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DIFFERENTIABLE MAPS WITH UNCOUNTABLE CRITICAL SETS

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Abstract. In this paper we provide some sufficient conditions on two manifolds which ensure uncountable critical sets for all smooth maps between the two manifolds. These sufficient conditions are given in terms of Stiefel-Whitney classes of the two manifolds and their duals.

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1. PRELIMINARY RESULTS

THEOREM 1.1. *If M is a connected differential manifold and $A \subset M$ is a closed countable set, then*

$$\pi_r(M, M\backslash A) \simeq 0, \text{ for all } r \in \{0, \dots, n-1\}.$$

In particular the inclusion $i : M\backslash A \hookrightarrow M$ is (n-1)-connected.

The proof of Theorem (1.1) relies on the homotopy sequences of the pairs

$$(M, M\backslash A)$$

and works along the same lines as in [5, Proposition 2.3]. By using the well-known Whitehead theorem [1, 8], we get the following:

COROLLARY 1.2. *If M is a connected differential manifold and $A \subset M$ is a closed countable set, then the induced homomorphism*

$$i_k : H_k(M\backslash A, \mathbb{Z}) \to H_k(M, \mathbb{Z})$$

by the inclusion $i : M\backslash A \hookrightarrow M$ is an isomorphism for $k < n$ and an epimorphism for $k = n$.

If we combine the following isomorphisms, which follow via the universal coefficient theorem,

$$H_k(M\backslash A, \mathbb{Z}_2) \cong H_k(M\backslash A, \mathbb{Z}) \otimes \mathbb{Z}_2,$$
$$H_k(M, \mathbb{Z}_2) \cong H_k(M, \mathbb{Z}) \otimes \mathbb{Z}_2,$$

by Corollary 1.2 we obtain the isomorphisms, induced by the inclusions i,

$$i_k : H_k(M\backslash A, \mathbb{Z}_2) \to H_k(M, \mathbb{Z}_2)$$

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for $k < n$ and an epimorphism for $k = n$. By using the equivalences stated in [1, p. 310] we obtain the isomorphisms, induced by the inclusions i,

$$i^k : H_k(M, \mathbb{Z}_2) \to H_k(M \backslash A, \mathbb{Z}_2),$$

for $k < n$ and a monomorphism for $k = n$.

REMARK 1.3. If A is a subset of a differential manifold M^m, then the equality $\bar{\omega}_i(M \setminus A) = j_A^*(\bar{\omega}_i(M))$ holds, where $j_A : M \setminus A \longrightarrow M$ is the inclusion, and $\bar{\omega}_i(M \setminus A)$ and $\bar{\omega}_i(M)$ stand for the i^{th} dual Stiefel-Whitney class of $M \setminus A$ and M, respectively. If A is additionally closed and countable, then $j_A^i : H^i(M, \mathbb{Z}_2) \longrightarrow H^i(M \setminus A, \mathbb{Z}_2)$ is an isomorphism for $i < m$ and a monomorphism for $i = m$.

Indeed, the dual Stiefel-Whitney classes can be expressd in terms of Stiefel-Whitney classes [6, p. 40] as follows:

$$\bar{\omega}_r = \omega_1 \bar{\omega}_{r-1} + \omega_2 \bar{\omega}_{r-2} + \cdots + \omega_{r-1} \bar{\omega}_1 + \omega_r \tag{1}$$

i.e.,

$$\begin{aligned} \bar{\omega}_1 &= \omega_1 \\ \bar{\omega}_2 &= \omega_1^2 + \omega_2 \\ \bar{\omega}_3 &= \omega_1^3 + \omega_3 \\ \bar{\omega}_4 &= \omega_1^4 + \omega_1^2 \omega_2 + \omega^2 + \omega_4 \\ &\vdots \end{aligned}$$

