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PERIOD OF BALANCING NUMBERS MODULO PRODUCT OF CONSECUTIVE PELL AND PELL-LUCAS NUMBERS

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Abstract. The period of balancing numbers modulo m, denoted by $\pi(m)$, is the least positive integer t such that $\{B_t, B_{t+1}\} \equiv \{0, 1\} \pmod{m}$, where B_t denotes the t-th balancing number. In this article, the periods of balancing numbers modulo product of consecutive Pell and Pell-Lucas numbers are examined.

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Key words. Balancing numbers, Lucas-balancing numbers, Pell numbers, Pell-Lucas numbers, periodicity.

1. INTRODUCTION

It is well known that the Lucas sequence in the general form is defined by the binary recurrence

$$x_{n+1} = Ax_n + Bx_{n-1}, \tag{1}$$

where $A, B \in \mathbb{Z}$ with $AB \neq 0$. This sequence provides two independent sequences, i.e., one sequence is not a constant multiple of the other, with the first sequence having initial values $x_0 = 0$ and $x_1 = 1$, while the second one has initial values $x_0 = 2$ and $x_1 = A$. It can also be seen that any other sequence obtained from (1) can be expressed as a linear combination of these two sequences. In particular, for $A = 2$ and $B = 1$ in (1), one can extract two independent sequences namely, the Pell and Pell-Lucas sequences, that are recursively defined by $P_{n+1} = 2P_n + P_{n-1}$, for $n \geq 1$, with $P_0 = 0$, $P_1 = 1$, and $Q_{n+1} = 2Q_n + Q_{n-1}$, for $n \geq 1$, with $Q_0 = 1$, $Q_1 = 1$. The product of Pell and Pell-Lucas sequences with similar indices gives another interesting sequence known as the sequence of balancing numbers [6]. As usual, a natural number n is said to be a *balancing number* with *balancer* r, if it is the solution of the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r). \tag{2}$$

Behera et al. have also shown that a natural number n is a balancing number if and only if n^2 is a triangular number [1]. From these two definitions, the sequence of balancing numbers $\{B_n\}$ can be listed as $\{0, 1, 6, 35, 204, \dots\}$. A sequence closely associated with balancing numbers is the sequence of Lucas-balancing numbers $\{C_n\}$, where $C_n = \sqrt{8B_n^2 + 1}$, for every natural number n [5]. Balancing and Lucas-balancing numbers have the same recurrence relations, but with different initial values. That is, $B_{n+1} = 6B_n - B_{n-1}$, for $n \geq 1$, with $B_0 = 0$, $B_1 = 1$, and $C_{n+1} = 6C_n - C_{n-1}$, for $n \geq 1$, with $C_0 = 1$,

$C_1 = 3$. The study of balancing numbers and their related sequences is now a source of attraction for many theorists because of their wonderful and amazing properties.

Panda et al. examined in [7] the periodicity of balancing numbers modulo primes. According to them, the period modulo m, denoted by $\pi(m)$, is the smallest positive integer t for which $\{B_t, B_{t+1}\} \equiv \{0, 1\} \pmod{m}$. They have also shown that $\pi(B_n) = 2n$. The rank of balancing numbers, $\alpha(n)$ of a natural number n, is defined as the smallest natural number k such that n divides B_k [8]. Patel et al. [8] developed some relations between period, rank and order of the balancing numbers. Among them, one such important relation is that the period of a sequence of balancing numbers is equal to the product of the rank of apparition and its order.

Marques established in [3, 4] some identities concerning the order of appearance of Fibonacci numbers modulo consecutive Fibonacci and Lucas numbers. In a subsequent paper [2] Khaochim et al. have extended Marques' ideas to study the period of Fibonacci numbers modulo consecutive Fibonacci numbers. In this article, we study the period of balancing numbers modulo product of consecutive Pell and Pell-Lucas numbers. Among other properties we will show that, for any natural number n,

$$\pi(P_n P_{n+1} P_{n+2}) = n(n+1)(n+2),$$

and

$$\pi(Q_n Q_{n+1} Q_{n+2} Q_{n+3}) = \begin{cases} n(n+1)(n+2)(n+3), & \text{if } n \not\equiv 0 \pmod{3} \\ \frac{n(n+1)(n+2)(n+3)}{3}, & \text{if } n \equiv 0 \pmod{3}. \end{cases}$$

2. PRELIMINARIES

In this section, we present some known results concerning the divisibility properties of balancing numbers. Throughout this article, for any two positive integers a and b, (a, b) and $[a, b]$ denote the greatest common divisor and the least common multiple of a and b, respectively.

