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SECOND ORDER DIFFERENTIAL SUBORDINATIONS AND SUPERORDINATIONS USING THE DZIOK-SRIVASTAVA LINEAR OPERATOR

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Abstract. By using the properties of the Dziok-Srivastava linear operator we obtain differential subordinations and superordinations by using functions from class A.

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1. INTRODUCTION AND PRELIMINARIES

Let U denote the unit disc of the complex plane $U = \{z \in \mathbb{C} : \ |z| < 1\}$ and $\overline{U} = \{z \in \mathbb{C} : \ |z| \leq 1\}$.

Let $\mathcal{H}(U)$ denote the space of holomorphic functions in U and let

$$A_n = \{f \in \mathcal{H}(U), \ f(z) = z + a_{n+1}z^{n+1} + \dots, \ z \in U\}$$

with $A_1 = A$.

Let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U), \ f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots, \ z \in U\},$$

$$S = \{f \in A; \ f \text{ is univalent in } U\}.$$

If f and g are analytic functions in U, then we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g[w(z)]$, for $z \in U$.

If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

The method of differential subordinations (also known as the admissible functions method) was introduced by P.T. Mocanu and S.S. Miller in 1978 [2] and 1981 [3] and developed in [4].

Let Ω and Δ be any sets in $\mathbb{C}$ and let p be an analytic function in the unit disk with $p(0) = a$ and let $\psi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$. The heart of this theory deals with generalizations of the following implication:

(i) $\{\psi(p(z), zp'(z), z^2p''(z); z) \mid z \in U\} \subset \Omega$ implies $p(U) \subset \Delta$.

Definition 1 ([4, p.16]). Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the (second-order) differential subordination

(ii) $\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z)$, $z \in U$,

then p is called a solution of the differential subordination.

The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (ii). A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (ii) is said to be the best dominant of (ii). (Note that the best dominant is unique up to a rotation of U.)

In [5] the authors introduced the dual problem of the differential subordination which they called differential superordination.

Definition 2 ([5]). Let $f, F \in \mathcal{H}(U)$ and let F be univalent in U. The function F is said to be superordinate to f, or f is subordinate to F, written $f \prec F$, if $f(0) = F(0)$ and $f(U) \subset F(U)$.

Let Ω and Δ be any sets in $\mathbb{C}$ and let p be an analytic function in the unit disc and function $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$. The heart of this theory deals with generalizations of the following implication:

(j) $\Omega \subset \{\varphi(p(z), zp'(z), z^2p''(z); z) \mid z \in U\}$ implies $\Delta \subset p(U)$.

Definition 3 ([5]). Let $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be analytic in U.

If p and $\varphi(p(z), zp'(z), z^2p''(z); z)$ are univalent in U and satisfy the (second-order) differential superordination

(jj) $h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z)$

then p is called a solution of the differential superordination.

An analytic function q is called a subordinant of the solution of the differential superordination, or more simply a subordinant if $q \prec p$ for all p satisfying (jj). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (jj) is said to be the best subordinant. (Note that the best subordinant is unique up to a rotation of U).

Definition 4 ([5, Definition 2.2b. p. 21]). We denote by Q the set of functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \left\{ \zeta \in \partial U;\ \lim_{z \to \zeta} f(z) = \infty \right\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$.

Definition 5 ([3, 4, 5, Definition 2.3.a, p. 27]). Let Ω be a set in $\mathbb{C}$, $q \in Q$ and n be a positive integer. *The class of admissible functions* $\psi_n[\Omega, q]$, consists of those functions $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the *admissibility condition*:

$$\text{(A)} \quad \psi(r, s, t; z) \notin \Omega$$

whenever $r = q(\zeta)$, $s = m\zeta q'(\zeta)$, $\operatorname{Re} \left(\frac{t}{s} + 1 \right) \geq m \operatorname{Re} \left[\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right]$, $z \in U$, $\zeta \in \partial U \setminus E(q)$ and $m \geq n$.

In particular $q(0) = 1$, $\Omega = \Delta = q(U) = \{w \in \mathbb{C} :\ \operatorname{Re} w > 0\}$ and we denote the class $\psi_n[\Omega, q]$ by $\psi_n\{1\}$. Condition of admissibility (A) becomes

(A') $\psi(\rho \mathrm{i}, \sigma, \mu + \nu i; z) \notin \Omega$, when $\rho, \sigma, \mu, \nu \in \mathbb{R}$, $\sigma \leq -\frac{n}{2}(1 + \rho^2)$, $\sigma + \mu \leq 0$, $z \in U$ and $n \geq 1$.