The equality $\bar{\omega}_i(M \setminus A) = j_A^*(\bar{\omega}_i(M))$ follows by using the representations (1) of the dual Stiefel-Whitney classes by taking into account that

$$j_A^i : H^i(M, \mathbb{Z}_2) \longrightarrow H^i(M \setminus A, \mathbb{Z}_2)$$

is a ring homomorphism and

$$\omega_i(M \setminus A) = \omega_i(\tau_{M \setminus A}) = \omega_i(j_A^*(\tau_M)) = j_A^*(\omega_i(M)),$$

for all $i \geq 0$. Note that $\tau_{M \setminus A}$ and τ_M stand for the tangent bundle of $M \setminus A$ and M, respectively. The property of $j_A^i : H^i(M, \mathbb{Z}_2) \longrightarrow H^i(M \setminus A, \mathbb{Z}_2)$ to be an isomorphism for $i < m$ and a monomorphism for $i = m$ follows from the above considerations.

2. MAPS WITH UNCOUNTABLE CRITICAL SETS

THEOREM 2.1. *Let M^m, N^n be smooth boundaryless manifolds such that $m < n < m + k$, for some $k \geq 1$. If $\bar{\omega}_k(M) \neq 0$ and and $\omega(N) = 1$, then the critical set of every smooth map from M to N is uncountable.*

Proof. Assume that the critical set $C(f)$ of the smooth map $f : M \to N$, which is obviously closed, is countable. Then the restriction

$$g := f\big|_{M \backslash C(f)} : M \backslash C(f) \to N$$

is an immersion and $g^*\tau_N \simeq \tau_{M\setminus C(f)} \oplus \nu_g$, where ν_g is the associated $(n-m)$-normal bundle of the immersion g. Therefore we have successively

$$\begin{aligned} 1 &= \omega_0(g^*\tau_N) = g^*(\omega_0(\tau_N)) = g^*(1) = g^*(\omega(N)) \\ &= \omega(g^*\tau_N) = \omega(\tau_{M\setminus C(f)} \oplus \nu_g) = \omega(\tau_{M\setminus C(f))}) \cup \omega(\nu_g) \\ &= \omega(M\setminus C(f))\omega(\nu_g), \end{aligned}$$

which shows that $\omega(\nu_g) = \bar{\omega}(M\setminus C(f))$. By using now Remark 1.3 we obtain

$$\omega_k(\nu_g) = \bar{\omega}_k(M\setminus C(f)) = j^k_{C(f)}(\bar{\omega}_k(M)) \neq 0,$$

as $\bar{\omega}_k(M) \neq 0$. On the other hand, $\omega_k(\nu_g) = 0$ as $\text{rank}(\nu_g) = n - m < k$. $\square$

Example 2.2. Let N be the sphere S^n or an n-dimensional Lie group. If $2 \leq m < n < 2m-1$, then the critical set of every smooth map $f : \mathbb{RP}^m \longrightarrow N$ is uncountable.

Theorem 2.3. *Let M^m, N^n be smooth manifolds such that M is compact and $m < n$. If M is immersible in N and $A \subseteq N$ is a closed countable subset of N, then M is immersible in $N\setminus A$, too. Moreover, the homotopy class of any immersion $f : M \to N$ contains an immersion g such that $g(M) \subseteq N\setminus A$.*

In order to prove Theorem 2.3, we need to prove first its local version.

Lemma 2.4. *Let M^m, N^n be smooth manifolds such that M is compact and $m < n$ and M is immersible in N. Let also $A \subseteq N$ be a closed countable subset of N. If $f : M \to N$ is an immersion, then, for each $q \in f(M) \cap A$, there exists an open neighborhood of q, say V_q, whose topological border ∂V_q avoids the set A, i.e., $\partial V_q \cap A = \emptyset$, and an immersion $g_q : M \to N$ such that*

$$g_q(f^{-1}(V_q)) \cap A = \emptyset,\ g_q\Big|_{M\setminus f^{-1}(V_q)} = f\Big|_{M\setminus f^{-1}(V_q)} \text{ and } f \simeq g_q\big(rel\ M\setminus f^{-1}(V_q)\big).$$