The following results are valid for any natural numbers m, n and can be found in [5].

Lemma 2.1. *B_m divides B_n if and only if m divides n.*

Lemma 2.2. $(B_m, B_n) = B_{(m,n)}$.

Lemma 2.3. $B_{m+n} = B_m C_n + C_m B_n$.

The following two results from [6] hold for any positive integers m and n.

Lemma 2.4. *P_m divides P_n if and only if m divides n.*

Lemma 2.5. *For all $n \geq 2$,*

$$(i)\ P_{n+1} P_{n-1} - P_n^2 = (-1)^n$$

$$(ii)\ Q_{n+1} Q_{n-1} - Q_n^2 = 2(-1)^{n-1}.$$

The following results concerning the periodicity of balancing numbers can be found in [7].

Lemma 2.6. *If m divides n, then $\pi(m)$ divides $\pi(n)$.*

Lemma 2.7. *If m divides B_n, then $\pi(m)$ divides $2n$.*

Lemma 2.8. *For any natural number n, $\pi(B_n) = 2n$.*

Lemma 2.9. *For any natural number n,*

$$\pi(P_n) = \begin{cases} n, & \text{if } n \text{ is even} \\ 2n, & \text{if } n \text{ is odd.} \end{cases}$$

Lemma 2.10. *For any natural number n,*

$$\pi(Q_n) = \begin{cases} 2n, & \text{if } n \text{ is even} \\ n, & \text{if } n \text{ is odd.} \end{cases}$$

The following results can be found in [8].

Lemma 2.11. *For any positive integers m and n, $\pi([m,n]) = [\pi(m), \pi(n)]$.*

Lemma 2.12. *If B_n divides m, then n divides $\pi(m)$.*

3. PERIOD MODULO PRODUCT OF CONSECUTIVE PELL NUMBERS

The following lemmas are useful while proving the subsequent theorems.

Lemma 3.1. *For all natural numbers n, $\pi(P_2P_n) = 2n$.*

Proof. Since a balancing number is the product of a Pell number and a Pell-Lucas number, we have $B_{2n} = P_{2n}Q_{2n}$, for any natural number n. Using the identity $P_{2n} = 2B_n$ (see [6]) and since $P_2 = 2$, we get $B_{2n} = P_2P_nQ_nQ_{2n} \equiv 0 \pmod{P_2P_n}$. Furthermore, using Lemma 2.3, $B_{2n+1} \equiv C_{2n} \pmod{P_2P_n}$. But $C_{2n} = C_n^2 + 8B_n^2$ (see [5]) and, since $C_n^2 = 1 + 8B_n^2$, $B_{2n+1} = 1 + 16B_n^2 \equiv 1 \pmod{P_2P_n}$. Therefore, $\pi(P_2P_n)$ divides $2n$. On the other hand, $B_n \not\equiv 0 \pmod{P_2P_n}$, because Q_n is odd, for all natural numbers n. Hence $\pi(P_2P_n) > n$. Since $n < \pi(P_2P_n) \leq 2n$, we conclude that $\pi(P_2P_n) = 2n$. □

Lemma 3.2. *For any natural number n,*

$$\pi(P_nP_{n+2}) = \begin{cases} 2n(n+2), & \text{if } n \equiv 0 \pmod{2} \\ n(n+2), & \text{if } n \equiv 1 \pmod{2}. \end{cases}$$

Proof. Consider the case that n is odd. Clearly, $P_{n+\epsilon}$ divides P_nP_{n+2}, when $\epsilon \in \{0, 2\}$. It follows from Lemma 2.9 that $2(n+\epsilon)$ divides $\pi(P_nP_{n+2})$. Since n is odd, we observe that 2, n and $n+2$ are pairwise co-prime and hence $2n(n+2)$ divides $\pi(P_nP_{n+2})$. Further, as $B_n = P_nQ_n$, P_n and P_{n+2} both divide $B_{n(n+2)}$. By Lemma 2.5, P_n and P_{n+2} are relatively prime, and thus, for any odd natural number n, P_nP_{n+2} divides $B_{n(n+2)}$. Consequently, $\pi(P_nP_{n+2})$ divides