Definition 6 ([5, Definition 3]). Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}[a,n]$ with $q'(z) \neq 0$. The class of admissible functions $\phi_n[\Omega, q]$, consists of those functions $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition:

(B) $\varphi(r,s,t;\zeta) \in \Omega$ whenever $r = q(z)$, $s = \dfrac{zq'(z)}{m}$, $\mathrm{Re}\left(\dfrac{t}{s}+1\right) \leq \dfrac{1}{m}\mathrm{Re}\left[\dfrac{zq''(z)}{q'(z)}+1\right]$, where $\zeta \in \partial U$, $z \in U$ and $m \geq n \geq 1$. When $n = 1$ we write $\phi_1[\Omega, q]$ as $\phi[\Omega, q]$

In the special case when h is an analytic mapping of U onto $\Omega \neq \mathbb{C}$, we denote this class $\phi_n[h(U), q]$ by $\phi_n[h, q]$.

In order to prove the new results we shall use the following lemmas:

Lemma A ([4, Lemma 2.2.d, p. 24]). *Let $q \in Q$ with $q(0) = a$, and let $p(z) = a + a_n z^n + \ldots$ be analytic in U with $p(z) \not\equiv a$ and $n \geq 1$. If p is not subordinate to q, then there exist points $z_0 = r_0 e^{i\theta_0} \in U$ and $\zeta_0 \in \partial U \setminus E(q)$, and an $m \geq n \geq 1$ for which $p(U_{r_0}) \subset q(U)$,*

(i) $p(z_0) = q(\zeta_0)$

(ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$ *and*

(iii) $\mathrm{Re}\ \dfrac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq m\mathrm{Re}\ \left[\dfrac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)}+1\right]$.

We will use the following Lemma A from the theory of differential subordination to determine subordinants of diferential superordination.

Lemma B ([5, Lemma A]). *Let $p \in Q(a)$ and let $q(z) = a + a_n z^n + \ldots$ be analytic in U with $q(z) \not\equiv a$ and $n \geq 1$. If q is not subordinate to p, then there exist points $z_0 = r_0 e^{i\theta_0} \in U$ and $\zeta_0 \in \partial U \setminus E(p)$, and an $m \geq n \geq 1$ for which $q(U_{r_0}) \subset p(U)$,*

(i) $q(z_0) = p(\zeta_0)$

(ii) $z_0 q'(z_0) = m\zeta_0 p'(\zeta_0)$ *and*

(iii) $\mathrm{Re}\ \dfrac{z_0 q''(z_0)}{q'(z_0)} + 1 \geq m\mathrm{Re}\ \left[\dfrac{\zeta_0 p''(\zeta_0)}{p'(\zeta_0)}+1\right]$.

In [1] the Dziok-Srivastava operator was defined:

$$H_m^l(\alpha_1,\alpha_2,\ldots,\alpha_l;\beta_1,\beta_2,\ldots,\beta_m)f(z) = z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}\ldots(\alpha_l)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}\ldots(\beta_m)_{n-1}} a_n \frac{z^n}{(n-1)!}, \tag{1}$$

$\alpha_i \in \mathbb{C}$, $i = 1,2,\ldots,l$, $\beta_j \in \mathbb{C} - \{0,-1,-2,\ldots\}$, $j = 1,2,\ldots,m$.

For simplicity, we write

$$H_m^l[\alpha_1]f(z) = H_m^l(\alpha_1,\alpha_2,\ldots,\alpha_l;\beta_1,\beta_2,\ldots,\beta_m)f(z). \tag{2}$$

For this operator we have the property

$$\alpha_1 H_m^l[\alpha_1+1]f(z) = z\{H_m^l[\alpha_1]f(z)\}' + (\alpha_1-1)H_m^l[\alpha_1]f(z). \tag{3}$$

