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POTENTIAL ANALYSIS FOR PSEUDODIFFERENTIAL MATRIX OPERATORS IN LIPSCHITZ DOMAINS ON RIEMANNIAN MANIFOLDS. APPLICATIONS TO BRINKMAN OPERATORS

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Abstract. In this paper we present the main properties of layer potentials associated to some pseudodifferential matrix type operators on Lipschitz domains in compact Riemannian manifolds of arbitrary dimension. We focus on a class of Brinkman operators and show compactness and invertibility results of associated layer potential operators, and well-posedness results for related transmission problems with the boundary data in some Sobolev spaces.

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1. INTRODUCTION

The layer potential theory has a significant role in the analysis of various elliptic boundary value problems. For example, Fabes, Kenig and Verchota [5] developed a layer potential analysis in the treatment of the L^2 Dirichlet problem for the Stokes system on Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$. Mitrea and Wright [18] used layer potential methods to prove the well-posedness of the main boundary value problems for the Stokes system in Lipschitz domains in $\mathbb{R}^n$, $n \geq 2$, with the boundary data in various function spaces. Hofmann, Mitrea and Taylor [6] studied boundary value problems for elliptic partial differential equations on (two-sided) NTA domains (in the sense of Jerison and Kenig [8]) with Ahlfors regular boundaries and small mean oscillations of the unit normals, by using layer potential methods. Escauriaza and Mitrea [4] shown the well-posedness of transmission problems for the Laplace operator on Lipschitz domains in $\mathbb{R}^n$ and boundary data in Lebesgue and Hardy spaces (see also [12]). Well-posedness of transmission problems for the Laplace-Beltrami operator in Sobolev or Besov spaces on Lipschitz domains in non-smooth manifolds have been obtained by Mitrea et al. [14]. Mitrea and Taylor [17] studied the L^2 Dirichlet problem for the Stokes system on arbitrary Lipschitz domains in compact Riemannian manifolds, by using a method based on single-layer potentials. Dindoš and Mitrea [3] developed a layer potential analysis for the

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Poisson problem associated to the Stokes system, as well as the Dirichlet problem for the Navier-Stokes equations on Lipschitz and C^1 domains in a smooth compact Riemannian manifold, when the boundary data belong to Sobolev or Besov spaces. We treated in [9, 10] transmission problems for the Stokes and Brinkman operators on Lipschitz domains of dimension ≤ 3, or on C^1 domains of arbitrary dimension in Riemannian manifolds, by employing layer potential techniques. The purpose of this paper is to obtain the main properties of layer potentials associated to some pseudodifferential matrix type operators on Lipschitz domains in compact Riemannian manifolds of arbitrary dimension. We focus on a class of Brinkman operators and show compactness and invertibility results of associated layer potential operators, and well-posedness results for related transmission problems with boundary data in some Sobolev spaces.

2. SPECIAL PSEUDODIFFERENTIAL MATRIX TYPE OPERATORS ON COMPACT RIEMANNIAN MANIFOLDS

In this section we show the invertibility property for a special class of pseudodifferential matrix type operators on compact Riemannian manifolds.

2.1. Preliminaries. Consider a smooth vector bundle $\mathcal{E}$ equipped with a C^∞-inner product $\langle\cdot,\cdot\rangle_{\mathcal{E}_x}$, $x \in M$. Thus, $\mathcal{E}$ is a Hermitian bundle. For sections u, $v \in C^\infty(M,\mathcal{E})$ one then defines the scalar product

$$\langle u, v\rangle_{\mathcal{E}} := \int_M \langle u(x), v(x)\rangle_{\mathcal{E}_x} \mathrm{dVol}(x). \tag{2.1}$$

Next, consider two smooth, Hermitian vector bundles $\mathcal{E}$, $\mathcal{F} \to M$, and a *differential operator* of order $k \geq 1$

$$D : C^\infty(M,\mathcal{E}) \to C^\infty(M,\mathcal{F}). \tag{2.2}$$

Then its formal adjoint D^* is defined by means of the inner products $\langle\cdot,\cdot\rangle_{\mathcal{E}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{F}}$, as $\langle Du, v\rangle_{\mathcal{F}} = \langle u, D^*v\rangle_{\mathcal{E}},\ \forall\ u \in C^\infty(M,\mathcal{E}),\ v \in C^\infty(M,\mathcal{F})$.

In particular, let (M, g) be a compact boundaryless Riemannian manifold of $\dim(M) := m \geq 2$ and let $g := g_{jk}dx_j \otimes dx_k$ be its smooth metric tensor. Hereafter one uses the summation convention rule and denote by (g^{jk}) the inverse of (g_{jk}). The tangent and cotangent bundles are $TM = \bigcup_{p\in M} T_pM$ and $T^*M = \bigcup_{p\in M} T_p^*M$, respectively, and $\mathfrak{X}(M)$ is the space of smooth vector fields on M. Also, Λ^1TM is the first exterior power bundle corresponding to TM. The adjoint of the exterior derivative $d : C^\infty(M) \to C^\infty(M,\Lambda^1TM)$ is usually denoted by δ, i.e., $\delta : C^\infty(M,\Lambda^1TM) \to C^\infty(M)$, and $\langle du, v\rangle = \langle u, \delta v\rangle$ for every $u \in C^\infty(M)$ and $v \in C^\infty(M,\Lambda^1TM)$.

Let ∇ be the Levi-Civita connection on M. If $X \in \mathfrak{X}(M)$, the symmetric part of the tensor field

$$\nabla X : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M),\ (\nabla X)(Y,Z) = \langle \nabla_Y X, Z\rangle,$$

is called the *deformation* of X and is denoted by Def X. Thus,

$$(\text{Def } X)(Y,Z) = \frac{1}{2}\{\langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle\}, \quad \forall\, Y,Z \in \mathfrak{X}(M). \tag{2.3}$$

A vector field $X \in \mathfrak{X}(M)$ such that Def $X = 0$ on M is called a *Killing field.* Further, assume that M *does not have any nontrivial Killing field* (see [17]).

By $OPS^{\ell}_{\rm cl}$ one denotes the class of classical pseudodifferential operators of order ℓ. The symbol $p(x,\xi)$ of such an operator P admits an asymptotic expansion of the form $p(x,\xi) \sim p_\ell(x,\xi) + p_{\ell-1}(x,\xi) + \cdots$, where $p_k(x,\xi)$ is smooth in x and ξ, positively homogeneous of degree k in $\xi \in \mathbb{R}^m$. The term $\sigma^0_P(x,\xi) := p_\ell(x,\xi)$ is called the *principal symbol* of P (for more details on pseudodifferential operators on smooth manifolds see [7, 19, 21]).

For $p \in (1,\infty)$ and $s \in \mathbb{R}$, denote by $L^p_s(M)$ the Sobolev scales on M. Also, $L^p_s(M,\mathcal{E}) := L^p_s(M)\otimes C^\infty(M,\mathcal{E})$ is the space of sections $u : M \to \mathcal{E}$ whose local representations have coefficients in $L^p_s(M)$. In particular, $L^2_s(M,\Lambda^1 TM) := L^2_s(M) \otimes \Lambda^1 TM$ are the Sobolev spaces of one forms, which, locally, have coefficients in $L^2_s(M)$ (for more details see e.g., [17, 19, 20, 21]).

Note that every $P \in OPS^{\ell}_{\rm cl}(M,\mathcal{E})$ extends to a linear and bounded operator $P : L^p_{s+\ell}(M,\mathcal{E}) \to L^p_s(M,\mathcal{E})$ for any $p \in (1,\infty)$ and $s \in \mathbb{R}$ (see e.g., [21, Theorem 8.38, Theorem 8.45]).

2.2. Special pseudodifferential matrix type operators. Let us consider the smooth, Hermitian vector bundles $\mathcal{E},\mathcal{F},G \to M$, and let

$$D : C^\infty(M,\mathcal{E}) \to C^\infty(M,\mathcal{F}),\ P_0 : C^\infty(M,G) \to C^\infty(M,\mathcal{E}) \tag{2.4}$$

be two first-order differential operators (note that $C^\infty(M) = C^\infty(M,\mathbb{R})$). With respect to the corresponding scalar products of the Hermitian vector bundles $\mathcal{E}$, $\mathcal{F}$ and G (see (2.1)), their adjoint operators

$$D^* : C^\infty(M,\mathcal{F}) \to C^\infty(M,\mathcal{E}),\ P_0^* : C^\infty(M,\mathcal{E}) \to C^\infty(M,G) \tag{2.5}$$

are pseudodifferential of order one, i.e., $D^* \in OPS^1_{\rm cl}(\mathcal{F},\mathcal{E})$, $P_0^* \in OPS^1_{\rm cl}(\mathcal{E},G)$.

Next, assume that second-order differential operator

$$\mathcal{L}_D := 2D^*D : C^\infty(M,\mathcal{E}) \to C^\infty(M,\mathcal{E}) \tag{2.6}$$

is *elliptic*, i.e., the principal symbol of $\mathcal{L}_D$ satisfies the condition

$$\sigma^0\left(D^*D; x,\xi\right) \text{ is invertible, } \forall\, x \in M,\ \xi \in T^*_x M \setminus \{0\}. \tag{2.7}$$

In addition, assuming that D is *one-to-one*, one finds that $\mathcal{L}_D$ is invertible.