$\pi(B_{n(n+2)}) = 2n(n+2)$. On the other hand, for n even, $(2n, 2(n+2)) = 4$ and $(P_n, P_{n+2}) = 2$. Therefore using 2.9 and 2.11, we have

$$\begin{aligned}\pi(P_nP_{n+2}) &= \pi([P_n, P_{n+2}](P_n, P_{n+2})) \\ &= \pi([P_n, P_{n+2}] \times 2) \\ &= [\pi(2P_n), \pi(2P_{n+2})] \\ &= [2n, 2(n+2)] \\ &= \frac{4n(n+2)}{(2n, 2(n+2))} \\ &= n(n+2),\end{aligned}$$

which ends the proof. □

The following results show that the periodicity of the sequence of balancing numbers modulo product of two or three consecutive Pell numbers equals the product of that consecutive natural numbers.

THEOREM 3.3. *For any natural number n, $\pi(P_nP_{n+1}) = n(n+1)$.*

Proof. According to Lemma 2.4, we have that P_n divides $P_{n(n+1)}$ and P_{n+1} divides $P_{n(n+1)}$. Since $(P_n, P_{n+1}) = 1$, P_nP_{n+1} divides $P_{n(n+1)}$, and hence $\pi(P_nP_{n+1})$ divides $\pi(P_{n(n+1)}) = n(n+1)$. On the other hand, for $\alpha \in \{0, 1\}$, $P_{n+\alpha}$ divides P_nP_{n+1}. It follows from Lemma 2.9 that, according to the parity of n, either $n+\alpha$ or $2(n+\alpha)$ divides $\pi(P_nP_{n+1})$. But, for any n, $(n, n+1) = 1$ and therefore $n(n+1)$ divides $\pi(P_nP_{n+1})$. This completes the proof. □

THEOREM 3.4. *For $n \geq 2$, $\pi(P_nP_{n+1}P_{n+2}) = n(n+1)(n+2)$.*

Proof. For any $\delta \in \{0, 1, 2\}$, $P_{n+\delta}$ divides $P_{n(n+1)(n+2)}$. Consider n is odd. Then P_n, P_{n+1} and P_{n+2} are pairwise co-prime. Therefore, $P_nP_{n+1}P_{n+2}$ divides $P_{n(n+1)(n+2)}$. By Lemma 2.5, $\pi(P_nP_{n+1}P_{n+2})$ divides $n(n+1)(n+2)$. On the other hand, $P_{n+\delta}$ divides $P_nP_{n+1}P_{n+2}$, for $\delta \in \{0, 1, 2\}$. Using Lemma 2.9, $2(n+\epsilon)$ divides $\pi(P_nP_{n+1}P_{n+2})$, for any $\epsilon \in \{0, 2\}$ and $(n+1)$ divides $\pi(P_nP_{n+1}P_{n+2})$, which implies that $(n+\delta)$ divides $\pi(P_nP_{n+1}P_{n+2})$. Since n is odd, n, $n+1$ and $n+2$ are pairwise co-prime. Therefore, $n(n+1)(n+2)$ divides $\pi(P_nP_{n+1}P_{n+2})$. Now consider n is even. Then $(2(n+1), 2n(n+2)) = 2$ and $(P_nP_{n+2}, P_{n+1}) = 1$. Using Lemma 2.9, 2.11 and 3.2, we have

$$\begin{aligned}\pi(P_nP_{n+1}P_{n+2}) &= \pi([P_{n+1}, P_nP_{n+2}]) \\ &= [\pi(P_{n+1}), \pi(P_nP_{n+2})] \\ &= [2(n+1), 2n(n+2)] \\ &= \frac{2n(n+1)(n+2)}{(2(n+1), n(n+2))} \\ &= n(n+1)(n+2),\end{aligned}$$

