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THE GENERALIZED NON-ABSOLUTE TYPE OF TRIPLE Γ^3 SEQUENCE SPACES DEFINED MUSIELAK-ORLICZ FUNCTION

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Abstract. In this paper, we introduce the notion of $\lambda_{mnk}-\Gamma^3$ and Λ^3 sequences. Further, we introduce the spaces $\left[\Gamma^{3\lambda}_f, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ and $\left[\Lambda^{3\lambda}_f, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$, which are of non-absolute type, and we prove that these spaces are linearly isomorphic to the spaces Γ^3 and Λ^3, respectively. Moreover, we establish some inclusion relations between these spaces.

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1. INTRODUCTION

A triple (real or complex) sequence can be defined as a function $x : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers, respectively. The different types of notions of triple sequence were introduced and investigated, at the beginning, by Sahiner et al. [10,11], Esi et al. [1–3], Datta et al. [4], Subramanian et al. [12], Debnath et al. [5] and many others.

A triple sequence $x = (x_{mnk})$ is said to be *triple analytic* if

$$\sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The space of all triple analytic sequences is usually denoted by Λ^3. A triple sequence $x = (x_{mnk})$ is called a *triple entire sequence*, if

$$|x_{mnk}|^{\frac{1}{m+n+k}} \to 0, \text{ as } m,n,k \to \infty.$$

The space of all triple entire sequences is usually denoted by Γ^3. The spaces Γ^3, Λ^3 are metric spaces with respect to the metric given by

$$d(x,y) = \sup_{m,n,k} \left\{ |x_{mnk} - y_{mnk}|^{\frac{1}{m+n+k}} : m,n,k : 1,2,3,... \right\},$$

for all $x = (x_{mnk})$ and $y = (y_{mnk})$ in Γ^3, Λ^3.

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The notion of difference sequence space (for single sequences) was introduced by Kizmaz [6] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\},$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$, for all $k \in \mathbb{N}$.

Let $w^3, \chi^3(\Delta_{mnk}), \Lambda^3(\Delta_{mnk})$ denote the spaces of all triple gai difference sequences and all triple analytic difference sequences, respectively. The space of difference triple sequences was introduced by Debnath et al. (see [5]) and is defined as

$$\begin{aligned}\Delta x_{mnk} =& x_{mnk} - x_{m,n+1,k} - x_{m,n,k+1} + x_{m,n+1,k+1} - x_{m+1,n,k} \\ &+ x_{m+1,n+1,k} + x_{m+1,n,k+1} - x_{m+1,n+1,k+1}\end{aligned}$$

and $\Delta^0 x_{mnk} = \langle x_{mnk} \rangle$.

2. DEFINITIONS AND PRELIMINARIES

Throughout the article $w^3, \chi^3(\Delta), \Lambda^3(\Delta)$ denote the spaces of all triple gai difference sequences and all triple analytic difference sequences, respectively.

For a triple sequence $x \in w^3$, Subramanian et al. introduced in [12] the spaces $\Gamma^3(\Delta), \Lambda^3(\Delta)$ as follows:

$$\Gamma^3(\Delta) = \left\{x \in w^3 : |\Delta x_{mnk}|^{1/m+n+k} \to 0 \text{ as } m,n,k \to \infty\right\}$$

$$\Lambda^3(\Delta) = \left\{x \in w^3 : \sup_{m,n,k} |\Delta x_{mnk}|^{1/m+n+k} < \infty\right\}.$$

The spaces $\Gamma^3(\Delta), \Lambda^3(\Delta)$ are metric spaces with respect to the metric given by

$$d(x,y) = \sup\nolimits_{m,n,k} \left\{|\Delta x_{mnk} - \Delta y_{mnk}|^{1/m+n+k} : m,n,k = 1,2,\cdots\right\},$$

for all $x = (x_{mnk})$ and $y = (y_{mnk})$ in $\Gamma^3(\Delta), \Lambda^3(\Delta)$.

Definition 2.1. An *Orlicz function* (see [7]) is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing, convex, with $M(0) = 0$, $M(x) > 0$, for $x > 0$, and $M(x) \to \infty$, as $x \to \infty$. If the convexity of the Orlicz function M is replaced by the condition $M(x+y) \le M(x) + M(y)$, then the function is called a *modulus function*.

Lindenstrauss and Tzafriri [8] used the idea of Orlicz function to construct Orlicz sequence spaces.