2. MAIN RESULTS

2.1. Differential subordinations in the right half-plane

THEOREM 1. *Let* $\psi \in \psi_n\{1\}$, $\alpha_1 > 0$, $f \in A$ *and let* $\frac{H_m^l[\alpha_1]f(z)}{z} \in \mathcal{H}[1,1]$, *where* $H_m^l[\alpha_1]f(z)$ *is the Dziok-Srivastava linear operator given by (1). Let*

$$\begin{aligned}&\psi\left(\frac{H_m^l[\alpha_1]f(z)}{z}, \frac{H_m^l[\alpha_1+1]f(z)}{z}, \frac{H_m^l[\alpha_1+2]f(z)}{z}; z\right)\\ &= \frac{H_m^l[\alpha_1]f(z)}{z} + \frac{H_m^l[\alpha_1+1]f(z)}{z} + \frac{H_m^l[\alpha_1+2]f(z)}{z}, \quad z \in U.\end{aligned} \tag{4}$$

If

$$\left\{\psi\left(\frac{H_m^l[\alpha_1]f(z)}{z}, \frac{H_m^l[\alpha_1+1]f(z)}{z}, \frac{H_m^l[\alpha_1+2]f(z)}{z}; z\right)\right\} \subset \Omega, \tag{5}$$

then

$$\frac{H_m^l[\alpha_1]f(z)}{z} \prec q(z), \quad z \in U.$$

Proof. If we let

$$\begin{aligned}p(z) &= \frac{H_m^l[\alpha_1]f(z)}{z}\\ &= \frac{z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}\dots(\alpha_l)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}\dots(\beta_m)_{n-1}} \cdot a_n \cdot \frac{z^n}{(n-1)!}}{z}\\ &= 1 + p_1 z + p_2 z^2 + \dots,\end{aligned} \tag{6}$$

then $p \in \mathcal{H}[1,1]$.

Differentiating (6), we obtain

$$\{H_m^l[\alpha_1]f(z)\}' = p(z) + zp'(z), \quad z \in U. \tag{7}$$

Using (3) and (6) and (7) we have

$$\frac{H_m^l[\alpha_1]f(z)}{z} = p(z) + \frac{zp'(z)}{\alpha_1}, \tag{8}$$

$$\{H_m^l[\alpha_1+1]f(z)\}' = p(z) + \frac{(\alpha_1+2)zp'(z)}{\alpha_1} + \frac{z^2p''(z)}{\alpha_1} \tag{9}$$

$$\frac{H_m^l[\alpha_1+2]f(z)}{z} = \frac{\alpha_1(\alpha_1+1)p(z) + (2\alpha_1+2)zp'(z) + z^2p''(z)}{\alpha_1(\alpha_1+1)}. \tag{10}$$

Using (6), (8) and (10), (5) becomes

$$\psi\left(\frac{H_m^l[\alpha_1]f(z)}{z},\frac{H_m^l[\alpha_1+1]f(z)}{z},\frac{H_m^l[\alpha_1+2]f(z)}{z};z\right) \tag{11}$$
$$= r(p(z),zp'(z),z^2p''(z);z) = 3p(z)+\frac{3zp'(z)}{\alpha_1}+\frac{z^2p''(z)}{\alpha_1(\alpha_1+1)}.$$

Then (5) becomes

$$\{r(p(z),zp'(z),z^2p''(z);z)\}\subset\Omega=\{w\in\mathbb{C}:\ \operatorname{Re} w>0\} \tag{12}$$

which is equivalent to

$$\operatorname{Re} r(p(z),zp'(z),z^2p''(z);z)>0,\quad z\in U. \tag{13}$$

Assume $p(z)\not\prec q(z)$. By Lemma A there exist points $z_0=r_0e^{i\theta_0}\in U$ and $\zeta_0\in\partial U\setminus E(q)$, and $m\geq n\geq 1$ that satisfy (i)-(iii) of Lemma A,

$$p(z_0)=q(\zeta_0)=\rho\mathrm{i},\quad z_0p'(z_0)=m\zeta_0q'(\zeta_0)=\sigma,\quad z_0^2p''(z_0)=\mu+\mathrm{i}\nu.$$