which completes the proof. □

THEOREM 3.5. *For* $n \not\equiv 0 \pmod 3$

$$\pi(P_nP_{n+1}P_{n+2}P_{n+3}) = n(n+1)(n+2)(n+3).$$

Proof. For $n \not\equiv 0 \pmod 3$, $(P_n, P_{n+3}) = 1$, and therefore

$$(P_nP_{n+2}, P_{n+1}P_{n+3}) = 1.$$

Using Lemma 3.2, $\pi(P_nP_{n+2}) = n(n+2)$ or $2n(n+2)$, and $\pi(P_{n+1}P_{n+3}) = (n+1)(n+3)$ or $2(n+1)(n+3)$, according to the parity of n. It follows that the greatest common divisor of $(2n(n+2), (n+1)(n+3))$ and $(n(n+2), 2(n+1)(n+3))$ equals 2. Now, by Lemma 2.11, we have

$$\pi(P_nP_{n+1}P_{n+2}P_{n+3}) = \pi([P_nP_{n+2}, P_{n+1}P_{n+3}]),$$

which implies that

$$\pi(P_nP_{n+1}P_{n+2}P_{n+3}) = [\pi(P_nP_{n+2}), \pi(P_{n+1}P_{n+3})].$$

Consequently,

$$\pi(P_nP_{n+1}P_{n+2}P_{n+3}) = \frac{2n(n+2)(n+1)(n+3)}{(2n(n+2), (n+1)(n+3))} \quad \text{or} \quad \frac{2n(n+2)(n+1)(n+3)}{(n(n+2), 2(n+1)(n+3))},$$

which implies $\pi(P_nP_{n+1}P_{n+2}P_{n+3}) = n(n+1)(n+2)(n+3)$. This ends the proof. □

4. PERIOD MODULO PRODUCT OF CONSECUTIVE PELL-LUCAS NUMBERS

The following result shows that the periodicity of the sequence of balancing numbers modulo product of two consecutive Pell-Lucas numbers equals twice the product of that consecutive natural numbers.

THEOREM 4.1. *For any natural number* n, $\pi(Q_nQ_{n+1}) = 2n(n+1)$.

Proof. For $\alpha \in \{0, 1\}$, $Q_{n+\alpha}$ divides Q_nQ_{n+1}. It follows from Lemma 2.10 that, according to the parity of n, either $2(n+\alpha)$ or $n+\alpha$ divides $\pi(Q_nQ_{n+1})$. But, for n odd and even, we have $(n, 2(n+1)) = 1$ and $(2n, n+1) = 1$, respectively. Therefore, $2n(n+1)$ divides $\pi(Q_nQ_{n+1})$. On the other hand, Q_n and Q_{n+1} both divide $B_{n(n+1)}$. Since $(Q_n, Q_{n+1}) = 1$, Q_nQ_{n+1} divides $B_{n(n+1)}$. By Lemma 2.8, $\pi(Q_nQ_{n+1})$ divides $2n(n+1)$. Consequently, $\pi(Q_nQ_{n+1}) = 2n(n+1)$. □

The following lemma is useful while proving the subsequent theorem.

LEMMA 4.2. *For any natural number* n, $\pi(Q_nQ_{n+2}) = n(n+2)$.

Proof. For an even n, Q_n and Q_{n+2} both divide Q_nQ_{n+2}. By Lemma 2.10, $2n$ and $2(n+2)$ both divide $\pi(Q_nQ_{n+2})$. Since n is even, for distinct a, $b \in \{-1,1\}$, $(2^an, 2^b(n+2)) = 1$ (the choice of a and b depends on the class of n modulo 4). Hence $2^{a+b}n(n+2)$ divides $\pi(Q_nQ_{n+2})$, which implies that $n(n+2)$ divides $\pi(Q_nQ_{n+2})$. On the other hand, Q_n and Q_{n+2} both divide $B_{n(n+2)/2}$. For all $n \in \mathbb{N}$, (Q_n, Q_{n+2}) are relatively prime, thus Q_nQ_{n+2} divides $B_{n(n+2)/2}$. By Lemmas 2.6 and 2.10, $\pi(Q_nQ_{n+2})$ divides $n(n+2)$. Now, for an odd n, both Q_n and Q_{n+2} divide Q_nQ_{n+2}, and thus n and $(n+2)$ both divide $\pi(Q_nQ_{n+2})$. Since n is odd, n and $n+2$ are relatively prime, thus $n(n+2)$ divides $\pi(Q_nQ_{n+2})$. Furthermore, Q_n and Q_{n+2} both divide $Q_{n(n+2)}$. Since $(Q_n, Q_{n+2}) = 1$, Q_nQ_{n+2} divides $Q_{n(n+2)}$. Therefore, $\pi(Q_nQ_{n+2})$ divides $n(n+2)$ and this is what had to be shown. □