A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = sup\{|v|\,u - (f_{mnk})(u) : u \ge 0\}, m,n,k = 1,2,\ldots,$$

is called the *complementary function of a Musielak-Orlicz function f*. For a given Musielak-Orlicz function f (see [9]), the Musielak-Orlicz sequence space t_f is defined as follows

$$t_f = \left\{x \in w^3 : I_f(|x_{mnk}|)^{1/m+n+k} \to 0 \text{ as } m,n,k \to \infty\right\},$$

where I_f is a convex modular defined by

$I_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk}(|x_{mnk}|)^{1/m+n+k}$, $x = (x_{mnk}) \in t_f$.

We consider t_f equipped with the Luxemburg metric, given by

$$d(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk}\left(\frac{|x_{mnk}|^{1/m+n+k}}{mnk}\right),$$

which is an extended real number.

Definition 2.2. Let X, Y be real vector spaces of dimension m, where $n \leq m$. A real-valued function $d_p(x_1, \ldots, x_n) = \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p$ on X is called the *p-product metric* of the Cartesian product of n metric spaces if it satisfies the following conditions (see [13]):

(i) $\|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p = 0$ if and only if $d_1(x_1, 0), \ldots, d_n(x_n, 0)$ are linearly dependent,

(ii) $\|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p$ is invariant under permutation,

(iii) $\|(\alpha d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p = |\alpha| \, \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p, \alpha \in \mathbb{R}$,

(iv) for $1 \leq p < \infty$ one has

$$d_p((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)) = \Big(d_X(x_1, x_2, \ldots, x_n)^p + d_Y(y_1, y_2, \ldots, y_n)^p\Big)^{1/p},$$

or

(v) for $x_1, x_2, \ldots, x_n \in X$ and $y_1, y_2, \ldots, y_n \in Y$ one has

$$d((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)) = \sup\{d_X(x_1, x_2, \ldots, x_n), d_Y(y_1, y_2, \ldots, y_n)\}.$$

Let $\eta = (\lambda_{mnk})$ be a non-decreasing sequence of positive real numbers tending to infinity with $\lambda_{111} = 1$ and $\lambda_{m+n+k+3} \leq \lambda_{m+n+k+3}+1$, for all $m, n, k \in \mathbb{N}$. The *generalized de la Vallée-Poussin mean* is defined by

$$t_{mnk}(x) = \lambda_{mnk}^{-1} \sum_{m,n,k \in I_{mnk}} x_{mnk}, \text{ where } I_{mnk} = [mnk - \lambda_{mnk} + 1, mnk].$$

A sequence $x = (x_{mnk})$ is said to be (V, λ)-*summable* to a number L, if $t_{mnk}(x) \to L$, as $mnk \to \infty$.

The notion of λ-triple entire and triple analytic sequences are defined as follows. Let $\lambda = (\lambda_{mnk})_{m,n,k=0}^{\infty}$ be a strictly increasing sequences of positive real numbers tending to infinity and consider

$$\begin{aligned} B_\eta^\mu(x) = &\frac{1}{\varphi_{rst}} \sum_{m \in I_{rst}} \sum_{n \in I_{rst}} \sum_{k \in I_{rst}} \lambda_{mnk} x_{mnk} - \lambda_{m,n+1,k} x_{m,n+1,k} \\ &- \lambda_{m,n,k+1} x_{m,n,k+1} + \lambda_{m,n+1,k+1} x_{m,n+1,k+1} - \lambda_{m+1,n,k} x_{m+1,n,k} \\ &+ \lambda_{m+1,n+1,k} x_{m+1,n+1,k} + \lambda_{m+1,n,k+1} x_{m+1,n,k+1} \\ &- \lambda_{m+1,n+1,k+1} x_{m+1,n+1,k+1}. \end{aligned}$$

Definition 2.3. We say that a sequence $x = (x_{mnk}) \in w^3$ is λ-*convergent* to a, if $B_\eta^\mu(x) \to a$, as $m, n, k \to \infty$, and we write $\lambda - lim(x) = a$.

Definition 2.4. A sequence $x = (x_{mnk}) \in w^3$ is said to be a λ-*triple entire sequence*, if $B^{\mu}_{\eta}(x) \to 0$, as $m, n, k \to \infty$.

Definition 2.5. A sequence $x = (x_{mnk}) \in w^3$ is said to be a λ-*triple analytic sequence*, if $\sup_{mnk} B^{\mu}_{\eta}(x) < \infty$.