Using these conditions with $r=q(\zeta_0)=\rho\mathrm{i}$, $\sigma=m\zeta_0q'(\zeta_0)$, $t=\mu+\mathrm{i}\nu$, $\rho,\sigma,\mu,\nu\in\mathbb{R}$ and $z=z_0$ in Definition 5, we obtain

$$\operatorname{Re} r(p(z_0),z_0p'(z_0),z_0^2p''(z_0);z_0)=\operatorname{Re} r(\rho\mathrm{i},\sigma,\mu+\mathrm{i}\nu;z_0)$$
$$=\operatorname{Re}\left[3\rho\mathrm{i}+\frac{3\sigma}{\alpha_1}+\frac{\mu+\mathrm{i}\nu}{\alpha_1(\alpha_1+1)}\right]=\frac{3\sigma}{\alpha_1}+\frac{\mu}{\alpha_1(\alpha_1+1)}\leq\frac{3\sigma}{\alpha_1}-\frac{\sigma}{\alpha_1(\alpha_1+1)}$$
$$=\frac{\sigma(3\alpha_1+3-1)}{\alpha_1(\alpha+1)}\leq-\frac{1}{2}(1+\rho^2)\cdot\frac{3\alpha_1+2}{\alpha_1(\alpha_1+1)}<0.$$

Since this contradicts (13) we must have $p(z)=\dfrac{H_m^l[\alpha_1]f(z)}{z}\prec q(z)$. □

REMARK 1. Upon examining the proof of Theorem 1, it is easy to see that the theorem also holds if condition (5) is replaced by

$$\left\{\psi\left(\frac{H_m^l[\alpha_1]f(w(z))}{w(z)},\frac{H_m^l[\alpha_1+1]f(w(z))}{w(z)},\frac{H_m^l[\alpha_1+2]f(w(z))}{w(z)};w(z)\right)\right\}\subset\Omega, \tag{14}$$

where $w(z)$ is any function mapping U into U.

We next consider the special situation when $\Omega=\{w\in\mathbb{C}:\ \operatorname{Re} w>0\}$ is a simply connected domain. In this case $\Omega=h(U)$, where h is a conformal mapping of U onto Ω and the class $\Psi_n[h(U),q]$ is written as $\Psi_n[h,q]$. The following result is an immediate consequence of Theorem 1.

COROLLARY 1. *Let $\psi\in\Psi_n[h,q]$ with $q(0)=1$, $\alpha_1>0$. If*

$$\frac{H_m^l[\alpha_1]f(z)}{z}\in\mathcal{H}[1,1],$$

$\psi\left(\frac{H_m^l[\alpha_1]f(z)}{z}, \frac{H_m^l[\alpha_1+1]f(z)}{z}, \frac{H_m^l[\alpha_1+2]f(z)}{z}; z\right)$ *is analytic in* U*, and*

$$\psi\left(\frac{H_m^l[\alpha_1]f(z)}{z}, \frac{H_m^l[\alpha_1+1]f(z)}{z}, \frac{H_m^l[\alpha_1+2]f(z)}{z}; z\right) \prec h(z), \tag{15}$$

then

$$\frac{H_m^l[\alpha_1]f(z)}{z} \prec q(z), \quad z \in U.$$

Example 1. Let

$$h(z) = \frac{1-z}{1+z}, \quad h(U) = \{w \in \mathbb{C} : \text{ Re } w > 0\} = \Omega.$$

If $\psi\left(\frac{H_m^l[\alpha_1]f(z)}{z}, \frac{H_m^l[\alpha_1+1]f(z)}{z}, \frac{H_m^l[\alpha_1+2]f(z)}{z}\right) \prec \frac{1-z}{1+z}$, then $\frac{H_m^l[\alpha_1]f(z)}{z} \prec q(z)$, $z \in U$, i.e. $1 + p_1 z + p_2 z^2 + \cdots \prec q(z)$, $z \in U$.

This result can be extended to those cases in which the behavior of q on the boundary of U is unknown by the following theorem.

Theorem 2. *Let* h *and* q *be univalent in* U*, with* $q(0) = 1$*, and set* $q_\rho(z) = q(\rho z)$ *and* $h_\rho(z) = h(\rho z)$*. Let* $\Psi : \mathbb{C}^3 \times U \to \mathbb{C}$ *satisfy one of the following conditions:*

(i) $\psi \in \Psi_n[h, q_\rho]$ *for some* $\rho \in (0, 1)$*, or*

(ii) there exists $\rho_0 \in (0, 1)$ *such that* $\psi \in \Psi_n[h_\rho, q_\rho]$*, for all* $\rho \in (\rho_0, 1)$*.*