Further, let $P \in OPS^0_{\rm cl}(\mathcal{E},\mathcal{E})$ be *self-adjoint and non-negative with respect to the* $L^2(M,\mathcal{E})$*-inner product*, i.e.,

$$\langle Pu, w\rangle_{\mathcal{E}} = \langle u, Pw\rangle_{\mathcal{E}},\ \ \langle Pu,u\rangle_{\mathcal{E}} \geq 0 \ \text{ for all } \ u,w \in L^2(M,\mathcal{E}). \tag{2.8}$$

By (2.8), the pseudodifferential operator (a zero-order perturbation of $\mathcal{L}_D$)

$$\mathcal{L}_{D,P} := \mathcal{L}_D + P = 2D^*D + P : C^\infty(M,\mathcal{E}) \to C^\infty(M,\mathcal{E}) \tag{2.9}$$

is elliptic, self-adjoint, and one-to-one, i.e., it is invertible. It extends to a Fredholm operator of index zero

$$\mathcal{L}_{D,P} : L^2_1(M,\mathcal{E}) \to L^2_{-1}(M,\mathcal{E}), \tag{2.10}$$

which is also one-to-one, and hence invertible. Next, assume that $\mathcal{L}_{D,P}$ is $L^2_1(M,\mathcal{E})$-*elliptic*, i.e., there is $\alpha_0 > 0$ such that (see e.g., [21])

$$\langle u, \mathcal{L}_{D,P} u\rangle_{L^2(M,\mathcal{E})} \geq \alpha_0 \|u\|^2_{L^2_1(M,\mathcal{E})}, \ \forall\, u \in L^2_1(M,\mathcal{E}). \tag{2.11}$$

Finally, assume that the 1^{th}-differential operator $P_0 : L^2(M,G) \to L^2_{-1}(M,\mathcal{E})$ *has closed range and finite-dimensional kernel*, i.e.,

$$\dim\left(\mathrm{Ker}\left(P_0 : L^2(M,G) \to L^2_{-1}(M,\mathcal{E})\right)\right) := n_0 < \infty. \tag{2.12}$$

Hence, $L^2_*(M,G)$ is closed in $L^2(M,G)$ and has the codimension n_0, where

$$L^2_*(M,G) := \left\{f \in L^2(M,G) : \langle f,\psi\rangle_{L^2(M,G)} = 0,\ \forall\, \psi \in \mathrm{Ker}\ P_0\right\}. \tag{2.13}$$

We now define the pseudodifferential matrix type operator

$$\begin{gathered} B_{D,P,P_0} : C^\infty(M,\mathcal{E}) \times C^\infty(M,G) \to C^\infty(M,\mathcal{E}) \times C^\infty(M,G) \\ B_{D,P,P_0} := \begin{pmatrix} \mathcal{L}_{D,P} & P_0 \\ P_0^* & 0 \end{pmatrix} = \begin{pmatrix} 2D^*D + P & P_0 \\ P_0^* & 0 \end{pmatrix} \end{gathered} \tag{2.14}$$

and its extension, denoted as before,

$$B_{D,P,P_0} : L^2_1(M,\mathcal{E}) \times L^2(M,G) \to L^2_{-1}(M,\mathcal{E}) \times L^2(M,G). \tag{2.15}$$

The restriction

$$\begin{gathered} B^0_{D,P,P_0} : L^2_1(M,\mathcal{E}) \times L^2_*(M,G) \to L^2_{-1}(M,\mathcal{E}) \times L^2_*(M,G), \\ B^0_{D,P,P_0} := B_{D,P,P_0}|_{L^2_1(M,\mathcal{E})\times L^2_*(M,G)}, \end{gathered} \tag{2.16}$$

is one-to-one, where $L^2_*(M,G)$ is given by (2.13). Taking into account (2.7), (2.11) and (2.12), and using similar arguments to those for [10, Theorem 3.1], one obtains the following main invertibility result:

THEOREM 2.1. *The operator B^0_{D,P,P_0}, given by* (2.16)*, is invertible, and*

$$(B^0_{D,P,P_0})^{-1} := \begin{pmatrix} \mathcal{A}_{D,P,P_0} & \mathcal{B}_{D,P,P_0} \\ \mathcal{C}_{D,P,P_0} & \mathcal{D}_{D,P,P_0} \end{pmatrix}, \tag{2.17}$$

where $\mathcal{A}_{D,P,P_0} \in OPS^{-2}_{\mathrm{cl}}(\mathcal{E},\mathcal{E})$, $\mathcal{B}_{D,P,P_0} \in OPS^{-1}_{\mathrm{cl}}(G,\mathcal{E})$, $\mathcal{C}_{D,P,P_0} \in OPS^{-1}_{\mathrm{cl}}(\mathcal{E},G)$ and $\mathcal{D}_{D,P,P_0} \in OPS^{0}_{\mathrm{cl}}(G,G)$.

The proof of Theorem 2.1 will be given in a forthcoming paper. In view of this result one finds that

$$\mathcal{L}_{D,P}\mathcal{A}_{D,P,P_0} + P_0\mathcal{C}_{D,P,P_0} = \mathbb{I},\ P_0^*\mathcal{A}_{D,P,P_0} = 0 \text{ on } M. \tag{2.18}$$

Thus, the Schwartz kernels $(\mathcal{G}_{D,P,P_0}(x,y), \Pi_{D,P,P_0}(x,y))$ of the operators $\mathcal{A}_{D,P,P_0}$ and $\mathcal{C}_{D,P,P_0}$ determine the fundamental solution of the operator B^0_{D,P,P_0}, i.e.,

$$(\mathcal{L}_D + P)\,\mathcal{G}_{D,P,P_0}(\cdot,y) + P_0\Pi_{D,P,P_0}(\cdot,y) = \mathrm{Dirac}_y, P_0^*\mathcal{G}_{D,P,P_0}(\cdot,y) = 0, \tag{2.19}$$

where Dirac_y is the Dirac distribution with mass at y.

In addition, one has the formula

$$B^0_{D,P,P_0} = B^0_{D,P_0} + \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}, \; B^0_{D,P_0} := \begin{pmatrix} \mathcal{L}_D & P_0 \\ P_0^* & 0 \end{pmatrix}. \tag{2.20}$$

The operators B^0_{D,P,P_0} and $B^0_{D,P_0} := B^0_{D,0,P_0}$ are invertible on the space $L^2_1(M,\mathcal{E}) \times L^2_*(M,G)$. Let us also consider the inverse of the operator B^0_{D,P_0},

$$(B^0_{D,P_0})^{-1} := \begin{pmatrix} \mathcal{A}_{D,P_0} & \mathcal{B}_{D,P_0} \\ \mathcal{C}_{D,P_0} & \mathcal{D}_{D,P_0} \end{pmatrix}. \tag{2.21}$$

In view of (2.20), one then obtains the relation

$$\mathcal{A}_{D,P,P_0} - \mathcal{A}_{D,P_0} = -\mathcal{A}_{D,P_0} P \mathcal{A}_{D,P,P_0} \in OPS^{-4}_{\mathrm{cl}}(\mathcal{E},\mathcal{E}), \tag{2.22}$$

which implies that $\tilde{\mathcal{V}}_{D,P,0,P_0} \in OPS^{-4}_{\mathrm{cl}}(M,\mathcal{E})$, where $\tilde{\mathcal{V}}_{D,P,0,P_0}$ is the Newtonian potential with the kernel $\mathcal{G}_{D,P,0,P_0} := \mathcal{G}_{D,P,P_0} - \mathcal{G}_{D,0,P_0}$, and $\mathcal{G}_{D,0,P_0}$ is the Schwartz kernel of the operator $\mathcal{A}_{D,P_0}$ (corresponding to $P = 0$).

2.3. Pseudodifferential matrix operator of type (2.14). Let us now consider the second-order partial differential operator

$$\mathfrak{L} : \mathfrak{X}(M) \to \mathfrak{X}(M), \quad \mathfrak{L} := 2\mathrm{Def}^*\mathrm{Def} = -\triangle + \mathrm{d}\delta - 2\mathrm{Ric}, \tag{2.23}$$

where Def^* is the adjoint of Def, $\triangle := -(\mathrm{d}\delta + \delta d)$ is the Hodge Laplacian and Ric is the Ricci tensor. Note that the role of the operator D from above is played here by the deformation operator Def. Recall that the deformation operator is indeed injective, due to the lack of non-trivial Killing vector fields on M. The operator $\mathfrak{L}$ is elliptic and extends to a Fredholm operator of index zero, $\mathfrak{L} : L^2_1(M,\Lambda^1TM) \to L^2_{-1}(M,\Lambda^1TM)$. Also consider a self-adjoint and non-negative operator $\mathcal{P} \in OPS^0_{\mathrm{cl}}(\Lambda^1TM,\Lambda^1TM)$ with respect to the $L^2(M,\Lambda^1TM)$ - inner product, i.e.,

$$\langle \mathcal{P}u, w\rangle = \langle u, \mathcal{P}w\rangle, \;\; \langle \mathcal{P}u, u\rangle \geq 0 \;\text{ for all }\; u, w \in L^2(M,\Lambda^1TM). \tag{2.24}$$

The *pseudodifferential Brinkman operator* [10]

$$\begin{aligned} &B_{\mathcal{P}} : C^\infty(M,\Lambda^1TM) \times C^\infty(M) \to C^\infty(M,\Lambda^1TM) \times C^\infty(M), \\ &B_{\mathcal{P}} := \begin{pmatrix} \mathfrak{L} & d \\ \delta & 0 \end{pmatrix} + \begin{pmatrix} \mathcal{P} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathfrak{L}_{\mathcal{P}} & d \\ \delta & 0 \end{pmatrix}, \; \mathfrak{L}_{\mathcal{P}} := \mathfrak{L} + \mathcal{P}, \end{aligned} \tag{2.25}$$

is of type (2.14), and satisfies the conditions (2.7), (2.11) and (2.12). Then, by Theorem 2.1, the following operator is invertible[1] (see also [10, Theorem 3.1]):

$$B_{\mathcal{P}} = \begin{pmatrix} \mathfrak{L}_{\mathcal{P}} & d \\ \delta & 0 \end{pmatrix} : L^2_1(M,\Lambda^1TM) \times L^2_*(M) \to L^2_{-1}(M,\Lambda^1TM) \times L^2_*(M), \tag{2.26}$$

[1] One uses the same notation $B_{\mathcal{P}}$ for both operators (2.25) and (2.26).

where $L^2_*(M) := \{\varphi \in L^2(M) : \langle \varphi, 1\rangle_{L^2(M)} = 0\}$. The inverse of $B_{\mathcal{P}}$ has the form

$$(B_{\mathcal{P}})^{-1} := \begin{pmatrix} \mathfrak{A}_{\mathcal{P}} & \mathfrak{B}_{\mathcal{P}} \\ \mathfrak{C}_{\mathcal{P}} & \mathfrak{D}_{\mathcal{P}} \end{pmatrix}, \tag{2.27}$$

and $\mathfrak{A}_{\mathcal{P}} \in OPS^{-2}_{\mathrm{cl}}(\Lambda^1 TM, \Lambda^1 TM)$, $\mathfrak{B}_{\mathcal{P}} \in OPS^{-1}_{\mathrm{cl}}(M, \mathbb{R})$, $\mathfrak{C}_{\mathcal{P}} \in OPS^{-1}_{\mathrm{cl}}(\Lambda^1 TM, \mathbb{R})$ and $\mathfrak{D}_{\mathcal{P}} \in OPS^{0}_{\mathrm{cl}}(M, \mathbb{R})$. The Schwartz kernels $(\mathcal{G}_{\mathcal{P}}, \Pi_{\mathcal{P}})$ of the operators $\mathfrak{A}_{\mathcal{P}}$ and $\mathfrak{C}_{\mathcal{P}}$ determine the fundamental solution of the operator $B_{\mathcal{P}}$. Hence,

$$(\mathfrak{L} + \mathcal{P})_x \mathcal{G}_{\mathcal{P}}(x, y) + d_x \Pi_{\mathcal{P}}(x, y) = \mathrm{Dirac}_y(x), \quad \delta_x \mathcal{G}_{\mathcal{P}}(x, y) = 0.$$

For $\mathcal{P} = 0$ one obtains the *Stokes operator* B_0.