We have

$$\lim_{m,n,k\to\infty} \left|B^{\mu}_{\eta}(x) - a\right| = \lim_{m,n,k\to\infty} \frac{1}{\varphi_{rst}} \sum_{m\in I_{rst}} \sum_{n\in I_{rst}} \sum_{k\in I_{rst}} \lambda_{mnk}x_{mnk}$$
$$- \lambda_{m,n+1,k}x_{m,n+1,k} - \lambda_{m,n,k+1}x_{m,n,k+1} + \lambda_{m,n+1,k+1}x_{m,n+1,k+1}$$
$$- \lambda_{m+1,n,k}x_{m+1,n,k} + \lambda_{m+1,n+1,k}x_{m+1,n+1,k}$$
$$+ \lambda_{m+1,n,k+1}x_{m+1,n,k+1} - \lambda_{m+1,n+1,k+1}x_{m+1,n+1,k+1} = a.$$

So we can say that $\lim_{m,n,k\to\infty} \left|B^{\mu}_{\eta}(x)\right| = a$. Hence x is $\lambda_{mnk}x_{mnk}$-convergent to a.

Lemma 2.6. *Every convergent sequence is λ_{mnk}-convergent to the same ordinary limit.*

Proof. Omitted. □

Lemma 2.7. *If a λ_{mnk}-Musielak-convergent sequence converges in the ordinary sense, then it must Musielak-converge to the same λ_{mnk}-limit.*

Proof. Let $x = (x_{mnk}) \in w^3$ and $m, n, k \geq 1$. We have

$$|\Delta^m x_{mnk}|^{1/m+n+k} - B^{\mu}_{\eta}(x) = |\Delta^m x_{mnk}|^{1/m+n+k}$$
$$- \frac{1}{\varphi_{rst}} \sum_{m\in I_{rst}} \sum_{n\in I_{rst}} \sum_{k\in I_{rst}} \lambda_{mnk}x_{mnk} - \lambda_{m,n+1,k}x_{m,n+1,k} - \lambda_{m,n,k+1}x_{m,n,k+1}$$
$$+ \lambda_{m,n+1,k+1}x_{m,n+1,k+1} - \lambda_{m+1,n,k}x_{m+1,n,k} + \lambda_{m+1,n+1,k}x_{m+1,n+1,k}$$
$$+ \lambda_{m+1,n,k+1}x_{m+1,n,k+1} - \lambda_{m+1,n+1,k+1}x_{m+1,n+1,k+1}.$$

Therefore, we have, for every $x = (x_{mnk}) \in w^3$, that

$$|\Delta^m x_{mnk}|^{1/m+n+k} - B^{\mu}_{\eta}(x) = S_{mnk}(x)\,(m, n, k \in \mathbb{N}),$$

where the sequence $S(x) = (S_{mnk}(x))^{\infty}_{m,n,k=0}$ is defined by $S_{000}(x) = 0$ and

$$S_{mnk}(x) = \frac{1}{\varphi_{rst}} \sum_{m\in I_{rst}} \sum_{n\in I_{rst}} \sum_{k\in I_{rst}} \lambda_{mnk} - \lambda_{m,n+1,k} - \lambda_{m,n,k+1} + \lambda_{m,n+1,k+1}$$
$$- \lambda_{m+1,n,k} + \lambda_{m+1,n+1,k} + \lambda_{m+1,n,k+1} - \lambda_{m+1,n+1,k+1}, m, n, k \geq 1.$$

□

Lemma 2.8. *A λ_{mnk}-Musielak-convergent sequence $x = (x_{mnk})$ converges if and only if* $S(x) \in \left[\Gamma^3_{fB^{\mu}_{\eta}}, \|(d(x_1, 0), d(x_2, 0), \cdots, d(x_{n-1}, 0))\|_p\right]$.

Proof. Let $x = (x_{mnk})$ be a λ_{mnk}-Musielak-convergent sequence. Then, by Lemma 2.7, we have that $x = (x_{mnk})$ converges to the same λ_{mnk}-limit. We obtain $S(x) \in \left[\Gamma^3_{fB^\mu_\eta}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$. Conversely, let $S(x) \in \left[\Gamma^3_{fB^\mu_\eta}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$. We have

$$\lim_{m,n,k\to\infty} |\Delta^m \lambda_{mnk} x_{mnk}|^{1/m+n+k} = \lim_{m,n,k\to\infty} B^\mu_\eta(x).$$

From the above equation, we deduce that the λ_{mnk}-convergent sequence $x = (x_{mnk})$ converges. □