Theorem 4.3. *For any natural number n,*

$$\pi(Q_nQ_{n+1}Q_{n+2}) = \begin{cases} n(n+1)(n+2), & \text{if } n \equiv 0 \pmod 2 \\ 2n(n+1)(n+2), & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

Proof. For $\delta \in \{0,1,2\}$, $Q_{n+\delta}$ divides $Q_nQ_{n+1}Q_{n+2}$. For $n \equiv 0 \pmod 2$, by Lemma 2.10, $2n$, $n+1$ and $2(n+2)$ each divides $\pi(Q_nQ_{n+1}Q_{n+2})$. For distinct a, $b \in \{-1,1\}$, $2^an, n+1$ and $2^b(n+2)$ are pairwise co-prime (the choice of a and b depends on the class of n modulo 4). Thus, $2^{a+b}n(n+1)(n+2)$ divides $\pi(Q_nQ_{n+1}Q_{n+2})$, which implies $n(n+1)(n+2)$ divides $\pi(Q_nQ_{n+1}Q_{n+2})$. Furthermore, for $\delta \in \{0,1,2\}$, $Q_{n+\delta}$ divides $B_{n(n+1)(n+2)/2}$. Since, Q_n, Q_{n+1} and Q_{n+2} are pairwise co-prime, therefore $Q_nQ_{n+1}Q_{n+2}$ divides $B_{n(n+1)(n+2)/2}$. Using Lemma 2.2, $\pi(Q_nQ_{n+1}Q_{n+2})$ divides $n(n+1)(n+2)$. On the other hand, for $\delta \in \{0,1,2\}$, $Q_{n+\delta}$ divides $B_{n(n+1)(n+2)}$. For $n \equiv 1 \pmod 2$, Q_n, Q_{n+1} and Q_{n+2} are pairwise co-prime. Thus, $Q_nQ_{n+1}Q_{n+2}$ divides $B_{n(n+1)(n+2)}$. By using Lemmas 2.6 and 2.10, $\pi(Q_nQ_{n+1}Q_{n+2})$ divides $2n(n+1)(n+2)$. Furthermore, for $\delta \in \{0,1,2\}$, $Q_{n+\delta}$ divides $Q_nQ_{n+1}Q_{n+2}$ and, using Lemma 2.10, we get that n, $2(n+1)$ and $n+2$ each divides $\pi(Q_nQ_{n+1}Q_{n+2})$. Since n is odd, n, $2(n+1)$ and $n+2$ are pairwise co-prime. Therefore $2n(n+1)(n+2)$ divides $\pi(Q_nQ_{n+1}Q_{n+2})$, which completes the proof. □

Theorem 4.4. *For any natural number n,*

$$\pi(Q_nQ_{n+1}Q_{n+2}Q_{n+3}) = \begin{cases} n(n+1)(n+2)(n+3), & \text{if } n \not\equiv 0 \pmod 3 \\ n(n+1)(n+2)(n+3)/3, & \text{if } n \equiv 0 \pmod 3. \end{cases}$$

Proof. For any natural number n, Q_nQ_{n+2} and $Q_{n+1}Q_{n+3}$ are pairwise co-prime. Let $n \not\equiv 0 \pmod 3$. Then $(n(n+2), (n+1)(n+3)) = 1$. By Lemma

2.10 and 4.2, we have

$$\begin{aligned}\pi(Q_nQ_{n+1}Q_{n+2}Q_{n+3}) &= \pi([Q_nQ_{n+2}, Q_{n+1}Q_{n+3}])\\ &= [\pi(Q_nQ_{n+2}), \pi(Q_{n+1}Q_{n+3})]\\ &= [n(n+2), (n+1)(n+3)]\\ &= \frac{n(n+2)(n+1)(n+3)}{(n(n+2), (n+1)(n+3))}\\ &= n(n+1)(n+2)(n+3).\end{aligned}$$

Let $n \equiv 0 \pmod 3$. Then $(n(n+2), (n+1)(n+3)) = 3$. Continuing as above, we get the desired result. □

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