.

THEOREM 2.9. *Let* $-\infty < -r_6 \leq r(t) \leq -r_7 \leq -1$, $r_6, r_7 > 0$, $t \in \mathbb{R}_+$ *and* $\tau > \sigma$. *Assume that*

(A_2) $H(uv) = H(u)H(v)$, $u, v \in \mathbb{R}$

and

(A_3) $\int_{\pm c}^{\pm\infty} \frac{\mathrm{d}x}{H(x)} < \infty$

hold. Then every solution of (1) *oscillates if and only if* (A_1) *holds.*

Proof. For the proofs of the necessary and sufficient parts of the theorem we refer to Theorem 2.6 and Theorem 3.2 in [13]. □

THEOREM 2.10. *Let* $-\infty < -r_6 \leq r(t) \leq -r_7 \leq -1$, $r_6, r_7 > 0$ *and* $t \in \mathbb{R}_+$. *Assume that* (A_2) *holds. Then every bounded solution of* (1) *oscillates if and only if* (A_1) *holds.*

Proof. For the proofs of the necessary and sufficient parts of the theorem we refer to Theorem 2.6 and Theorem 3.3 from [13]. □

THEOREM 2.11. *Let* $-1 < -r_5 \leq r(t) \leq 0$, $r_5 > 0$ *and* $t \in \mathbb{R}_+$. *Assume that* (A_2) *hold. Furthermore assume that*

(A_4) $\int_{c_1}^{\pm c_2} \frac{\mathrm{d}x}{H(x)} < \infty$

hold. Then every solution of the system (1) *oscillates if and only if* (A_1) *hold.*

Proof. For the proofs of the necessary and sufficient parts of the theorem we refer to Theorem 2.5 and Theorem 3.4 [13], respectively. □

3. EXAMPLES

EXAMPLE 3.1. Consider

$$\big(x(t) + e^{-\pi}x(t-\pi)\big)' + e^{-2\pi}x(t-2\pi) + e^{-3\pi}x(t-3\pi) = 0, \tag{7}$$

where $r(t) = e^{-\pi}$, $\phi_1(t) = e^{-2\pi}$, $\phi_2(t) = e^{-3\pi}$, $\tau = \pi$, $m = 2$, $\sigma_1 = 2\pi$, $\sigma_2 = 3\pi$ and $H(x) = x$. Clearly,

$$\int_0^\infty \big[\phi_1(t) + \phi_2(t)\big]\mathrm{d}t = \infty.$$

Hence, by Theorem 2.3, every solutions of (7) converges to zero as $t \to \infty$. Indeed, $x(t) = e^{-t}\sin t$ is such a solution of (7).

EXAMPLE 3.2. Consider

$$\big(x(t) + e^{-\pi}x(t-\pi)\big)' + e^{-6\pi}x^3(t-2\pi) + e^{-9\pi}x^3(t-3\pi) = 0, \tag{8}$$

where $r(t) = e^{-\pi}$, $\phi_1(t) = e^{-6\pi}$, $\phi_2(t) = e^{-9\pi}$, $\tau = \pi$, $m = 2$, $\sigma_1 = 2\pi$, $\sigma_2 = 3\pi$ and $H(x) = x^3$. Clearly, $\displaystyle\int_0^\infty [\phi_1(t) + \phi_2(t)]\mathrm{d}t = \infty$. Hence, by Theorem 2.3, every solution of (8) converges to zero as $t \to \infty$. Indeed, $x(t) = e^{-t}\sin t$ is such a solution of (8).

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Sambalpur University
Department of Mathematics
Sambalpur 768019, India
E-mail: `shyam01.math@gmail.com`