Proof. Let $f : M \to N$ be an immersion. For $q \in f(M) \cap A$, we observe that $k := \#[f^{-1}(q)]$ is finite, and we consider an open connected neighborhood V'_q of q, small enough such that its inverse image $f^{-1}(V'_q)$ has k connected components $U'_1, \ldots, U'_k$. Consider a smaller connected open neighborhood $V_q \subseteq V'_q$ which is also bounded with respect to a certain Riemannian metric on N such that $\bar{V}_q \subseteq V'_q$, $\partial V_q \cap A = \emptyset$ whose inverse image $f^{-1}(V_q)$ has the connected components $U_1, \ldots, U_k$ that obviously satisfy $\bar{U}_1 \subseteq U'_1, \ldots, \bar{U}_k \subseteq U'_k$. We can also assume that $U'_1, \ldots, U'_k$ are the domains of certain local charts $(U'_1, \varphi_1), \ldots, (U'_q, \varphi_k)$ and that V'_q is the common domain of some local charts $(V'_q, \psi_1), \ldots, (V'_q, \psi_k)$ such that $(\psi_i \circ f \circ \varphi_i^{-1})(x_1, \ldots, x_m) = (x_1, \ldots, x_m, \underbrace{0, \ldots, 0}_{n\text{-}m \text{ times}})$, for all $i \in \{1, \ldots, k\}$ and all $x = (x_1, \ldots, x_m) \in \varphi_i(U_i)$.

Consider a smooth real positive function $\theta : N \to \mathbb{R}$ such that $\theta^{-1}(0) = N\backslash V_q$ and the smooth vector fields $X_1, \ldots, X_k$ which are defined on N by

$$X_i(z) = \begin{cases} \theta(z)\frac{\partial^{\psi_i}}{\partial x_n}\Big|_z, & \text{if } z \in V'_q \\ 0, & \text{if } z \in N\backslash \bar{V}_q. \end{cases}$$

Obviously the norms $||X_1||, \ldots, ||X_k||$ of the fields $X_1, \ldots, X_k$ are bounded with respect to the considered Riemannian metric on N, namely they are completely integrable (see [3, pp. 183]). Denote by α_t^i the global flow induced by X_i and consider the projection $\pi_n : \mathbb{R}^n \to \mathbb{R}$, $\pi_n(x_1, \ldots, x_n) = x_n$. Observe that

$(\pi_n \circ \psi_i \circ f \circ \varphi_i^{-1})(x_1, \ldots, x_m) = 0$ for all $x = (x_1, \ldots, x_m) \in \varphi_i(U_i)$, namely

$$(\pi_n \circ \psi_i \circ f)(x) = 0, \text{ for all } x \in U_i.$$

One can easily prove that

$$\pi_n(\psi_i(\alpha_t^i(f(z))) = \int_0^t \theta_i(\alpha_s^i(f(z)))\mathrm{d}s,$$

for all $t \neq 0$ and all $z \in U_i$.

$$\psi_i((f(\varphi^{-1}(x_1, \ldots, x_m))) = \psi_i(\alpha_0^i(f(\varphi^{-1}(x_1, \ldots, x_m))) = (c_1, \ldots, c_{n-1}, c_n)$$

we have that $c_{m+1} = \cdots = c_n = 0$. Consequently it implies that

$$\psi_i(\alpha_t^i(y)) = (c_1, \ldots, c_m, 0, \ldots, \int_0^t \theta_i(\alpha_s^i(y))\mathrm{d}s),$$

and, for $t \neq 0$ and $z \in U_i$, we have

$$\pi_n(\psi_i(\alpha_t^i(f(z))) = \int_0^t \theta_i(\alpha_s^i(f(z)))\mathrm{d}s \neq 0.$$

Consequently, $t \neq 0 \Rightarrow \alpha_t^i(f(U_i)) \cap f(U_i) = \emptyset$ or equivalently, $t \neq t' \Rightarrow \alpha_t^i(f(U_i)) \cap \alpha_{t'}^i(f(U_i) = \emptyset$.