2.4. Sobolev spaces of sections in vector bundles. Le $\Omega_+ := \Omega \subset M$ be a Lipschitz domain (i.e., the boundary $\partial\Omega$ of Ω can be described in appropriate local coordinates by means of graphs of Lipschitz functions) and assume that $\Omega_- := M \setminus \overline{\Omega}$ is connected. Fix $\kappa = \kappa(\partial\Omega) > 1$, sufficiently large, and define the *non-tangential maximal operator* $\mathcal{N} := \mathcal{N}_\kappa$ by

$$\mathcal{N}(u)(x) := \sup\{|u(y)| : y \in \gamma_\pm(x)\}, \ x \in \partial\Omega, \tag{2.28}$$

for arbitrary $u : \Omega_\pm \to \mathbb{R}$, where

$$\gamma_\pm(x) := \{y \in \Omega_\pm : \mathrm{dist}(x, y) < \kappa \ \mathrm{dist}\,(y, \partial\Omega)\}, \quad x \in \partial\Omega, \tag{2.29}$$

are non-tangential approach regions (lying in Ω_+ and Ω_-, respectively). Denote by $\mathrm{Tr}^\pm$ the *non-tangential boundary trace operators* on $\partial\Omega$, given by

$$(\mathrm{Tr}^\pm u)(x) := \lim_{\gamma_\pm(x) \ni y \to x} u(y), \ x \in \partial\Omega, \tag{2.30}$$

$$\mathrm{Tr}^\pm : C^0(\overline{\Omega}_\pm) \to C^0(\partial\Omega), \quad \mathrm{Tr}^\pm u = u|_{\partial\Omega}. \tag{2.31}$$

Also, for $p \in (1, \infty)$ and $s \geq 0$, consider the Sobolev spaces of functions

$$L^p_s(\Omega_\pm) := \{f|_{\Omega_\pm} : f \in L^p_s(M)\}, \quad \tilde{L}^p_s(\Omega_\pm) := \{f \in L^p_s(M) : \mathrm{supp} f \subseteq \overline{\Omega}_\pm\},$$

and denote by $L^p_{-s}(\Omega_\pm) = \left(\tilde{L}^q_s(\Omega_\pm)\right)^*$ the dual of the space $\tilde{L}^q_s(\Omega_\pm)$, where $q \in (1, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1$. Recall that for a smooth, Hermitian vector bundle $\mathcal{E} \to M$, the set of smooth sections of $\mathcal{E}$ on M is denoted by $C^\infty(M, \mathcal{E})$. Then

$$L^p_s(\Omega_\pm, \mathcal{E}) := L^p_s(\Omega_\pm) \otimes C^\infty(M, \mathcal{E}), \quad \tilde{L}^p_s(\Omega_\pm, \mathcal{E}) := \tilde{L}^p_s(\Omega_\pm) \otimes C^\infty(M, \mathcal{E})$$

are the Sobolev spaces of sections $u : \Omega_\pm \to \mathcal{E}$ having their coefficients in $L^p_s(\Omega_\pm)$ and $\tilde{L}^p_s(\Omega_\pm)$, respectively, and $L^p_{-s}(\Omega_\pm, \mathcal{E}) = \left(\tilde{L}^q_s(\Omega_\pm, \mathcal{E})\right)^*$.

For any $p \in (1, \infty)$ and $s \in [0, 1]$, the boundary Sobolev space $L^p_s(\partial\Omega)$ can be obtained by using the Euclidean space $L^p_s(\mathbb{R}^{m-1})$, a partition of unity and pull-back, and $L^p_s(\partial\Omega, \mathcal{E}|_{\partial\Omega}) := L^p_s(\partial\Omega) \otimes C^\infty(M, \mathcal{E})|_{\partial\Omega}$.

Now, for any $s \in (0, 1)$, define the spaces of sections

$$\tilde{L}^2_{s-\frac{3}{2}}(\Omega_\pm, \mathcal{E}) := \left\{\mathbf{f} \in L^2_{s-\frac{3}{2}}(M, \mathcal{E}) : \mathrm{supp}\ \mathbf{f} \subseteq \overline{\Omega}_\pm\right\}, \tag{2.32}$$

$$L^2_{s+\frac{1}{2}}(\Omega_\pm, \mathcal{L}_{D,P,P_0}) := \big\{(\mathbf{u}, \pi, \mathbf{f}) : \mathbf{u} \in L^2_{s+\frac{1}{2}}(\Omega_\pm, \mathcal{E}),\ \pi \in L^p_{s-\frac{1}{2}}(\Omega_\pm, G),$$
$$\mathbf{f} \in \tilde{L}^2_{s-\frac{3}{2}}(\Omega_\pm, \mathcal{E}) \ \text{ such that } \mathcal{L}_{D,P,P_0}(\mathbf{u}, \pi) = \mathbf{f}|_{\Omega_\pm} \text{ and } P_0^*\mathbf{u} = 0 \text{ in } \Omega_\pm\big\},$$

$$\mathcal{L}_{D,P,P_0}(\mathbf{u}, \pi) := (\mathcal{L}_D + P)\mathbf{u} + P_0\pi. \tag{2.33}$$

The non-tangential boundary trace operator has the following property (see e.g., [1, 3, 18]):

Lemma 2.2. *For every* $s \in \left(\frac{1}{2}, \frac{3}{2}\right)$*, the restriction operator to the boundary,* $C^\infty(\overline{\Omega}_\pm, \Lambda^1TM) \ni u \mapsto u|_{\partial\Omega_\pm}$*, extends to a linear and bounded operator* $\mathrm{Tr}^\pm : L^2_s(\Omega_\pm, \Lambda^1TM) \to L^2_{s-\frac{1}{2}}(\partial\Omega_\pm, \Lambda^1TM)$*, which is onto, having a bounded right inverse* $\mathcal{Z}^\pm : L^2_{s-\frac{1}{2}}(\partial\Omega_\pm, \Lambda^1TM) \to L^2_s(\Omega_\pm, \Lambda^1TM)$*. For* $s > \frac{3}{2}$*,* $\mathrm{Tr}^\pm : L^2_s(\Omega_\pm, \Lambda^1TM) \to L^2_1(\partial\Omega_\pm, \Lambda^1TM)$ *is also bounded.*

Remark 2.3. Lemma 2.2 can be extended to Sobolev spaces of sections, as: For any $r \in (0, 1)$ the trace operator $\mathrm{Tr}^\pm : L^2_{r+\frac{1}{2}}(\Omega_\pm, \mathcal{E}) \to L^2_r(\partial\Omega, \mathcal{E})$ is bounded and onto, and has a right inverse $\mathcal{Z}^\pm : L^2_r(\partial\Omega, \mathcal{E}) \to L^2_{r+\frac{1}{2}}(\Omega_\pm, \mathcal{E})$, which is bounded as well (see e.g., [3] in the case of one forms).

2.5. The conormal derivative operator on Lipschitz boundaries. Let $r \in [0, 1]$ and $\nu \in L^2_{-r}(\partial\Omega, \Lambda^1TM)$ be the outward unit conormal to $\partial\Omega$, which is defined with respect to the $L^2(\partial\Omega, \Lambda^1TM)$-inner product and the outward unit normal field $\mathbf{n} \in L^\infty(\partial\Omega, TM)$. Note that $\mathbf{n}$ is defined a.e., with respect to the surface element $\mathrm{d}\sigma$, on $\partial\Omega$. The next result extends the notion of the conormal derivative operator, given by Mitrea and Wright [18] for the Stokes system on Lipschitz domains in $\mathbb{R}^n$ to the matrix type operator (2.14) on Sobolev spaces of sections in Riemannian manifolds (see also [3, 9, 10] for the Stokes or Brinkman systems in the context of compact Riemanian manifolds):

Lemma 2.4. *For* $p \in (1, \infty)$ *and* $s \in (0, 1)$*, the conormal derivative operator* $\partial^+_{\nu;D,P,P_0} : L^2_{s+\frac{1}{2}}(\Omega, \mathcal{L}_{D,P,P_0}) \to L^2_{s-1}(\partial\Omega, \mathcal{E})$*, given for any* $\Psi \in L^2_{1-s}(\partial\Omega, \mathcal{E})$ *by*

$$\langle \partial^+_{\nu;D,P,P_0}(\mathbf{u}, \pi, \mathbf{f}), \Psi\rangle_{\partial\Omega} := 2\int_\Omega \langle D\mathbf{u}, D(\mathcal{Z}^+\Psi)\rangle \mathrm{dVol} + \int_\Omega \langle P\mathbf{u}, \mathcal{Z}^+\Psi\rangle \mathrm{dVol}$$
$$+ \int_\Omega \big\langle \pi, P_0^*(\mathcal{Z}^+\Psi)\big\rangle\, \mathrm{dVol} - \langle \mathbf{f}|_\Omega, \mathcal{Z}^+\Psi\rangle_\Omega, \tag{2.34}$$

is well defined and bounded. In addition, for any $(\mathbf{u}, \pi, \mathbf{f}) \in L^2_{s+\frac{1}{2}}(\Omega, \mathcal{L}_{D,P,P_0})$ *and* $\mathbf{w} \in L^2_{\frac{3}{2}-s}(\Omega, \mathcal{E})$*, one has the Green formula:*

$$\langle \partial^+_{\nu;D,P,P_0}(\mathbf{u}, \pi, \mathbf{f}), \mathrm{Tr}^+\ \mathbf{w}\rangle_{\partial\Omega} - 2\int_\Omega \langle D\mathbf{u}, D\mathbf{w}\rangle \mathrm{dVol} - \int_\Omega \langle P\mathbf{u}, \mathbf{w}\rangle \mathrm{dVol} =$$
$$\int_\Omega \langle \pi, P_0^*\mathbf{w}\rangle\, \mathrm{dVol} - \langle \mathbf{f}|_\Omega, \mathbf{w}\rangle_\Omega. \tag{2.35}$$

Proof. Let us observe that all duality pairings in the right-hand side of (2.34) are well defined. This shows that $\partial^+_{\nu;D,P,P_0}(\mathbf{u},\pi,\mathbf{f}) \in L^2_{s-1}(\partial\Omega,\mathcal{E})$ and, in addition, $\|\partial^+_{\nu;D,P,P_0}(\mathbf{u},\pi,\mathbf{f})\|_{L^2_{s-1}(\partial\Omega_\pm,\mathcal{E})} \leq c\|(\mathbf{u},\pi,\mathbf{f})\|_{L^2_{s+\frac{1}{2}}(\Omega,\mathcal{L}_{D,P,P_0})}$ with some constant $c > 0$ and for every $(\mathbf{u},\pi,\mathbf{f}) \in L^2_{s+\frac{1}{2}}(\Omega,\mathcal{L}_{D,P,P_0})$. This shows the well posedness and boundedness of the operator (2.34). The Green formula (2.35) follows with similar arguments to those for [10, Lemma 2.2]. □