If $\frac{H_m^l[\alpha_1]f(z)}{z} \in \mathcal{H}[1, 1]$*,*

$$\psi\left(\frac{H_m^l[\alpha_1]f(z)}{z}, \frac{H_m^l[\alpha_1+1]f(z)}{z}, \frac{H_m^l[\alpha_1+2]f(z)}{z}; z\right)$$

is analytic in U*, and*

$$\psi\left(\frac{H_m^l[\alpha_1]f(z)}{z}, \frac{H_m^l[\alpha_1+1]f(z)}{z}, \frac{H_m^l[\alpha_1+2]f(z)}{z}; z\right) \prec h(z), \tag{16}$$

then

$$\frac{H_m^l[\alpha_1]f(z)}{z} \prec q(z), \quad z \in U.$$

Proof. Case (i). By applying Theorem 1 we obtain $\frac{H_m^l[\alpha_1]f(z)}{z} \prec q_\rho(z)$.

Since $q_\rho(z) \prec q(z)$, we deduce $\frac{H_m^l[\alpha_1]f(z)}{z} \prec q(z), \quad z \in U.$

Case (ii). If we let

$$\frac{H_m^l[\alpha_1]f(\rho z)}{\rho z} = p(\rho z) = p_\rho(z),$$

then

$$\psi(p_\rho(z), zp'_\rho(z), z^2 p''_\rho(z); \rho z) = \psi(p(\rho z), \rho z p'(\rho z), \rho^2 z^2 p''(\rho z); \rho z) \in h_\rho(U).$$

By using Theorem 1 and the comment associated with (14), with $w(z) = \rho z$, we obtain $p_\rho(z) \prec q_\rho(z)$, for $\rho \in (\rho_0, 1)$.

By letting $\rho \to 1$ we obtain $\frac{H_m^l[\alpha_1]f(z)}{z} \prec q(z), \quad z \in U.$ □

The next two theorems yield best dominants of the differential subordination (16).

THEOREM 3. *Let h be univalent in U and let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$. Suppose that the differential equation*

$$\psi(q(z), zq'(z), z^2q''(z); z) = h(z), \tag{17}$$

has a solution q, with $q(0) = 1$, and one of the following conditions is satisfied:

(i) *$q \in Q$ and $\psi \in \Psi[h, q]$.*

(ii) *q is univalent in U and $\psi \in \Psi[h, q_\rho]$, for some $\rho \in (0, 1)$, or*

(iii) *q is univalent in U and there exists $\rho_0 \in (0, 1)$ such that $\psi \in \Psi[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

If

$$\frac{H_m^l[\alpha_1]f(z)}{z} \in \mathcal{H}[1, 1]$$

and $\psi\left(\frac{H_m^l[\alpha_1]f(z)}{z}, \frac{H_m^l[\alpha_1+1]f(z)}{z}, \frac{H_m^l[\alpha_1+2]f(z)}{z}; z\right)$ *is analytic in U, and if*

$$\psi\left(\frac{H_m^l[\alpha_1]f(z)}{z}, \frac{H_m^l[\alpha_1+1]f(z)}{z}, \frac{H_m^l[\alpha_1+2]f(z)}{z}; z\right) \prec h(z), \tag{18}$$

then

$$\frac{H_m^l[\alpha_1]f(z)}{z} \prec q(z),$$

and q is best dominant.

Proof. By applying Corollary 1 and Theorem 2 we deduce that q is a dominant of (18). Since q satisfies (17) it is a solution of (18) and therefore q will be dominated by all dominants of (18). Hence q will be the best dominant of (18). □

2.2. Differential superordinations in the right half-plane

THEOREM 4. *Let $\Omega \subset \{w \in \mathbb{C} : \text{ Re } w > 0\}$, $\alpha_1 > 0$ let $q \in \mathcal{H}[1, n]$ and let $\varphi \in \phi_n[\Omega, q]$. If $\frac{H_m^l[\alpha_1]f(z)}{z} \in Q(1)$ and*

$$\begin{aligned}&\varphi\left(\frac{H_m^l[\alpha_1]f(z)}{z}, \frac{H_m^l[\alpha_1+1]f(z)}{z}, \frac{H_m^l[\alpha_1+2]f(z)}{z}; z\right)\\ &= \frac{H_m^l[\alpha_1]f(z)}{z} + \frac{H_m^l[\alpha_1+1]f(z)}{z} + \frac{H_m^l[\alpha_1+2]f(z)}{z}\end{aligned} \tag{19}$$

is univalent in U*, then*

$$\Omega \subset \left\{\varphi\left(\frac{H_m^l[\alpha_1]f(z)}{z}, \frac{H_m^l[\alpha_1+1]f(z)}{z}, \frac{H_m^l[\alpha_1+2]f(z)}{z}; z\right)\right\} \tag{20}$$

implies

$$q(z) \prec \frac{H_m^l[\alpha_1]f(z)}{z}, \quad z \in U,$$

where $H_m^l[\alpha_1]f(z)$ *is given by* (1).