Indeed, assuming that $\alpha_t^i(f(U_i)) \cap f(U_i) \neq \emptyset$, for some $t \neq 0$, one can consider p, $p' \in U_i$ such that $\alpha_t^i(f(p)) = f(p')$. But such an equality is impossible, because it would imply the following relations: $0 \neq \pi_n(\psi_i(\alpha_1^i(f(p))) = \pi_n(\psi_i(f(p')) = 0$. Therefore there exists $t_i > 0$ such that $\alpha_{t_i}^i(f(U_i))) \cap A = \emptyset$, for all $i \in \{1, \ldots, k\}$, since A is countable. Define the map g_q in the following way:

$$g_q(x) = \begin{cases} \alpha_{t_1}^1(f(x)), & \text{if } x \in U'_1 \\ \vdots & \vdots \\ \alpha_{t_k}^k(f(x)), & \text{if } x \in U'_k \\ f(x), & \text{if } x \in M\backslash \bigcup_{i=1}^{p} \bar{U}_i. \end{cases}$$

Because $\alpha^1_{t_i}, \ldots, \alpha^k_{t_k}$ are diffeomorphisms and f is an immersion, it follows that g_q is also an immersion that has the additional property $g_q(U_1 \cup \cdots \cup U_k) \cap A = \emptyset$. Therefore, for the fixed $q \in f(M) \cap A$, we have constructed a neighborhood V_q and an immersion $g_q : M \to N$ such that

$$g_q(f^{-1}(V_q)) \cap A = \emptyset \text{ and } g_q\Big|_{M\setminus f^{-1}(V_q)} = f\Big|_{M\setminus f^{-1}(V_q)}.$$

Finally, let us observe that that $H : [0,1] \times M \to N$ given by

$$H_q(t,x) = \begin{cases} \alpha^1_{t\cdot t_1}(f(x)), & \text{if } x \in U'_1 \\ \vdots & \vdots \\ \alpha^k_{t\cdot t_k}(f(x)), & \text{if } x \in U'_k \\ f(x), & \text{if } x \in M\setminus \bigcup_{i=1}^{p} \bar{U}_i, \end{cases}$$

is a homotopy between f and g_q relative to the subset $M\setminus f^{-1}(V_q)$. □

Proof of Theorem 2.3. If A is finite, then Theorem 2.3 follows by applying successively Lemma 2.4. Otherwise the open sets $\{V_q\}_{q\in f(M)\cap A}$ constructed in Lemma 2.4 form an open covering of the compact set $f(M) \cap A$ and there exist $q_1, \ldots, q_r \in f(M) \cap A$ with the property $f(M) \cap A \subseteq V_{q_1} \cup \cdots \cup V_{q_r}$. We can obviously assume that $V_{q_i} \setminus \bigcup_{\substack{j=1 \\ j\neq i}}^{p} V_{q_j} \neq \emptyset$ and that $q_i \in V_{q_i} \setminus \bigcup_{\substack{j=1 \\ j\neq i}}^{p} V_{q_j}$. By applying the same procedure like in Lemma 2.4, we can successively reconstruct the immersions $g_{q_2}, \ldots, g_{q_r}$ in such a way that the following properties hold:

$$\begin{aligned} & g_{q_1}(f^{-1}(V_{q_1})) \cap A = \emptyset, \\ & g_{q_2}(f^{-1}(V_{q_2} \setminus \bar{V}_{q_1})) \cap A = \emptyset, \\ & g_{q_3}(f^{-1}(V_{q_3} \setminus (\bar{V}_{q_1} \cup \bar{V}_{q_2}))) \cap A = \emptyset, \\ & \vdots \\ & g_{q_r}(f^{-1}(V_{q_r} \setminus (\bar{V}_{q_1} \cup \cdots \cup \bar{V}_{q_{r-1}}))) \cap A = \emptyset \end{aligned}$$