Lemma 2.9. *Every triple analytic sequence is λ_{mnk}-triple analytic.*

Proof. Let $S(x) \in \left[\Lambda^3_{fB^\mu_\eta}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$. Then there exists $M > 0$ such that

$$\sup_{mnk} |\Delta^m \lambda_{mnk} x_{mnk}|^{1/m+n+k} = \sup_{mnk} B^\mu_\eta(x) < M.$$

From the above equation, we deduce that the sequence $x = (x_{mnk})$ is λ_{mnk}-triple analytic. □

Lemma 2.10. *A λ_{mnk}-Musielak-analytic sequence $x = (x_{mnk})$ is analytic if and only if $S(x) \in \left[\Lambda^3_{fB^\mu_\eta}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$.*

Proof. The assertion follows from Lemma 2.9, $S_{000}(x) = 0$ and

$$S_{mnk}(x) = \frac{1}{\varphi_{rst}} \sum_{m\in I_{rst}} \sum_{n\in I_{rst}} \sum_{k\in I_{rst}} \lambda_{mnk} x_{mnk} - \lambda_{m,n+1,k} x_{m,n+1,k}$$
$$- \lambda_{m,n,k+1} x_{m,n,k+1} + \lambda_{m,n+1,k+1} x_{m,n+1,k+1} - \lambda_{m+1,n,k} x_{m+1,n,k}$$
$$+ \lambda_{m+1,n+1,k} x_{m+1,n+1,k} + \lambda_{m+1,n,k+1} x_{m+1,n,k+1} - \lambda_{m+1,n+1,k+1} x_{m+1,n+1,k+1},$$

for $m, n, k \geq 1$. □

3. THE SPACES OF λ_{MNK}-TRIPLE ENTIRE AND TRIPLE ANALYTIC SEQUENCES

In this section we introduce the sequence spaces:

$$\left[\Gamma^3_{f\Delta^\lambda_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right],$$
$$\left[\Lambda^3_{f\Delta^\lambda_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$$

i.e.,

$$\left[\Gamma^3_{f\Delta^\lambda_{mnk}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p\right]$$
$$= \lim_{m,n,k\to\infty} \left[B^\mu_\eta, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p\right] = 0.$$
$$\left[\Lambda^3_{f\Delta^\lambda_{mnk}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p\right]^{I(F)}$$

$$= \sup_{mnk} \left[B_\eta^\mu, \left\| \left(d\left(x_1, 0\right), d\left(x_2, 0\right), \ldots, d\left(x_{n-1}, 0\right) \right) \right\|_p \right] < \infty.$$

THEOREM 3.1. *The sequence spaces*

$$\left[\Gamma^3_{f\Delta^\lambda_{mnk}}, \left\| \left(d\left(x_1, 0\right), d\left(x_2, 0\right), \ldots, d\left(x_{n-1}, 0\right) \right) \right\|_p \right]$$

and

$$\left[\Lambda^3_{f\Delta^\lambda_{mnk}}, \left\| \left(d\left(x_1, 0\right), d\left(x_2, 0\right), \ldots, d\left(x_{n-1}, 0\right) \right) \right\|_p \right]$$

are isomorphic to the spaces

$$\left[\Gamma^3_f, \left\| \left(d\left(x_1, 0\right), d\left(x_2, 0\right), \ldots, d\left(x_{n-1}, 0\right) \right) \right\|_p \right]$$

and

$$\left[\Lambda^3_f, \left\| \left(d\left(x_1, 0\right), d\left(x_2, 0\right), \ldots, d\left(x_{n-1}, 0\right) \right) \right\|_p \right].$$

Proof. We consider only the case

$$\left[\Gamma^3_{f\Delta^\lambda_{mnk}}, \left\| \left(d\left(x_1, 0\right), d\left(x_2, 0\right), \ldots, d\left(x_{n-1}, 0\right) \right) \right\|_p \right]$$
$$\cong \left[\Gamma^3_f, \left\| \left(d\left(x_1, 0\right), d\left(x_2, 0\right), \ldots, d\left(x_{n-1}, 0\right) \right) \right\|_p \right].$$

The case

$$\left[\Lambda^3_{f\Delta^\lambda_{mnk}}, \left\| \left(d\left(x_1, 0\right), d\left(x_2, 0\right), \ldots, d\left(x_{n-1}, 0\right) \right) \right\|_p \right]$$
$$\cong \left[\Lambda^3_f, \left\| \left(d\left(x_1, 0\right), d\left(x_2, 0\right), \ldots, d\left(x_{n-1}, 0\right) \right) \right\|_p \right]$$

can be treated similarly.