Proof. Using (6), (8), (10), we have

$$\begin{aligned}&\varphi\left(\frac{H_m^l[\alpha_1]f(z)}{z}, \frac{H_m^l[\alpha_1+1]f(z)}{z}, \frac{H_m^l[\alpha_1+2]f(z)}{z}; z\right)\\ &= v(p(z), zp'(z), z^2p''(z); z) = 3p(z) + \frac{3zp'(z)}{\alpha_1} + \frac{z^2p''(z)}{\alpha_1(\alpha_1+1)}.\end{aligned} \tag{21}$$

Then (20) becomes

$$\Omega \subset v(p(z), zp'(z), z^2p''(z); z), \quad z \in U. \tag{22}$$

Assume

$$q(z) \not\prec p(z) = \frac{H_m^l[\alpha_1]f(z)}{z}, \quad p \in \mathcal{H}[1,1].$$

By Lemma B there exist points $z_0 = r_0e^{i\theta_0} \in U$ and $\zeta_0 \in \partial U \setminus E(q)$ and $m \geq n \geq 1$, that satisfy conditions (i)-(iii) of Lemma B. Using these conditions with $r = q(z_0) = p(\zeta_0)$, $s = z_0q'(z_0) = m\zeta_0p'(\zeta_0)$, $t = \zeta_0^2p''(\zeta_0)$ and $\zeta = \zeta_0$ in Definition 6, we obtain

$$\begin{aligned}&\varphi\left(\frac{H_m^l[\alpha_1]f(z_0)}{z_0}, \frac{H_m^l[\alpha_1+1]f(z_0)}{z_0}, \frac{H_m^l[\alpha_1+2]f(z_0)}{z_0}; \zeta_0\right)\\ &= v(p(\zeta_0), \zeta_0p'(\zeta_0), \zeta_0^2p''(\zeta_0); \zeta_0) \in \Omega.\end{aligned} \tag{23}$$

Since (23) contradicts (22), we must have $q(z) \prec \dfrac{H_m^l[\alpha_1]f(z)}{z}, \quad z \in U.$ □

We next consider the special situation when h is analytic in U and $h(U) = \Omega \neq \mathbb{C}$. In this case, the class $\phi_n[h(U), q]$ is written as $\phi_n[h, q]$ and the following result is an immediate consequence of Theorem 4.

THEOREM 5. *Let* $q \in \mathcal{H}[1,n]$*,* h *be analytic in* U*,* $\alpha_1 > 0$ *and* $\varphi \in \phi_n[h,q]$*. If*

$$\frac{H_m^l[\alpha_1]f(z)}{z} \in Q(1),$$

and $\varphi\left(\dfrac{H_m^l[\alpha_1]f(z)}{z}, \dfrac{H_m^l[\alpha_1+1]f(z)}{z}, \dfrac{H_m^l[\alpha_1+2]f(z)}{z}; z\right)$ *is univalent in* U*, then*

$$h(z) \prec \varphi\left(\frac{H_m^l[\alpha_1]f(z)}{z}, \frac{H_m^l[\alpha_1+1]f(z)}{z}, \frac{H_m^l[\alpha_1+2]f(z)}{z}; z\right) \tag{24}$$

implies

$$q(z) \prec \frac{H_m^l[\alpha_1]f(z)}{z}, \quad z \in U,$$

where $H_m^l[\alpha_1]f(z)$ *is given by* (1).