and

$$\begin{aligned} & g_{q_1}\Big|_{M\setminus f^{-1}(V_{q_1})} = f\Big|_{M\setminus f^{-1}(V_{q_1})}, \\ & g_{q_2}\Big|_{M\setminus f^{-1}(V_{q_2}\setminus \bar{V}_{q_1})} = g_{q_1}\Big|_{M\setminus f^{-1}(V_{q_2}\setminus \bar{V}_{q_1})}, \\ & g_{q_3}\Big|_{M\setminus f^{-1}(V_{q_3}\setminus(\bar{V}_{q_1}\cup \bar{V}_{q_2}))} = g_{q_2}\Big|_{M\setminus f^{-1}(V_{q_3}\setminus(\bar{V}_{q_1}\cup \bar{V}_{q_2}))}, \\ & \vdots \\ & g_{q_r}\Big|_{M\setminus f^{-1}(V_{q_r}\setminus(\bar{V}_{q_1}\cup\cdots\cup \bar{V}_{q_{r-1}}))} = g_{q_{r-1}}\Big|_{M\setminus f^{-1}(V_{q_r}\setminus(\bar{V}_{q_1}\cup\cdots\cup \bar{V}_{q_{r-1}}))} \end{aligned}$$

and $f \simeq g_{q_1} \simeq g_{q_2} \simeq \ldots \simeq g_{q_r}$. In order to show that the immersion g_{q_r} has the property $g_{q_r}(M) \subseteq N \backslash A$, we first show, by mathematical induction on i, that $g_{q_i}\big(f^{-1}(V_{q_1} \cup \cdots \cup V_{q_i})\big) \cap A = \emptyset$. Indeed, for $i = 1$ the statement is obviously true such that we assume that it is true for i and prove its validity for $i + 1$. In this respect, we have

$$g_{q_{i+1}}\big(f^{-1}(V_{q_1} \cup \cdots \cup V_{q_i} \cup V_{q_{i+1}})\big) =$$

$$= g_{q_{i+1}}\big(f^{-1}(V_{q_1} \cup \cdots \cup V_{q_i})\big) \cup g_{q_{i+1}}\big(f^{-1}(V_{q_{i+1}} \backslash (V_{q_1} \cup \cdots \cup V_{q_i})\big).$$

But, since the inclusion $f^{-1}(V_{q_1} \cup \cdots \cup V_{q_i}) \subseteq M \backslash f^{-1}(V_{q_{i+1}} \backslash (\bar{V}_{q_1} \cup \cdots \cup \bar{V}_{q_i}))$ is obvious, it follows that

$$g_{q_{i+1}}\big(f^{-1}(V_{q_1} \cup \cdots \cup V_{q_i})\big) \cap A = g_{q_i}\big(f^{-1}(V_{q_1} \cup \cdots \cup V_{q_i})\big) \cap A = \emptyset,$$

such that it remains only to prove that $g_{q_{i+1}}\big(f^{-1}(V_{q_{i+1}} \backslash (V_{q_1} \cup \cdots \cup V_{q_i})\big) \cap A = \emptyset$. Indeed, otherwise we may consider $q \in g_{q_{i+1}}\big(f^{-1}(V_{q_{i+1}} \backslash (V_{q_1} \cup \cdots \cup V_{q_i})\big) \cap A$, namely $q = g_{q_{i+1}}(p)$, for some p such that $f(p) \in V_{q_{i+1}} \backslash (V_{q_1} \cup \cdots \cup V_{q_i})$. If $f(p) \notin \bar{V}_{q_j}$, for all $j \in \{1, \ldots, i\}$, then $p \in f^{-1}(V_{q_{i+1}} \backslash (\bar{V}_{q_1} \cup \cdots \cup \bar{V}_{q_i}))$, which shows that