Consider the transformation T defined by

$$Tx = B_\eta^\mu \in \left[\Gamma^3_f, \left\| \left(d\left(x_1, 0\right), d\left(x_2, 0\right), \ldots, d\left(x_{n-1}, 0\right) \right) \right\|_p \right],$$

for every $x \in \left[\Gamma^3_{f\Delta^\lambda_{mnk}}, \left\| \left(d\left(x_1, 0\right), d\left(x_2, 0\right), \ldots, d\left(x_{n-1}, 0\right) \right) \right\|_p \right]$. The linearity of T is obvious. It is trivial that $x = 0$, whenever $Tx = 0$, and hence T is injective. To show that T is surjective, we define the sequence $x = \{x_{mnk}(\lambda)\}$ by

$$B_\eta^\mu(x) = \frac{1}{\varphi_{rst}} \sum_{m \in I_{rst}} \sum_{n \in I_{rst}} \sum_{k \in I_{rst}} \left(\Delta \lambda_{mnk} x_{mnk} \right) = y_{mnk}. \tag{1}$$

We can say that $B_\eta^\mu(x) = y_{mnk}$ from (1) and

$$x \in \left[\Gamma^3_f, \left\| \left(d\left(x_1, 0\right), d\left(x_2, 0\right), \ldots, d\left(x_{n-1}, 0\right) \right) \right\|_p \right],$$

hence

$$B_\eta^\mu(x) \in \left[\Gamma^3_f, \left\| \left(d\left(x_1, 0\right), d\left(x_2, 0\right), \ldots, d\left(x_{n-1}, 0\right) \right) \right\|_p \right].$$

We deduce that $x \in \left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ and $Tx = y$. Hence T is surjective. We have, for every

$$x \in \left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p\right],$$

that $d(Tx,0)_{\chi^3} = d(Tx,0)_{\Lambda^3} = d(x,0)_{\left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p\right]}$.

Hence, $\left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p\right]^{I(F)}$ and $\left[\Gamma^3_{f}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p\right]$ are isomorphic. Similarly we obtain the isomorphism of the other sequence spaces. □

Theorem 3.2. *The inclusion*

$$\left[\Gamma^3_{f\Delta_{mnk}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p\right] \subset \left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p\right]$$

holds.

Proof. Let $\left[\Gamma^3_{f\Delta_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$. Then we deduce that

$$\frac{1}{\varphi_{rst}} \sum_{m\in I_{rst}} \sum_{n\in I_{rst}} \sum_{k\in I_{rst}} (\Delta\lambda_{mnk} x_{mnk}) \leq \frac{1}{\varphi_{rst}} \lim_{m,n,k\to\infty} \sum_{m\in I_{rst}} \sum_{n\in I_{rst}} \sum_{k\in I_{rst}} (\Delta\lambda_{mnk} x_{mnk}) = \lim_{m,n,k\to\infty} |\Delta^m \lambda_{mnk} x_{mnk}|^{1/m+n+k} = 0.$$

Hence, $x \in \left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p\right]$. □

Theorem 3.3. *The inclusion*

$$\left[\Lambda^3_{f\Delta_{mnk}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p\right] \subset \left[\Lambda^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p\right]$$

holds.

Proof. It is obvious. Therefore we omit the proof. □

Theorem 3.4. *The inclusion*

$$\left[\Gamma^3_{f}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p\right] \subset \left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p\right]$$

holds. Furthermore, the equality holds if and only if

$$S(x) \in \left[\Gamma^3_{f}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p\right],$$

for every sequence $x \in \left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p\right]$.

Proof. The inclusion

$$(2) \quad \left[\Gamma^3_f, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p\right] \subset \left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p\right]$$

is obvious from Lemma 2.9. Then we have, for every

$$x \in \left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p\right],$$

that

$$x \in \left[\Gamma^3_f, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p\right],$$

and hence

$$S(x) \in \left[\Gamma^3_f, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p\right],$$

by Lemma 2.10. Conversely, let

$$x \in \left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p\right].$$

Then we have that

$$S(x) \in \left[\Gamma^3_f, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p\right].$$

Thus, it follows by Lemma 2.9 and then by Lemma 2.10, that

$$x \in \left[\Gamma^3_f, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p\right].$$

We get

$$(3) \quad \left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \subset \left[\Gamma^3_f, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p\right]^{I(F)}.$$

From the equations (2) and (3) we get

$$\left[\Gamma^3_{f\Delta^{\lambda}_{mnk}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] = \left[\Gamma^3_f, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p\right].$$

□

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