EXAMPLE 2. Let

$$h(z) = \frac{1 + (1 - 2\alpha)z}{1 - z}, \quad 0 \leq \alpha < 1,\ h(U) = \{w \in \mathbb{C};\ \mathrm{Re}\ w > \alpha\}.$$

From Theorem 5, if

$$\frac{1 + (1 - 2\alpha)z}{1 - z} \prec \varphi\left(\frac{H_m^l[\alpha_1]f(z)}{z}, \frac{H_m^l[\alpha_1 + 1]f(z)}{z}, \frac{H_m^l[\alpha_1 + 2]f(z)}{z}; z\right),$$

then $q(z) \prec \dfrac{H_m^l[\alpha_1]f(z)}{z} = 1 + p_1 z + p_2 z^2 + \ldots,\ z \in U.$

Theorem 4 and Theorem 5 can only be used to obtain subordinants of a differential superordination of the form (20) and (24). The following theorem proves the existence of the best subordinant of (24) for certain φ and also provides a method for finding the best subordinant.

THEOREM 6. *Let h be analytic in U and let $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$. Suppose that the differential equation*

$$\varphi(q(z), zq'(z); z^2 q''(z); z) = h(z) \tag{25}$$

has a solution $q \in Q(1)$. If $\varphi \in \phi[h, q]$, $\dfrac{H_m^l[\alpha_1]f(z)}{z} \in Q(1)$ and $\varphi\left(\dfrac{H_m^l[\alpha_1]f(z)}{z}, \dfrac{H_m^l[\alpha_1 + 1]f(z)}{z}, \dfrac{H_m^l[\alpha_1 + 2]f(z)}{z}; z\right)$ is univalent in U, then

$$h(z) \prec \varphi\left(\frac{H_m^l[\alpha_1]f(z)}{z}, \frac{H_m^l[\alpha_1 + 1]f(z)}{z}, \frac{H_m^l[\alpha_1 + 2]f(z)}{z}; z\right) \tag{26}$$

implies

$$q(z) \prec \frac{H_m^l[\alpha_1]f(z)}{z}, \quad z \in U,$$

where $H_m^l[\alpha_1]f(z)$ *is given by* (1) *and q is the best dominant.*

Proof. Since $\varphi \in \phi[h, q]$, by applying Theorem 5 we deduce that q is a subordinant of (26). Since q also satisfies (25) it is also a solution of the differential equation (26) and therefore all subordinants of (26) will be subordinate to q. Hence q will be the best subordinant of (26).

REMARK. The conclusion of the theorem can be written in the symmetric form $\varphi(q(z), zq'(z), z^2 q''(z); z) \prec \varphi(q(z), zq'(z), z^2 q''(z); z)$ implies $q(z) \prec p(z)$, $z \in U$.

EXAMPLE 3. If $\alpha_1 = 1$, $m = 0$, $l = 1$ we have

$$H_0^1[1]f(z) = f(z), \quad H_0^1[2]f(z) = zf'(z), \quad H_0^1[3]f(z) = \frac{z^2 f''(z)}{2} + zf'(z).$$

Then

$$\frac{H_0^1[1]f(z)}{z} = \frac{f(z)}{z}, \quad \frac{H_0^1[2]f(z)}{z} = f'(z), \quad \frac{H_0^1[3]f(z)}{z} = \frac{zf''(z)}{2} + f'(z)$$

and

$$\varphi\left(\frac{H_0^1[1]f(z)}{z}, \frac{H_0^1[2]f(z)}{z}, \frac{H_0^1[3]f(z)}{z}; z\right) = \frac{f(z)}{z} + 2f'(z) + \frac{zf''(z)}{2}.$$

Let $z = \cos\alpha + i\sin\alpha$, $q(z) = 1 + z$ from which we deduce

$$q(\cos\alpha + i\sin\alpha) = 1 + \cos\alpha + i\sin\alpha, \quad q(U) \subset \{w \in \mathbb{C};\ \mathrm{Re}\ w > 0\}$$

and

$$h(z) = \frac{q(z)}{z} + 2q'(z) + \frac{zq''(z)}{z} = \frac{1+z}{z} + 2 = \frac{1+3z}{z},$$

$$h(\cos\alpha + i\sin\alpha) = \cos\alpha + 3 - i\sin\alpha,$$

$$h(U) = \{w \in \mathbb{C};\ \mathrm{Re}\ w > 3 + \cos\alpha\} \subset \{w \in \mathbb{C};\ \mathrm{Re}\ w > 0\}.$$

If $\frac{1+3z}{z} \prec \frac{f(z)}{2} + 2f'(z) + \frac{zf''(z)}{z}$, then we deduce

$$q(z) = 1 + z \prec 1 + a_2 z + a_3 z^2 + \dots, \quad z \in U.$$

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