$$q = g_{q_{i+1}}(p) \in g_{q_{i+1}}\Big(f^{-1}(V_{q_{i+1}} \backslash (\bar{V}_{q_1} \cup \cdots \cup \bar{V}_{q_i}))\Big) \cap A,$$

a contradiction with the construction of $g_{q_{i+1}}$. Consequently, $f(p) \in \bar{V}_{q_j}$, for some $j \in \{1, \ldots, i\}$. But, since

$$f(p) \in V_{q_{i+1}} \backslash (V_{q_1} \cup \cdots \cup V_{q_i}) \subseteq M \backslash (V_{q_1} \cup \cdots \cup V_{q_i}) \subseteq M \backslash V_{q_j} = \overline{M \backslash V}_{q_j},$$

it follows that $f(p) \in \partial V_{q_j}$. Because $f(p) \in \bar{V}_{q_j} \subseteq \bar{V}_{q_1} \cup \cdots \cup \bar{V}_{q_i}$, it follows that $f(p) \notin V_{q_{i+1}} \backslash (\bar{V}_{q_1} \cup \cdots \cup \bar{V}_{q_i})$, i.e., $p \in M \backslash f^{-1}\big(V_{q_{i+1}} \backslash (\bar{V}_{q_1} \cup \cdots \cup \bar{V}_{q_i})\big)$, which means that $q = g_{q_{i+1}}(p) = g_{q_i}(p)$. Since $f(p) \in V_{q_{i+1}} \backslash (V_{q_1} \cup \cdots \cup V_{q_i})$ it follows that $f(p) \notin V_{q_k}$ for all $k \in \{1, \ldots, i\}$ which, in particular, implies that $f(p) \notin V_{q_k} \backslash (V_{q_1} \cup \cdots \cup V_{q_{k-1}})$ for all $k \in \{1, \ldots, i\}$. Consequently

$$p \in M \backslash f^{-1}\big(V_{q_k} \backslash (\bar{V}_{q_1} \cup \cdots \cup \bar{V}_{q_{k-1}})\big), \forall k \in \{1, \ldots, i\},$$

which means that $q = g_{q_i}(p) = \cdots = g_{q_1}(p) = f(p) \in \partial V_{q_j}$, a contradiction with the construction of V_{q_j}. The inductive proof is now completely done.

In what follows, we will show that the immersion g_{q_r} satisfies the desired condition, i.e., its image avoids the set A.

$$g_{q_r}(M) \cap A = [g_{q_r}\big(M \backslash f^{-1}\big(V_{q_1} \cup \cdots \cup V_{q_r}\big)\big) \cap A] \cup [g_{q_r}\big(f^{-1}\big(V_{q_1} \cup \cdots \cup V_{q_r}\big)\big) \cap A] =$$

$$= g_{q_r}\big(M \backslash f^{-1}\big(V_{q_1} \cup \cdots \cup V_{q_r}\big)\big) \cap A.$$

From the obvious inclusions

$$V_{q_i} \backslash (\bar{V}_{q_1} \cup \cdots \cup \bar{V}_{q_{i-1}}) \subseteq V_{q_1} \cup \cdots \cup V_{q_r}, \ \forall i \in \{1, \ldots, r\},$$

we obtain

$$f^{-1}\big(V_{q_i}\backslash(\bar{V}_{q_1}\cup\cdots\cup\bar{V}_{q_{i-1}})\big)\subseteq f^{-1}\big(V_{q_1}\cup\cdots\cup V_{q_r}\big)$$

$$\Leftrightarrow M\backslash f^{-1}\big(V_{q_1}\cup\cdots\cup V_{q_r}\big)\subseteq M\backslash f^{-1}\big(V_{q_i}\backslash(\bar{V}_{q_1}\cup\cdots\cup\bar{V}_{q_{i-1}})\big), \tag{2}$$