Remark 2.5. By considering the Brinkman operator (2.25) and choosing

$$L^2_{s+\frac{1}{2}}(\Omega_\pm,\mathcal{L}_\mathcal{P}) := \big\{(\mathbf{u},\pi) \in L^2_{s+\frac{1}{2}}(\Omega_\pm,\Lambda^1TM)\times L^p_{s-\frac{1}{2}}(\Omega_\pm) : \mathcal{L}_\mathcal{P}(\mathbf{u},\pi) = \mathbf{0} \text{ and } \delta\mathbf{u} = 0 \text{ in } \Omega_\pm\big\}, \tag{2.36}$$

one obtains the conormal derivative $\partial^\pm_{\nu;\mathcal{P}} : L^2_{s+\frac{1}{2}}(\Omega_\pm,\mathcal{L}_\mathcal{P}) \to L^2_{s-1}(\partial\Omega,\Lambda^1TM)$. In addition, the Green formula (2.35) becomes (see [10])

$$\pm\langle\partial^\pm_{\nu;\mathcal{P}}(\mathbf{u},\pi),\mathrm{Tr}^\pm\,\mathbf{w}\rangle_{\partial\Omega} = 2\int_{\Omega_\pm}\langle\mathrm{Def}\,\mathbf{u},\mathrm{Def}\,\mathbf{w}\rangle\mathrm{dVol} + \int_{\Omega_\pm}\langle\mathcal{P}\mathbf{u},\mathbf{w}\rangle\mathrm{dVol} + \int_{\Omega_\pm}\langle\pi,\delta\mathbf{w}\rangle\,\mathrm{dVol},\ \ \mathbf{w}\in L^2_{1-s}(\Omega_\pm,\Lambda^1TM) \tag{2.37}$$

for any $(\mathbf{u},\pi) \in L^2_{s+\frac{1}{2}}(\Omega_\pm,\mathcal{L}_\mathcal{P})$, where $\mathcal{L}_\mathcal{P}(u,\pi) := (\mathfrak{L}+\mathcal{P})u + d\pi$. From now on we will use the notation $\partial^\pm_\nu$ instead of $\partial^\pm_{\nu;\mathcal{P}}$ whenever the operator $B_\mathcal{P}$ is involved.

3. LAYER POTENTIAL THEORY FOR THE OPERATOR B^0_{D,P,P_0}

Let us now present the main properties of the layer potentials associated to the pseudodifferential matrix operator B^0_{D,P,P_0} given by (2.16). As in the previous section, $\Omega \subset M$ is a Lipschitz domain. Also let $\mathbf{f} : \partial\Omega \to \mathcal{E}$ and $\mathbf{h} : \partial\Omega \to \mathcal{E}$ be given sections. Then one defines the layer potentials $\mathbf{V}_{D,P,P_0;\partial\Omega}f : M\setminus\partial\Omega \to \mathcal{E}$ and $\mathcal{Q}^s_{D,P,P_0;\partial\Omega}\mathbf{f} : M\setminus\partial\Omega \to G$ as

$$\left.\begin{aligned}(\mathbf{V}_{D,P,P_0;\partial\Omega}\mathbf{f})(x) &:= \int_{\partial\Omega}\langle\mathcal{G}_{D,P,P_0}(x,y),\mathbf{f}(y)\rangle_y\mathrm{d}\sigma(y)\\(\mathcal{Q}^s_{D,P,P_0;\partial\Omega}\mathbf{f})(x) &:= \int_{\partial\Omega}\langle\Pi_{D,P,P_0}(x,y),\mathbf{f}(y)\rangle_y\mathrm{d}\sigma(y)\end{aligned}\right\},\ x\in M\setminus\partial\Omega,$$

where $\mathbf{V}_{D,P,P_0;\partial\Omega}\mathbf{f}$ is the *single-layer potential* with density $\mathbf{f}$.

Let $\mathbf{W}_{D,P,P_0;\partial\Omega}\mathbf{h} : M\setminus\partial\Omega \to \mathcal{E}$ and $\mathcal{Q}^d_{D,P,P_0;\partial\Omega}\mathbf{h} : M\setminus\partial\Omega \to G$ be the layer potentials given on $M\setminus\partial\Omega$ by

$$\mathbf{W}_{D,P,P_0;\partial\Omega}\mathbf{h} := \int_{\partial\Omega}\Big\langle\partial_{\nu_y;D,P,P_0}\Big(\mathcal{G}_{D,P,P_0}(\cdot,y),(\Pi_{D,P,P_0})^\top(y,\cdot)\Big),\mathbf{h}(y)\Big\rangle_y\mathrm{d}\sigma,$$

$$\mathcal{Q}^d_{D,P,P_0;\partial\Omega}\mathbf{h} := \int_{\partial\Omega}\big\langle\partial_{\nu_y;D,P,P_0}\left(\Pi_{D,P,P_0}(\cdot,y),-\Xi_{D,P,P_0}(\cdot,y)\right),\mathbf{h}(y)\big\rangle_y\mathrm{d}\sigma,$$

where $\partial_{\nu;D,P,P_0} := \partial^+_{\nu;D,P,P_0}$. Also, $\Xi_{D,P,P_0}(x,y)$ is the Schwartz kernel of the operator $\left(-\mathcal{D}_{D,P,P_0}\right)^\top \in OPS^0_{\mathrm{cl}}(G,G)$, and satisfies the relation (see [9, 10] in the context of one forms)

$$(\mathcal{L}_{D,P})_x \left(\Pi_{D,P,P_0}\right)^\top (y,x) = (P_0)_x \Xi_{D,P,P_0}(x,y). \tag{3.1}$$

Note that $\mathbf{W}_{D,P,P_0;\partial\Omega}\mathbf{h} : M \setminus \partial\Omega \to \mathcal{E}$ is called the *double-layer potential* with the density $\mathbf{h}$. The corresponding principal value version is denoted by $\mathbf{K}_{D,P,P_0;\partial\Omega}\mathbf{h}$ and is given a.e. on $\partial\Omega$ by

$$\mathbf{K}_{D,P,P_0;\partial\Omega}\mathbf{h} := \text{p.v.} \int_{\partial\Omega} \left\langle \partial_{\nu_y;D,P,P_0} \left(\mathcal{G}_{D,P,P_0}(\cdot,y), (\Pi_{D,P,P_0})^\top(y,\cdot) \right), f(y) \right\rangle_y \mathrm{d}\sigma.$$

In view of (2.19), $\left(\mathbf{V}_{D,P,P_0;\partial\Omega}\mathbf{f}, \mathcal{Q}^s_{D,P,P_0;\partial\Omega}\mathbf{f}\right)$ satisfies on $M \setminus \partial\Omega$ the equations

$$(\mathcal{L}_D + P)\, \mathbf{V}_{D,P,P_0;\partial\Omega}\mathbf{f} + P_0 \mathcal{Q}^s_{D,P,P_0;\partial\Omega}\mathbf{f} = 0,\ P_0^*(\mathbf{V}_{D,P,P_0;\partial\Omega}\mathbf{f}) = 0. \tag{3.2}$$

Similarly, by (2.19) and (3.1) one obtains on $M \setminus \partial\Omega$ the equations

$$(\mathcal{L}_D + P)\mathbf{W}_{D,P,P_0;\partial\Omega}\mathbf{h} + P_0 \mathcal{Q}^d_{D,P,P_0;\partial\Omega}\mathbf{h} = 0,\ P_0^*\mathbf{W}_{D,P,P_0;\partial\Omega}\mathbf{h} = 0. \tag{3.3}$$

Now, using the theory developed in [15] (see also the corresponding results for the Stokes system in [17, Proposition 3.3, Theorem 3.1], [3, Theorem 2.1]), one obtains the following property:

Theorem 3.1. *Let $\Omega \subset M$ be a Lipschitz domain. Also let $s \in [0,1]$. If $\mathbf{h} \in L^2_s(\partial\Omega,\mathcal{E})$ and $\mathbf{f} \in L^2_{s-1}(\partial\Omega,\mathcal{E})$, then one has a.e. on $\partial\Omega$*

$$\mathrm{Tr}^+(\mathbf{V}_{D,P,P_0;\partial\Omega} f) = \mathrm{Tr}^-(\mathbf{V}_{D,P,P_0;\partial\Omega}\mathbf{f}) := \mathcal{V}_{D,P,P_0;\partial\Omega}\mathbf{f}, \tag{3.4}$$

$$\mathrm{Tr}^\pm(\mathbf{W}_{D,P,P_0;\partial\Omega}\mathbf{h}) = \Big(\pm \frac{1}{2}\mathbb{I} + \mathbf{K}_{D,P,P_0;\partial\Omega} \Big)\mathbf{h}, \tag{3.5}$$

$$\partial^\pm_{\nu;D,P,P_0}(\mathbf{V}_{D,P,P_0;\partial\Omega}\mathbf{f}, \mathcal{Q}^s_{D,P,P_0;\partial\Omega}\mathbf{f}) = \Big(\mp \frac{1}{2}\mathbb{I} + \mathbf{K}^*_{D,P,P_0;\partial\Omega} \Big)\mathbf{f}, \tag{3.6}$$

$$\mathcal{H}^+_{D,P,P_0;\partial\Omega}\mathbf{h} - \mathcal{H}^-_{D,P,P_0;\partial\Omega}\mathbf{h} \in \mathrm{Ker}\mathcal{V}_{D,P,P_0;\partial\Omega}, \tag{3.7}$$

where $\mathbf{K}^*_{D,P,P_0;\partial\Omega}$ is the formal transpose of $\mathbf{K}_{D,P,P_0;\partial\Omega}$, and

$$\mathcal{H}^\pm_{D,P,P_0;\partial\Omega} := \partial^\pm_{\nu;D,P,P_0}(\mathbf{W}_{D,P,P_0;\partial\Omega}, \mathcal{Q}^d_{D,P,P_0;\partial\Omega}).$$