and the relation (2) is obviously satisfied for all $i\in\{1,\ldots,r\}$. Therefore, we obtain

$$\begin{aligned} g_{q_r}\left(M\backslash f^{-1}\big(V_{q_1}\cup\cdots\cup V_{q_r}\big)\right)\cap A &= g_{q_{r-1}}\left(M\backslash f^{-1}\big(V_{q_1}\cup\cdots\cup V_{q_r}\big)\right)\cap A\\ &= g_{q_{r-2}}\left(M\backslash f^{-1}\big(V_{q_1}\cup\cdots\cup V_{q_r}\big)\right)\cap A\\ &\ \ \vdots\\ &= f\big(M\backslash f^{-1}\big(V_{q_1}\cup\cdots\cup V_{q_r}\big)\big)\cap A\\ &= \emptyset, \end{aligned}$$

and the emptiness of $f\big(M\backslash f^{-1}\big(V_{q_1}\cup\cdots\cup V_{q_r}\big)\big)\cap A$ follows, as $\{V_{q_1},\ldots,V_{q_r}\}$ is a covering of $f(M)\cap A$. □

COROLLARY 2.5. *Let M^m, N^n, E^e be smooth manifolds such that $m<e\le n<m+k$. Assume that M is immersible in E and $\omega(N)=1$. If $\bar{\omega}_k(M)\neq 0$ for some $k\ge 1$, then then the critical set of every smooth map $f:E\longrightarrow N$ is uncountable.*

Proof. Assume that there exists a smooth map $f:E\to N$ with countably many critical points, i.e., its critical set $C(f)$ is a closed countable subset of E. If $h:M\to E$ is an immersion such that $h(N)\cap C(f)=\emptyset$, whose existence is ensured by Theorem 2.3, then $f\circ h:M\to N$ is obviously an immersion and also $\omega_k(\nu_{f\circ h})=\bar{\omega}_k(M)\neq 0$, where $\nu_{f\circ h}$ is the associated $(n-m)$-normal bundle of the immersion $f\circ h$, impossible since $0<n-m<k$. □

COROLLARY 2.6. *Let N^n be a differential manifold and let $\pi:E^e\to M^m$ be a smooth fibration which admits a smooth cross-section $s:M\to E$, i.e. $\pi\circ s=id_M$. If $\omega(N)=1$, $\bar{\omega}_k(M)\neq 0$, for some $k\ge 1$ and $m<e\le n<m+k$, then the critical set of every smooth map $f:E\longrightarrow N$ is uncountable.*

Proof. We only need to observe that M is immersible in E, as

$$(\mathrm{d}\pi)_{s(x)}\circ \mathrm{d}s_x = id_{T_x(M)},\ \forall x\in M,$$

which shows that s is an immersion. □

Particular pairs of manifolds satisfying the hypothesis of Corollary 2.6 are provided by the next example.

EXAMPLE 2.7. Let X^k be a differential manifold, let n be a natural number such that $n+1$ is not a power of 2, and let r be the integer $[\log_2 n]+1$, where $[r]$ stands for the the greatest integer smaller than or equal to r. If M^m is a Lie group or a sphere such that $2n+k<m<2^{r+1}-3$, then the critical set of any smooth map $f:G_{2,n}\times X\longrightarrow M$ is uncountable. In particular, the critical set of any smooth map $G_{2,n}\times X\longrightarrow\mathbb{R}^m$ or $G_{2,n}\times X\longrightarrow T^m$ is uncountable.

REMARK 2.8. The lists provided by examples 2.2 and 2.7 can be considerably extended by using the non-triviality of the dual Stiefel-Whitney classes

$$\bar{\omega}_{3(s-n)-3}(G_{3,n}), \bar{\omega}_{s+3}(G_{3,s-2}), \bar{\omega}_{2s}(G_{3,s-1}), \bar{\omega}_{3(2s-n)-3}(G_{3,n}),$$

proved in [2].

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