3.1. Single- and double-layer potentials for the Brinkman operator. We now refer to the Brinkman operator $B_{\mathcal{P}}$ given by (2.26). With respect to the one forms $\mathbf{f} \in L^2_{r-1}(\partial\Omega, \Lambda^1 TM)$ and $\mathbf{h} \in L^2_r(\partial\Omega, \Lambda^1 TM)$, $r \in [0,1]$, the associated single- and double-layer potentials are given a.e. on $M \setminus \partial\Omega$ by

$$\begin{aligned}
&\mathbf{V}_{\mathcal{P};\partial\Omega}\mathbf{f} := \int_{\partial\Omega} \langle \mathcal{G}_{\mathcal{P}}(\cdot,y), \mathbf{f}(y) \rangle \mathrm{d}\sigma(y),\ \mathcal{Q}^s_{\mathcal{P};\partial\Omega}\mathbf{f} := \int_{\partial\Omega} \langle \Pi_{\mathcal{P}}(\cdot,y), \mathbf{f}(y) \rangle \mathrm{d}\sigma(y),\\
&\mathbf{W}_{\mathcal{P};\partial\Omega}\mathbf{h} := \int_{\partial\Omega} \langle \Pi^\top_{\mathcal{P}}(y,\cdot)\nu(y) - 2\mathrm{Def}_y\, \mathcal{G}_{\mathcal{P}}(\cdot,y)\nu(y), \mathbf{h}(y) \rangle \mathrm{d}\sigma(y),\\
&\mathcal{Q}^{\mathrm{d}}_{\mathcal{P};\partial\Omega}\mathbf{h} := \int_{\partial\Omega} \langle -2\mathrm{Def}_y\, \Pi_{\mathcal{P}}(\cdot,y)\nu(y) - \Xi_{\mathcal{P}}(x,y)\nu(y), \mathbf{h}(y) \rangle \mathrm{d}\sigma(y),
\end{aligned} \tag{3.8}$$

where $\Xi_{\mathcal{P}}(x,y)$ is the Schwartz kernel of the operator $(-\mathfrak{D}_{\mathcal{P}})^{\top} \in OPS^0_{\mathrm{cl}}(M,\mathbb{R})$, and satisfies the relation (see [9, 10])

$$(\mathfrak{L}_{\mathcal{P}})_x (\Pi_{\mathcal{P}})^{\top}(y,x) = d_x \Xi_{\mathcal{P}}(x,y). \tag{3.9}$$

These layer potentials satisfy the pseudodifferential equations

$$\begin{array}{l}(\mathfrak{L}+\mathcal{P})\mathbf{V}_{P;\partial\Omega}\mathbf{f} + \mathrm{d}\mathcal{Q}^s_{P;\partial\Omega}\mathbf{f} = 0,\ \delta(\mathbf{V}_{P;\partial\Omega}\mathbf{f}) = 0\\ (\mathfrak{L}+\mathcal{P})\mathbf{W}_{P;\partial\Omega}\mathbf{h} + \mathrm{d}\mathcal{Q}^d_{P;\partial\Omega}\mathbf{h} = 0,\ \delta\mathbf{W}_{P;\partial\Omega}\mathbf{h} = 0,\end{array} \quad \text{on } M\setminus\partial\Omega. \tag{3.10}$$

Also, the principal value of $\mathbf{W}_{P;\partial\Omega}\mathbf{h}$ is given a.e. on $M\setminus\partial\Omega$ by

$$\mathbf{K}_{\mathcal{P};\partial\Omega}\mathbf{h} := \mathrm{p.v.}\int_{\partial\Omega} \big\langle \Pi^{\top}_{\mathcal{P}}(y,\cdot)\nu(y) - 2\mathrm{Def}_y\, \mathcal{G}_{\mathcal{P}}(\cdot,y)\nu(y), \mathbf{h}(y)\big\rangle \mathrm{d}\sigma(y). \tag{3.11}$$

In addition, the relations (3.4)-(3.7) remain valid for the layer potential operators associated to the Brinkman operator $B_{\mathcal{P}}$, i.e., one has a.e. on $\partial\Omega$ (see [10], [17, Proposition 3.3, Theorem 3.1], [3, Theorem 2.1])
(3.12)

$$\mathrm{Tr}^+(\mathbf{V}_{\mathcal{P};\partial\Omega}f) = \mathrm{Tr}^-(\mathbf{V}_{\mathcal{P};\partial\Omega}\mathbf{f}) = \mathcal{V}_{\mathcal{P};\partial\Omega}\mathbf{f},\ \mathrm{Tr}^{\pm}(\mathbf{W}_{\mathcal{P};\partial\Omega}\mathbf{h}) = \Big(\pm\tfrac{1}{2}\mathbb{I} + \mathbf{K}_{\mathcal{P};\partial\Omega}\Big)\mathbf{h},$$
$$\partial^{\pm}_{\nu;\mathcal{P}}(\mathbf{V}_{\mathcal{P};\partial\Omega}\mathbf{f}, \mathcal{Q}^s_{\mathcal{P};\partial\Omega}\mathbf{f}) := \Big(\mp\tfrac{1}{2}\mathbb{I} + \mathbf{K}^*_{\mathcal{P};\partial\Omega}\Big)\mathbf{f},\ \mathcal{H}^+_{\mathcal{P};\partial\Omega}\mathbf{h} - \mathcal{H}^-_{\mathcal{P};\partial\Omega}\mathbf{h} \in \mathbb{R}\nu,$$

where $\mathcal{H}^{\pm}_{\mathcal{P};\partial\Omega}\mathbf{h} := \partial^{\pm}_{\nu}(\mathbf{W}_{\mathcal{P};\partial\Omega}\mathbf{h}, \mathcal{Q}^d_{\mathcal{P};\partial\Omega}\mathbf{h})$.

3.2. Compactness of the complementary layer potential operators associated to B^0_{D,P,P_0} on the sphere S^m. One of the main results of the layer potential theory is the compactness of the complementary layer potential operators. We show this result in the case of the m-dimensional unit sphere S^m.

THEOREM 3.2. *Let $\Omega\subset S^m$ be a Lipschitz domain. Then for any $s\in(0,1)$ the following complementary layer potential operators are compact:*

$$\begin{array}{l}\mathcal{V}_{D,P,0,P_0;\partial\Omega} := \mathcal{V}_{D,P,P_0;\partial\Omega} - \mathcal{V}_{D,0,P_0;\partial\Omega} : L^2_{s-1}(\partial\Omega,\Lambda^1TS^m)\to L^2_s(\partial\Omega,1TS^m),\\ \mathbf{K}_{D,P,0,P_0;\partial\Omega} := \mathbf{K}_{D,P,P_0;\partial\Omega} - \mathbf{K}_{D,0,P_0;\partial\Omega} : L^2_s(\partial\Omega,\Lambda^1TS^m)\to L^2_s(\partial\Omega,\Lambda^1TS^m),\\ \mathbf{K}^*_{D,P,0,P_0;\partial\Omega} : L^2_{-s}(\partial\Omega,\Lambda^1TS^m)\to L^2_{-s}(\partial\Omega,\Lambda^1TS^m),\\ \mathcal{H}_{D,P,0,P_0;\partial\Omega} := \mathcal{H}_{D,P,P_0;\partial\Omega} - \mathcal{H}_{D,0,P_0;\partial\Omega} : L^2_s(\partial\Omega,\Lambda^1TS^m)\to L^2_{s-1}(\partial\Omega,\Lambda^1TS^m).\end{array}$$

Proof. First, one shows that for any $r\in\big(\frac{1}{2},1\big]$, the complementary single-layer potential operator $\mathcal{V}_{D,P,0,P_0;\partial\Omega} : L^2_{\frac{1}{2}-r}(\partial\Omega,\Lambda^1TS^m)\to L^2_{\frac{3}{2}-s}(\partial\Omega,\Lambda^1TS^m)$ is compact. To this aim, note that this operator can be written as

$$\mathcal{V}_{D,P,0,P_0;\partial\Omega} = i_{L^2_1(\partial\Omega,\Lambda^1TS^m),L^2_{\frac{3}{2}-s}(\partial\Omega,\Lambda^1TS^m)} \circ \big(\mathbf{Tr}\circ\tilde{\mathcal{V}}_{D,P,0,P_0}\circ\mathbf{Tr}^*\big), \tag{3.13}$$

where $\mathbf{Tr} : L^2_r(M,\Lambda^1TS^m)\to L^2_{r-\frac{1}{2}}(\partial\Omega,\Lambda^1TS^m)$ is the non-tangential trace operator acting on one forms and $\mathbf{Tr}^* : L^2_{\frac{1}{2}-r}(\partial\Omega,\Lambda^1TS^m)\to L^2_{-r}(M,\Lambda^1TS^m)$ is its adjoint. These operators are bounded for any $r\in(\frac{1}{2},1]$ (see [13]). In addition, $\tilde{\mathcal{V}}_{D,P,0,P_0}$ is the complementary Newtonian potential operator, i.e.,

$(\tilde{\mathcal{V}}_{D,P,0,P_0}\mathbf{u})(x) := \langle (\mathcal{G}_{D,P,0,P_0}(x,\cdot)), \mathbf{u}\rangle_{L^2(M,TS^m)}$, and $\mathcal{G}_{D,P,0,P_0} := \mathcal{G}_{D,P,P_0} - \mathcal{G}_{D,0,P_0}$. Note that $\tilde{\mathcal{V}}_{D,P,0,P_0} \in OPS^{-4}_{\rm cl}(M, \Lambda^1 TS^m)$, as follows from (2.22). Also, $i_{L^2_1(\partial\Omega,\Lambda^1TS^m), L^2_{\frac{3}{2}-r}(\partial\Omega,\Lambda^1TS^m)}$ is the compact imbedding operator of the space $L^2_1(\partial\Omega, \Lambda^1 TS^m)$ into $L^2_{\frac{3}{2}-r}(\partial\Omega, \Lambda^1 TS^m)$. Consequently, the operator (3.13) is compact for any $r \in (\frac{1}{2}, 1]$. Moreover, by using an extrapolation result of Cwikel [2] about compactness on complex interpolation scales of Banach spaces, we conclude that the complementary single-layer potential operator $\mathcal{V}_{D,P,0,P_0;\partial\Omega} : L^2_{\frac{1}{2}-r}(\partial\Omega, \Lambda^1 TS^m) \to L^2_{\frac{3}{2}-s}(\partial\Omega, \Lambda^1 TS^m)$ is compact as well, for any $r \in \left[\frac{1}{2}, \frac{3}{2}\right]$. The compactness of the other complementary layer potential operators can be similarly obtained. For brevity, we omit the details, but they will be given in a forthcoming paper. □

3.3. Compactness of complementary layer potential operators associated to the pseudodifferential Brinkman operator $B_{\mathcal{P}}$. Next, we show the compactness of the complementary layer potential operators associated to the Brinkman and Stokes operators $B_{\mathcal{P}}$ and B_0, when $\mathcal{P} = \lambda \mathbb{I}$ and $\lambda > 0$ is a constant. The more general case corresponding to an operator $\mathcal{P} \in OPS^0_{\rm cl}(M, \Lambda^1 TM)$ of the form $V\mathbb{I}$, where $V \in C^\infty(M)$ is an arbitrary non-negative function, will be treated in a forthcoming paper. For brevity, we replace the subscript $\mathcal{P}$ by λ, and obtain (see also the compactness results in [9, 10] obtained for $m = 2, 3$):

Theorem 3.3. *If M is a boundaryless compact Riemannian manifold of dimension $m \geq 2$, $\Omega \subset M$ is a Lipschitz domain and $\lambda > 0$ is a given constant, then for any $s \in [0, 1]$ the following layer potential operators are compact:*

(a) *The complementary single- and double-layer potential operators*

$$\begin{aligned} &\mathcal{V}_{\lambda,0;\partial\Omega} := \mathcal{V}_{\lambda;\partial\Omega} - \mathcal{V}_{0;\partial\Omega} : L^2_{s-1}(\partial\Omega, \Lambda^1 TM) \to L^2_s(\partial\Omega, \Lambda^1 TM),\\ &\mathbf{K}_{\lambda,0;\partial\Omega} := \mathbf{K}_{\lambda;\partial\Omega} - \mathbf{K}_{0;\partial\Omega} : L^2_s(\partial\Omega, \Lambda^1 TM) \to L^2_s(\partial\Omega, \Lambda^1 TM),\\ &\mathbf{K}_{\lambda,0;\partial\Omega} : L^2_{s;\nu}(\partial\Omega, \Lambda^1 TM) \to L^2_{s;\nu}(\partial\Omega, \Lambda^1 TM), \end{aligned} \tag{3.14}$$

where $L^2_{s;\nu}(\partial\Omega, \Lambda^1 TM) := \{\mathbf{h} \in L^2_s(\partial\Omega, \Lambda^1 TM) : \langle \nu, \mathbf{h}\rangle_{L^2(\partial\Omega)} = 0\}$.

(b) *The adjoint of the complementary layer potential operator*

$$\mathbf{K}^*_{\lambda,0;\partial\Omega} := \mathbf{K}^*_{\lambda;\partial\Omega} - \mathbf{K}^*_{0;\partial\Omega} : L^2_{s-1}(\partial\Omega, \Lambda^1 TM) \to L^2_{s-1}(\partial\Omega, \Lambda^1 TM) \tag{3.15}$$

(c) *The complementary hypersingular layer potential operator*

$$\mathcal{H}_{\lambda,0;\partial\Omega} := \mathcal{H}_{\lambda;\partial\Omega} - \mathcal{H}_{0;\partial\Omega} : L^2_s(\partial\Omega, \Lambda^1 TM) \to L^2_{s-1}(\partial\Omega, \Lambda^1 TM). \tag{3.16}$$

Proof. We use a localization technique due to Mitrea et al. in [15, Chapter 10]. To this aim, let $\{U_j : j = 1, \ldots, N\}$ be an open, finite covering of $\partial\Omega$ with domains of coordinate charts in M, each of them being homeomorphic with the unit ball in $\mathbb{R}^m$. Let us embed isometrically U_j in a compact boundaryless Riemannian manifold M_j of dimension m, $j = 1, \ldots, N$. Each M_j can be obtained by taking two copies of U_j with opposite orientation and gluing them together along their boundaries. The result is an m-dimensional (possibly exotic) sphere. In addition, for each $j = 1, \dots, N$, select a Lipschitz domain Ω_j in M_j such that $\{\partial\Omega_j \cap \partial\Omega : j = 1, \dots, N\}$ is an open covering of $\partial\Omega$.

Next, for each j, we may define a pseudodifferential Brinkman-type operator $B_\lambda^{(j)} : C^\infty(M_j, \Lambda^1 TM_j) \times C^\infty(M_j) \to C^\infty(M_j, \Lambda^1 TM_j) \times C^\infty(M_j)$, as in (2.25). Indeed, we may choose a suitable Riemannian structure on M_j to avoid the non-trivial Killing vector fields. Thus, we may construct the corresponding complementary layer potential operators, and, by Theorem 3.2, they are compact. For example, the complementary single-layer potential operator

$$\mathcal{V}^{(j)}_{\lambda,0;\partial\Omega_j} := \mathcal{V}^{(j)}_{\lambda;\partial\Omega_j} - \mathcal{V}^{(j)}_{0;\partial\Omega_j} : L^2_{s-1}(\partial\Omega_j, \Lambda^1 TM_j) \to L^2_s(\partial\Omega_j, \Lambda^1 TM_j) \tag{3.17}$$

is compact, for any $s \in [0, 1]$. Let $\{\xi_j : j = 1, \dots, N\}$ be a partition of unity, by Lipschitz functions, which is subordinated to the covering $\{U_j : j = 1, \dots, N\}$ of $\partial\Omega$ and satisfies the relations $\partial\Omega \cap \operatorname{supp} \xi_j \subseteq \partial\Omega_j$ for each j. Using these data, one obtains the following decomposition of the complementary single-layer potential operator $\mathcal{V}_{\lambda,0;\partial\Omega}$:

$$\mathcal{V}_{\lambda,0;\partial\Omega}\mathbf{f} = \sum_{j=1}^{N}\sum_{k=1}^{N} \xi_k|_{\partial\Omega\cap\partial\Omega_k} \mathcal{V}^{(j)}_{\lambda,0;\partial\Omega_j}(\xi_j \mathbf{f})|_{\partial\Omega\cap\partial\Omega_j}. \tag{3.18}$$

Since the compactness of a linear and bounded operator on a Banach space is equivalent to its sequential compactness, consider a bounded sequence $\{\mathbf{\Phi}_n\}$ in $L^2_{s-1}(\partial\Omega, \Lambda^1 TM)$, which determines the bounded sequences $\{(\xi_j \mathbf{\Phi}_n)|_{\partial\Omega\cap\partial\Omega_j}\}$ in $L^2_{s-1}(\partial\Omega_j, \Lambda^1 TM_j)$, for each j. In view of the compactness of the operator $\mathcal{V}^{(j)}_{\lambda,0;\partial\Omega_j}$ on $L^2_{s-1}(\partial\Omega_j, \Lambda^1 TM_j)$, we get a subsequence $\{\mathbf{\Phi}_{n_k}\}$ of $\{\mathbf{\Phi}_n\}$ such that $\{\mathcal{V}^{(j)}_{\lambda,0;\partial\Omega_j}(\xi_j \mathbf{\Phi}_{n_k})|_{\partial\Omega\cap\partial\Omega_j}\}$ converges to an element $\mathbf{\Phi}^{(j)} \in L^2_s(\partial\Omega_j, \Lambda^1 TM_j)$, for each j. Finally, in view of (3.18), one finds that the sequence $\{\mathcal{V}_{\lambda,0;\partial\Omega}\mathbf{\Phi}_{n_k}\}$ converges to $\mathbf{\Phi} := \sum_{j=1}^{N}\sum_{k=1}^{N} \xi_k|_{\partial\Omega\cap\partial\Omega_k}\mathbf{\Phi}^{(j)} \in L^2_s(\partial\Omega, \Lambda^1 TM)$. This shows the compactness of $\mathcal{V}_{\lambda,0;\partial\Omega} : L^2_{s-1}(\partial\Omega, \Lambda^1 TM) \to L^2_s(\partial\Omega, \Lambda^1 TM)$. The compactness of the other complementary layer potential operators in (3.14)-(3.16) can be similarly obtained. □

3.4. Invertible layer potential operators for the Brinkman system.

Let us mention the following useful invertibility property (see also [6, 18]):

THEOREM 3.4. *Under the hypothesis of Theorem* 3.3, *the operators*

$$\tilde{\mathbf{K}}^{\pm}_{\lambda;\partial\Omega;\mu} := \mp\frac{1}{2}\frac{1+\mu}{1-\mu}\mathbb{I} + \mathbf{K}_{\lambda;\partial\Omega} : L^2_{s;\nu}(\partial\Omega, \Lambda^1 TM) \to L^2_{s;\nu}(\partial\Omega, \Lambda^1 TM) \tag{3.19}$$

are invertible for any $\mu \in (0, 1)$ *and* $s \in \{0, 1\}$.

Proof. First, we show the Fredholm and zero index properties of (3.19) on $L^2(\partial\Omega, \Lambda^1 TM)$. For this aim, note that $\tilde{\mathbf{K}}^{\pm}_{\lambda;\partial\Omega;\mu} = \tilde{\mathbf{K}}^{\pm}_{0;\partial\Omega;\mu} + \mathbf{K}_{\lambda,0;\partial\Omega}$. By Theorem 3.3, the operators $\mathbf{K}_{\lambda,0;\partial\Omega} : L^2(\partial\Omega, \Lambda^1 TM) \to L^2(\partial\Omega, \Lambda^1 TM)$

and $\mathbf{K}_{\lambda,0;\partial\Omega} : L^2_1(\partial\Omega, \Lambda^1TM) \to L^2_1(\partial\Omega, \Lambda^1TM)$ are compact. Note that the operators

$$(3.20)\qquad \tilde{\mathbf{K}}^{\pm}_{0;\partial\Omega;\mu} := \mp\frac{1}{2}\frac{1+\mu}{1-\mu}\mathbb{I} + \mathbf{K}_{0;\partial\Omega} : L^2(\partial\Omega, \Lambda^1TM) \to L^2(\partial\Omega, \Lambda^1TM),$$

corresponding to the Stokes system ($\lambda = 0$), are bounded from below modulo compact operators, as we will show in a forthcoming paper by using the localization technique developed in [15, Chapter 10]. This means that there exists a constant $C > 0$ such that

$$\|\mathbf{f}\|_{L^2(\partial\Omega,\Lambda^1TM)} \le C\|\tilde{\mathbf{K}}^{\pm}_{0;\partial\Omega;\mu}\mathbf{f}\|_{L^2(\partial\Omega,\Lambda^1TM)} + \|\mathrm{Comp}_{\pm}(\mathbf{f})\|,$$

i.e., $\tilde{\mathbf{K}}^{\pm}_{0;\partial\Omega;\mu} : L^2(\partial\Omega, \Lambda^1TM) \to L^2(\partial\Omega, \Lambda^1TM)$ are semi-Fredholm. For μ sufficiently close to 1, they are invertible, by means of a Neumann series. Then, combining this semi-Fredholm property with the homotopic invariance of the index, we conclude that these operators are Fredholm with index zero for any $\mu \in (0, 1)$ (see [18, Corollary 11.38]). By using [18, Corollary 11.38], we conclude that the operators

$$(3.21)\qquad \tilde{\mathbf{K}}^{\pm}_{0;\partial\Omega;\mu} := \mp\frac{1}{2}\frac{1+\mu}{1-\mu}\mathbb{I} + \mathbf{K}_{0;\partial\Omega} : L^2_\nu(\partial\Omega, \Lambda^1TM) \to L^2_\nu(\partial\Omega, \Lambda^1TM)$$

are Fredholm with index zero as well.

Next, we show the Fredholm and zero index properties of the operators

$$(3.22)\quad \tilde{\mathbf{K}}^{\pm}_{0;\partial\Omega;\mu} := \mp\frac{1}{2}\frac{1+\mu}{1-\mu}\mathbb{I} + \mathbf{K}_{0;\partial\Omega} : L^2_{1;\nu}(\partial\Omega, \Lambda^1TM) \to L^2_{1;\nu}(\partial\Omega, \Lambda^1TM).$$

This property follows from the relation (see e.g. [16, (7.41)])

$$\pm\frac{1}{2}\frac{1+\mu}{1-\mu}\mathbb{I} + \mathbf{K}_{0;\partial\Omega} = \mathcal{V}_{0;\partial\Omega}\left(\pm\frac{1}{2}\frac{1+\mu}{1-\mu}\mathbb{I} + \mathbf{K}^*_{0;\partial\Omega}\right)\mathcal{V}^{-1}_{0;\partial\Omega} \text{ on } L^2_{1,\nu}(\partial\Omega, \Lambda^1TM),$$

the invertibility of $\mathcal{V}_{0;\partial\Omega} : L^2_\nu(\partial\Omega, \Lambda^1TM) \to L^2_{1,\nu}(\partial\Omega, \Lambda^1TM)$ (see [17, Theorem 6.1]) and and the fact that the operators

$$\mathcal{V}_{0;\partial\Omega} : L^2(\partial\Omega, \Lambda^1TM) \to L^2_{1,\nu}(\partial\Omega, \Lambda^1TM),$$

$$\pm\frac{1}{2}\frac{1+\mu}{1-\mu}\mathbb{I} + \mathbf{K}^*_{0;\partial\Omega} : L^2_\nu(\partial\Omega, \Lambda^1TM) \to L^2(\partial\Omega, \Lambda^1TM)$$

are Fredholm of opposite index. It remains to show that the operators (3.19) are injective. Let $\mathbf{h} \in \mathrm{Ker}\big(\tilde{\mathbf{K}}^{+}_{\lambda;\partial\Omega;\mu} : L^2_{1;\nu}(\partial\Omega, \Lambda^1TM) \to L^2_{1;\nu}(\partial\Omega, \Lambda^1TM)\big)$. By repeated applications of the Green formulas (2.37) to the double-layer potential $\mathbf{W}_{\lambda;\partial\Omega}\mathbf{h}$ and its associated pressure potential $\mathcal{Q}^d_{\lambda;\partial\Omega}\mathbf{h}$, one obtains that $\mathbf{h} = \mathbf{0}$, i.e., the operator $\tilde{\mathbf{K}}^{+}_{\lambda;\partial\Omega;\mu} : L^2_{1;\nu}(\partial\Omega, \Lambda^1TM) \to L^2_{1;\nu}(\partial\Omega, \Lambda^1TM)$ is indeed injective. The injectivity of $\tilde{\mathbf{K}}^{+}_{\lambda;\partial\Omega;\mu} : L^2_\nu(\partial\Omega, \Lambda^1TM) \to L^2_\nu(\partial\Omega, \Lambda^1TM)$ follows from [18, Lemma 11.40] and the density of the continuous imbedding $L^2_{1;\nu}(\partial\Omega, \Lambda^1TM) \hookrightarrow L^2_\nu(\partial\Omega, \Lambda^1TM)$. For brevity we omit the details, but they will be given in a forthcoming paper. This completes the proof. $\square$

4. APPLICATIONS OF THE LAYER POTENTIAL THEORY

In this section we apply the layer potential theory developed in Section 3 to show the well-posedness of some transmission problems for the Stokes Brinkman operator B_0 and B_λ, $\lambda > 0$, on Lipschitz domains on a compact Riemannian manifold of dimension ≥ 2, with boundary data in L^2 spaces. Recall that $\mathfrak{L} = 2\mathrm{Def}^*\mathrm{Def}$ and $\mathcal{N}$ is the non-tangential maximal operator (see (2.23) and (2.28)). First, we show the following well-posedness result:

THEOREM 4.1. *Let M be a compact boundaryless Riemannian manifold, $\dim(M) \geq 2$, $\Omega_+ := \Omega \subset M$ be a Lipschitz domain and $\Omega_- := M \setminus \overline{\Omega}$. Also let $\lambda > 0$ be a given constant. Then for any $\mu \in (0,1)$ the transmission problem*[2]

$$\left\{\begin{array}{l} \delta \mathbf{u}_+ = 0,\ \mathfrak{L}\mathbf{u}_+ + \lambda \mathbf{u}_+ + d\pi_+ = 0 \ \textit{in}\ \Omega_+, \\ \delta \mathbf{u}_- = 0,\ \mathfrak{L}\mathbf{u}_- + \lambda \mathbf{u}_- + d\pi_- = 0 \ \textit{in}\ \Omega_-, \\ \mathcal{N}(\nabla \mathbf{u}_\pm) \in L^2(\partial\Omega),\ \mathcal{N}(\pi_\pm) \in L^2(\partial\Omega), \\ \mu \mathrm{Tr}^+\mathbf{u}_+ - \mathrm{Tr}^-\mathbf{u}_- = \mathcal{U} \in L^2_{1;\nu}(\partial\Omega, \Lambda^1 TM) \ \textit{on}\ \partial\Omega, \\ \partial_\nu^+(\mathbf{u}_+, \pi_+) - \partial_\nu^-(\mathbf{u}_-, \pi_-) = \mathcal{F} \in L^2(\partial\Omega, \Lambda^1 TM) \ \textit{on}\ \partial\Omega, \end{array}\right. \tag{4.1}$$

is well-posed, i.e., it has a unique solution

$$((\mathbf{u}_+, \pi_+), (\mathbf{u}_-, \pi_-)) \in (C^2(\Omega_+, \Lambda^1 TM) \times C^1(\Omega_+)) \times (C^2(\Omega_-, \Lambda^1 TM) \times C^1(\Omega_-))$$

(*up to a constant pressure*), *and there exists a constant $c > 0$ such that*

$$\|\mathcal{N}(\nabla \mathbf{u}_\pm)\|_{L^2(\partial\Omega)} + \|\mathcal{N}(\pi_\pm)\|_{L^2(\partial\Omega)} \leq c\big(\|\mathcal{U}\|_{L^2_1(\partial\Omega,\Lambda^1 TM)} + \|\mathcal{F}\|_{L^2(\partial\Omega,\Lambda^1 TM)}\big). \tag{4.2}$$

Proof. Let us consider the layer potentials

$$\mathbf{u}_\pm = \mathbf{W}_{\lambda;\partial\Omega}\mathbf{h} + \mathbf{V}_{P;\partial\Omega}\mathbf{f},\ \pi_\pm = \mathcal{Q}_{\lambda;\partial\Omega}\mathbf{h} + \mathcal{Q}_{P;\partial\Omega}\mathbf{f} \text{ in } \Omega_\pm, \tag{4.3}$$

with the unknown densities $\mathbf{h} \in L^2_{1;\nu}(\partial\Omega, \Lambda^1 TM)$ and $\mathbf{f} \in L^2(\partial\Omega, \Lambda^1 TM)$. In view of (3.10), these layer potentials satisfy the Brinkman equations of (4.1). In addition, the general theory developed in [15, Chapters 1,2] show that (4.3) satisfy the necessary conditions in (4.1), required to have a meaningful formulated problem, i.e., the conditions $\mathcal{N}(\nabla \mathbf{u}_\pm)$, $\mathcal{N}(\pi_\pm) \in L^2(\partial\Omega)$.

Now, by imposing the transmission conditions of (4.1) to the layer potentials (4.3) and using the formulas (3.12), one obtains the equations

$$\begin{array}{l} \left(-\dfrac{1}{2}\dfrac{1+\mu}{1-\mu}\mathbb{I} + \mathbf{K}_{\lambda;\partial\Omega}\right)\mathbf{h} + \mathcal{V}_{\lambda;\partial\Omega}\mathbf{f} = -\dfrac{1}{1-\mu}\mathcal{U} \\ \left(\mathcal{H}^+_{\lambda;\partial\Omega} - \mathcal{H}^-_{\lambda;\partial\Omega}\right)\mathbf{h} - \mathbf{f} = \mathcal{F} \end{array} \quad \text{a.e. on } \partial\Omega, \tag{4.4}$$

where $\mathcal{H}^\pm_{\lambda;\partial\Omega}\mathbf{h} := \partial^\pm_{\nu_{\partial\Omega}}(\mathbf{W}_{\lambda;\partial\Omega}\mathbf{h}, \mathcal{Q}_{\lambda;\partial\Omega}\mathbf{h})$. Since $\left(\mathcal{H}^+_{\lambda;\partial\Omega} - \mathcal{H}^-_{\lambda;\partial\Omega}\right)\mathbf{h} \in \mathbb{R}\nu$ (see also (3.12)), the first equation in (4.4) takes the form

$$\left(-\frac{1}{2}\frac{1+\mu}{1-\mu}\mathbb{I} + \mathbf{K}_{\lambda;\partial\Omega}\right)\mathbf{h} = -\frac{1}{1-\mu}\mathcal{U} + \mathcal{V}_{\lambda;\partial\Omega}\mathcal{F} \text{ a.e. on } \partial\Omega, \tag{4.5}$$

[2]In the context of L^2 boundary spaces, one has $\partial_\nu^+(\mathbf{u}, \pi) = (-\pi\mathbb{I} + 2\mathrm{Def}\mathbf{u})\,\nu$.

where the right-hand side belongs to $L^2_{1;\nu}(\partial\Omega;\Lambda^1TM)$, due to the property $\mathcal{V}_{\lambda;\partial\Omega}\mathcal{F} \in L^2_{1;\nu}(\partial\Omega;\Lambda^1TM)$ (for any $\mathcal{F} \in L^2(\partial\Omega;\Lambda^1TM)$). In addition, by Theorem 3.4, the operator

$$-\frac{1}{2}\frac{1+\mu}{1-\mu}\mathbb{I} + \mathbf{K}_{\lambda;\partial\Omega} : L^2_{1;\nu}(\partial\Omega;\Lambda^1TM) \to L^2_{1;\nu}(\partial\Omega;\Lambda^1TM)$$

is invertible. Thus, there exists a unique solution $\mathbf{h} \in L^2_{1;\nu}(\partial\Omega,\Lambda^1TM)$ of the equation (4.5). Moreover, the density $\mathbf{f}$ is also unique, as it is given by the second equation in (4.4), i.e.,

$$\mathbf{f} = \left(\mathcal{H}^+_{\lambda;\partial\Omega} - \mathcal{H}^-_{\lambda;\partial\Omega}\right)\mathbf{h} - \mathcal{F} \in L^2(\partial\Omega,\Lambda^1TM).$$

Consequently, the layer potentials (4.3) determine a solution to the transmission problem (4.1), which satisfies an estimate of type (4.2). Indeed, in view of [18, Proposition 4.5, Proposition 4.10] and the boundedness properties of the operators

$$-\frac{1}{2}\frac{1+\mu}{1-\mu}\mathbb{I} + \mathbf{K}_{\lambda;\partial\Omega} : L^2_{1;\nu}(\partial\Omega,\Lambda^1TM) \to L^2_{1;\nu}(\partial\Omega,\Lambda^1TM),$$
$$\mathcal{H}^+_{\lambda;\partial\Omega} - \mathcal{H}^-_{\lambda;\partial\Omega} : L^2_{1;\nu}(\partial\Omega,\Lambda^1TM) \to L^2(\partial\Omega,\Lambda^1TM),$$

one has successively

$$\begin{aligned} \|\mathcal{N}(\nabla\mathbf{u}_\pm)\|_{L^2(\partial\Omega)} + \|\mathcal{N}(\pi_\pm)\|_{L^2(\partial\Omega)} &\le c\left(\|\mathbf{h}\|_{L^2_1(\partial\Omega,\Lambda^1TM)} + \|\mathbf{f}\|_{L^2(\partial\Omega,\Lambda^1TM)}\right) \\ &\le c\left(\|\mathcal{U}\|_{L^2_1(\partial\Omega,\Lambda^1TM)} + \|\mathcal{F}\|_{L^2(\partial\Omega,\Lambda^1TM)}\right), \end{aligned} \tag{4.6}$$

with some constant $c > 0$. Now, we show that the solution of the transmission problem (4.1) is unique (up to a constant pressure). To this aim, suppose that the pairs $(\tilde{\mathbf{u}}_\pm, \tilde{\pi}_\pm)$ satisfies the homogeneous version of (4.1). Taking into account the representations (see e.g., [3, (3.7)] in the case of the Stokes system)

$$\left.\begin{aligned} \tilde{\mathbf{u}}_+ &= \mathbf{W}_{\lambda;\partial\Omega}(\mathrm{Tr}^+\tilde{\mathbf{u}}_+) - \mathbf{V}_{\lambda;\partial\Omega}\left(\partial^+_\nu(\tilde{\mathbf{u}}_+,\tilde{\pi}_+)\right) \\ \mathbf{0} &= -\mathbf{W}_{\lambda;\partial\Omega}(\mathrm{Tr}^-\tilde{\mathbf{u}}_-) + \mathbf{V}_{\lambda;\partial\Omega}\left(\partial^-_\nu(\tilde{\mathbf{u}}_-,\tilde{\pi}_-)\right) \end{aligned}\right\} \text{ in } \Omega_+, \tag{4.7}$$

and the transmission conditions in (4.1), one finds $\tilde{\mathbf{u}}_+ = (1-\mu)\mathbf{W}_{\lambda;\partial\Omega}(\mathrm{Tr}^+\tilde{\mathbf{u}}_+)$ in Ω_+. If we apply the non-tangential boundary trace Tr^+ to both sides of this formula and use Theorem 3.4, one obtains the uniquely solvable equation

$$\left(-\frac{1}{2}\frac{1+\mu}{1-\mu}\mathbb{I} + \mathbf{K}_{\lambda;\partial\Omega}\right)\mathrm{Tr}^+\tilde{\mathbf{u}}_+ = \mathbf{0}$$

in the space $L^2_{1;\nu}(\partial\Omega,\Lambda^1TM)$. Consequently, $\mathrm{Tr}^+\tilde{\mathbf{u}}_+ = \mathbf{0}$. Therefore, the pair $(\tilde{\mathbf{u}}_+,\tilde{\pi}_+)$ is a solution of the homogeneous Dirichlet problem for the Brinkman system in Ω_+. Finally, taking into account by [10, Theorem 5.4], we conclude that $\tilde{\mathbf{u}}_+ = \mathbf{0}$ and $\tilde{\pi}_+ = 0$ (up to a constant) in Ω_+. Similar arguments as before imply that $\tilde{\mathbf{u}}_- = \mathbf{0}$ and $\tilde{\pi}_- = 0$ (up to a constant) in Ω_-. □

Next, we show the well-posedness of a transmission problem associated to the Stokes and Brinkman operators B_0 and B_λ, $\lambda > 0$:

THEOREM 4.2. *Let M be a compact boundaryless Riemannian manifold, $\dim(M) \geq 2$, $\Omega_+ := \Omega \subset M$ be a Lipschitz domain and $\Omega_- := M \setminus \overline{\Omega}$. Also let $\lambda > 0$ be a given constant. Then for any $\mu \in (0,1)$ the transmission problem*

$$\begin{cases} \delta \mathbf{u}_+ = 0, \ \mathfrak{L}\mathbf{u}_+ + \lambda \mathbf{u}_+ + d\pi_+ = 0 \ \text{in } \Omega_+, \\ \delta \mathbf{u}_- = 0, \ \mathfrak{L}\mathbf{u}_- + d\pi_- = 0 \ \text{in } \Omega_-, \\ \mathcal{N}(\nabla \mathbf{u}_\pm), \ \mathcal{N}(\pi_\pm) \in L^2(\partial\Omega), \\ \mu \mathrm{Tr}^+ \mathbf{u}_+ - \mathrm{Tr}^- \mathbf{u}_- = \mathcal{U} \in L^2_{1;\nu}(\partial\Omega, \Lambda^1 TM) \ \text{on } \partial\Omega, \\ \partial_\nu^+(\mathbf{u}_+, \pi_+) - \partial_\nu^-(\mathbf{u}_-, \pi_-) = \mathcal{F} \in L^2(\partial\Omega, \Lambda^1 TM) \ \text{on } \partial\Omega, \end{cases} \tag{4.8}$$

has a unique solution

$$((\mathbf{u}_+, \pi_+), (\mathbf{u}_-, \pi_-)) \in (C^2(\Omega_+, \Lambda^1 TM) \times C^1(\Omega_+)) \times (C^2(\Omega_-, \Lambda^1 TM) \times C^1(\Omega_-))$$

(up to a constant pressure), and, for some $C > 0$,

$$\|\mathcal{N}(\nabla \mathbf{u}_\pm)\|_{L^2(\partial\Omega)} + \|\mathcal{N}(\pi_\pm)\|_{L^2(\partial\Omega)} \leq C\big(\|\mathcal{U}\|_{L^2_1(\partial\Omega,\Lambda^1 TM)} + \|\mathcal{F}\|_{L^2(\partial\Omega,\Lambda^1 TM)}\big). \tag{4.9}$$

Proof. First, note that, in view of Theorem 3.4, the operator

$$\mathfrak{T}_\lambda : L^2_{1;\nu}(\partial\Omega, \Lambda^1 TM) \times L^2(\partial\Omega, \Lambda^1 TM) \to L^2_{1;\nu}(\partial\Omega, \Lambda^1 TM) \times L^2(\partial\Omega, \Lambda^1 TM),$$

$$\mathfrak{T}_\lambda := \begin{pmatrix} (\mu - 1)\left(-\frac{1}{2}\frac{1+\mu}{1-\mu}\mathbb{I} + \mathbf{K}_{\lambda;\partial\Omega}\right) & (\mu-1)\mathcal{V}_{\lambda;\partial\Omega} \\ \mathbf{0} & -\mathbb{I} \end{pmatrix}, \tag{4.10}$$

is Fredholm with index zero. In addition, by Theorem 3.3, the operator

$$\mathfrak{C}_{\lambda;0} : L^2_{1;\nu}(\partial\Omega, \Lambda^1 TM) \times L^2(\partial\Omega, \Lambda^1 TM) \to L^2_{1;\nu}(\partial\Omega, \Lambda^1 TM) \times L^2(\partial\Omega, \Lambda^1 TM),$$

$$\mathfrak{C}_{\lambda;0} := \begin{pmatrix} \mathbf{K}_{\lambda;0;\partial\Omega} & \mathcal{V}_{\lambda;0;\partial\Omega} \\ \mathcal{H}_{\lambda,0;\partial\Omega} & \mathbf{K}^*_{\lambda,0;\partial\Omega} - \alpha\langle \mu_{\partial\Omega}, \cdot\rangle_{\partial\Omega}\nu \end{pmatrix} \tag{4.11}$$

is compact, for any constant $\alpha \in \mathbb{R}$, where $\mu_{\partial\Omega} \in L^2(\partial\Omega, \Lambda^1 TM)$ is chosen such that $\langle \nu, \mu_{\partial\Omega}\rangle_{\partial\Omega} = 1$. Therefore, the operator

$$\mathfrak{T}_{\lambda;0} : L^2_{1;\nu}(\partial\Omega, \Lambda^1 TM) \times L^2(\partial\Omega, \Lambda^1 TM) \to L^2_{1;\nu}(\partial\Omega, \Lambda^1 TM) \times L^2(\partial\Omega, \Lambda^1 TM),$$

$$\mathfrak{T}_{\lambda;0} := \mathfrak{T}_\lambda + \mathfrak{C}_{\lambda;0} \tag{4.12}$$

is Fredholm with index zero too.

Now, choosing $\alpha \neq \zeta - 1$, where (see e.g., [3, 10])

$$\mathbf{K}^*_{\lambda,0;\partial\Omega}\nu = \zeta\nu \text{ on } \partial\Omega,$$

and using similar arguments to those for Theorem 4.1, one obtains that the operator (4.12) is injective, and hence invertible. Consequently the equation

$$\mathfrak{T}_{\lambda;0}\begin{pmatrix} \mathbf{h} \\ \mathbf{f} \end{pmatrix} = \begin{pmatrix} \mathcal{U} \\ \mathcal{F} \end{pmatrix} \in L^2_{1;\nu}(\partial\Omega, \Lambda^1 TM) \times L^2(\partial\Omega, \Lambda^1 TM) \tag{4.13}$$

has a unique solution $(\mathbf{h}, \mathbf{f})^\top \in L^2_{1;\nu}(\partial\Omega, \Lambda^1 TM) \times L^2(\partial\Omega, \Lambda^1 TM)$. Then the layer potentials

$$\mathbf{u}_+ = \mathbf{W}_{\lambda;\partial\Omega}\mathbf{h} + \mathbf{V}_{\lambda;\partial\Omega}\mathbf{f}, \quad \pi_+ = \mathcal{Q}^d_{\lambda;\partial\Omega}\mathbf{h} + \mathcal{Q}^s_{\lambda;\partial\Omega}\mathbf{f} \text{ in } \Omega_+, \tag{4.14}$$

$$\mathbf{u}_- = \mathbf{W}_{0;\partial\Omega}\mathbf{h} + \mathbf{V}_{0;\partial\Omega}\mathbf{f}, \ \pi_- = \mathcal{Q}^d_{0;\partial\Omega}\mathbf{h} + \mathcal{Q}^s_{0;\partial\Omega}\mathbf{f} + \alpha\langle \mu_{\partial\Omega}, \mathbf{f}\rangle_{\partial\Omega} \text{ in } \Omega_-, \tag{4.15}$$

determine a solution $((\mathbf{u}_+, \pi_+), (\mathbf{u}_-, \pi_-))$ of the transmission problem (4.8). Note that the system of equations that follow by applying the transmission conditions in (4.8) to the layer potentials (4.14) and (4.15) is equivalent to the matrix type equation (4.13), as can be observed by means of the formulas (3.12). In addition, the solution $((\mathbf{u}_+, \pi_+), (\mathbf{u}_-, \pi_-))$ is unique (up to a constant pressure) and satisfy the estimate (4.9). For brevity, we omit the details, but they will be given in a forthcoming paper. □

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